Research Article

Cyril Imbert, Tianling Jin* and Luis Silvestre

Hölder gradient estimates for a class of singular or degenerate parabolic equations

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Abstract: We prove interior Hölder estimates for the spatial gradients of the viscosity solutions to the singular or degenerate parabolic equation

$$u_t = |\nabla u|^\kappa \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

where $p \in (1, \infty)$ and $\kappa \in (1-p, \infty)$. This includes the from L^∞ to $C^{1,\alpha}$ regularity for parabolic p-Laplacian equations in both divergence form with $\kappa = 0$, and non-divergence form with $\kappa = 2-p$.

Keywords: Regularity, p-Laplacian, degenerate parabolic equations, singular parabolic equations

MSC 2010: 35B65, 35K92, 35K65, 35K67

1 Introduction

Let $1 < p < \infty$ and $\kappa \in (1-p, \infty)$. We are interested in the regularity of solutions of

$$u_t = |\nabla u|^\kappa \operatorname{div}(|\nabla u|^{p-2}\nabla u). \tag{1.1}$$

When $\kappa = 0$, this is the classical parabolic p-Laplacian equation in divergence form. This is the natural case in the context of gradient flows of Sobolev norms. Hölder estimates for the spatial gradient of their weak solutions (in the sense of distribution) were obtained by DiBenedetto and Friedman in [8] (see also Wiegner [26]).

When $\kappa = 2-p$, equation (1.1) is a parabolic homogeneous p-Laplacian equation. This is the most relevant case for applications to tug-of-war-like stochastic games with white noise; see Peres and Sheffield [22]. This equation has been studied by Garofalo [10], Banerjee and Garofalo [3–5], Does [9], Manfredi, Parviainen and Rossi [19, 20], Rossi [23], Juutinen [15], Kawohl, Krömer and Kurtz [16], Liu and Schikorra [18], Rudd [24] as well as the second and third authors of this paper [14]. Hölder estimates for the spatial gradient of their solutions were proved in [14]. The solution of this equation is understood in the viscosity sense. The toolbox of methods that one can apply are completely different to the variational techniques used classically for p-Laplacian problems.

Equation (1.1) can be rewritten as

$$u_t = |\nabla u|^\gamma(\Delta u + (p-2)|\nabla u|^{-2}u_i u_j u_{ij}), \tag{1.2}$$

where $\gamma = p + \kappa - 2 > -1$. In this paper, we prove Hölder estimates for the spatial gradients of viscosity solutions to (1.2) for $1 < p < \infty$ and $\gamma \in (-1, \infty)$. Therefore, it provides a unified approach for all those γ and p, including the two special cases $\gamma = 0$ and $\gamma = p-2$ mentioned above.

Cyril Imbert: Department of Mathematics and Applications, CNRS & École Normale Supérieure (Paris), 45 rue d'Ulm, 75005 Paris, France, e-mail: cyril.imbert@ens.fr
***Corresponding author: Tianling Jin:** Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, e-mail: tianlingjin@ust.hk
Luis Silvestre: Department of Mathematics, The University of Chicago, 5734 S. University Avenue, Chicago, IL 60637, USA, e-mail: luis@math.uchicago.edu

The viscosity solutions to (1.2) with $\gamma > -1$ and $p > 1$ fall into the general framework studied by Ohnuma and Sato in [21], which is an extension of the work of Barles and Georgelin [6] and Ishii and Souganidis [13] on the viscosity solutions of singular/degenerate parabolic equations. We postpone the definition of viscosity solutions of (1.2) to Section 5. For $r > 0$, by Q_r we denote $B_r \times (-r^2, 0]$, where $B_r \subset \mathbb{R}^n$ is the ball of radius r centered at the origin.

Theorem 1.1. *Let u be a viscosity solution of (1.2) in Q_1, where $1 < p < \infty$ and $\gamma \in (-1, \infty)$. Then there exist two constants $\alpha \in (0,1)$ and $C > 0$, both of which depend only on n, γ, p and $\|u\|_{L^\infty(Q_1)}$, such that*

$$\|\nabla u\|_{C^\alpha(Q_{1/2})} \leq C.$$

Also, the following Hölder regularity in time holds:

$$\sup_{(x,t),(x,s)\in Q_{1/2}} \frac{|u(x,t) - u(x,s)|}{|t-s|^{(1+\alpha)/(2-\alpha\gamma)}} \leq C.$$

Note that $(1+\alpha)/(2-\alpha\gamma) > \frac{1}{2}$ for every $\alpha > 0$ and $\gamma > -1$.

Our proof in this paper follows a similar structure as in [14], with some notable differences that we explain below. We use non-divergence techniques in the context of viscosity solutions. The classical variational methods can only be used for $\gamma = p - 2$, when the equation is in divergence form. Theorem 1.1 tells us that our techniques are in some sense stronger when dealing with the regularity of scalar p-Laplacian-type equations. The weakness of our methods (at least as of now) is that they are ineffective for systems.

The result in [14] has recently been extended to allow for a bounded right-hand side of the equation by Attouchi and Parviainen in [1]. We have not explored the possibility of adding a right-hand side for arbitrary values of the exponent κ.

The greatest difficulty extending the result in [14] to Theorem 1.1 comes from the lack of uniform ellipticity. When $\gamma = 0$, equation (1.2) is a parabolic equation in non-divergence form with uniformly elliptic coefficients (depending on the solution u). Because of this, in [14], we use the theory developed by Krylov and Safonov, and other classical results, to get some basic uniform a priori estimates. This fact is no longer true for other values of γ. The first step in our proof is to obtain a Lipschitz modulus of continuity. That step uses the uniform ellipticity very strongly in [14]. In this paper, we take a different approach using the method of Ishii and Lions [12] (see also [11, Theorem 5]). Another step where the uniform ellipticity plays a strong role is in a lemma which transfers an oscillation bound in space, for every fixed time, to a space-time oscillation. In this paper, that is achieved through Lemmas 4.4 and 4.5, which are considerably more difficult than their counterpart in [14]. Other, more minor, difficulties include the fact that the non-homogeneous right-hand side forces us to work with a different scaling (see the definition of Q_r^ρ by the beginning of Section 4).

In order to avoid some of the technical difficulties caused by the non-differentiability of viscosity solutions, we first consider the regularized problem (1.3) below, and then obtain uniform estimates so that we can pass to the limit in the end. For $\varepsilon \in (0,1)$, let u be smooth and satisfy

$$\partial_t u = (|\nabla u|^2 + \varepsilon^2)^{\frac{\gamma}{2}} \Big(\delta_{ij} + (p-2)\frac{u_i u_j}{|\nabla u|^2 + \varepsilon^2}\Big) u_{ij}. \tag{1.3}$$

We are going to establish Lipschitz estimates and Hölder gradient estimates for u, which will be independent of $\varepsilon \in (0,1)$, in Sections 2, 3 and 4. Then, in Section 5, we recall the definition of viscosity solutions to (1.2) as well as their several useful properties, and prove Theorem 1.1 via approximation arguments. This idea of approximating the problem with a smoother one and proving uniform estimates is very standard.

2 Lipschitz estimates in the spatial variables

The proof of Lipschitz estimates in [14] for $\gamma = 0$ is based on a calculation that $|\nabla u|^p$ is a subsolution of a uniformly parabolic equation. We are not able to find a similar quantity for other nonzero γ. The proof we give here is completely different. It makes use of the Ishii–Lions' method [12]. However, we need to apply this method twice: first we obtain log-Lipschitz estimates, and then use this log-Lipschitz estimates and the Ishii–Lions' method again to prove Lipschitz estimates. Moreover, the Lipschitz estimates holds for $\gamma > -2$ instead of $\gamma > -1$.

Lemma 2.1 (Log-Lipschitz estimate). *Let u be a smooth solution of (1.3) in Q_4 with $\gamma > -2$ and $\varepsilon \in (0,1)$. Then there exist two positive constants L_1 and L_2 depending only on n, p, γ and $\|u\|_{L^\infty(Q_4)}$ such that for every $(t_0, x_0) \in Q_1$ we have*

$$u(t,x) - u(t,y) \le L_1|x-y|\big|\log|x-y|\big| + \frac{L_2}{2}|x-x_0|^2 + \frac{L_2}{2}|y-x_0|^2 + \frac{L_2}{2}(t-t_0)^2$$

for all $t \in [t_0 - 1, t_0]$ and $x, y \in B_1(x_0)$.

Proof. Without loss of generality, we assume $x_0 = 0$ and $t_0 = 0$. It is sufficient to prove that

$$M := \max_{-1\le t\le 0,\, x,y\in \overline{B_1}} \Big\{u(t,x) - u(t,y) - L_1\phi(|x-y|) - \frac{L_2}{2}|x|^2 - \frac{L_2}{2}|y|^2 - \frac{L_2}{2}t^2\Big\}$$

is non-positive, where

$$\phi(r) = \begin{cases} -r\log r & \text{for } r \in [0, e^{-1}],\\ e^{-1} & \text{for } r \ge e^{-1}.\end{cases}$$

We assume this is not true and we will exhibit a contradiction. In the rest of the proof, $t \in [-1, 0]$ and $x, y \in \overline{B}_1$ denote the points realizing the maximum defining M.

Since $M \ge 0$, we have

$$L_1\phi(|x-y|) + \frac{L_2}{2}(|x|^2 + |y|^2 + t^2) \le 2\|u\|_{L^\infty(Q_4)}.$$

In particular,

$$\phi(\delta) \le \frac{2\|u\|_{L^\infty(Q_4)}}{L_1}, \quad \text{where } \delta = |a| \text{ and } a = x - y, \tag{2.1}$$

and

$$|t| + |x| + |y| \le 6\sqrt{\frac{\|u\|_{L^\infty(Q_4)}}{L_2}}. \tag{2.2}$$

Hence, for L_2 large enough depending only on $\|u\|_{L^\infty(Q_4)}$, we can ensure that $t \in (-1, 0]$ and $x, y \in B_1$. We choose L^2 here and fix it for the rest of the proof. Thus, from now on L_2 is a constant depending only on $\|u\|_{L^\infty}$.

Choosing L_1 large, we can ensure that δ ($< e^{-2}$) is small enough to satisfy

$$\phi(\delta) \ge 2\delta.$$

In this case, (2.1) implies

$$\delta \le \frac{\|u\|_{L^\infty(Q_4)}}{L_1}.$$

Since $t \in [-1, 0]$ and $x, y \in B_1$ realize the supremum defining M, we have that

$$\begin{aligned}
\nabla u(t,x) &= L_1\phi'(\delta)\hat{a} + L_2 x,\\
\nabla u(t,y) &= L_1\phi'(\delta)\hat{a} - L_2 y,\\
u_t(t,x) - u_t(t,y) &= L_2 t,\\
\begin{bmatrix} \nabla^2 u(t,x) & 0\\ 0 & -\nabla^2 u(t,y)\end{bmatrix} &\le L_1 \begin{bmatrix} Z & -Z\\ -Z & Z\end{bmatrix} + L_2 I,
\end{aligned} \tag{2.3}$$

where

$$Z = \phi''(\delta)\hat{a}\otimes\hat{a} + \frac{\phi'(\delta)}{\delta}(I - \hat{a}\otimes\hat{a}) \quad \text{and} \quad \hat{a} = \frac{a}{|a|} = \frac{x-y}{|x-y|}.$$

For $z \in \mathbb{R}^n$, we let

$$A(z) = I + (p-2)\frac{z_i z_j}{|z|^2 + \varepsilon^2},$$

$q = L_1\phi'(\delta)\hat{a}$, $X = \nabla^2 u(t,x)$ and $Y = \nabla^2 u(t,y)$. By evaluating the equation at (t,x) and (t,y), we have

$$L_2 t \le (|q + L_2 x|^2 + \varepsilon^2)^{\frac{\gamma}{2}} \operatorname{Tr}(A(q + L_2 x)X) - (|q - L_2 y|^2 + \varepsilon^2)^{\frac{\gamma}{2}} \operatorname{Tr}(A(q - L_2 y)Y). \tag{2.4}$$

Whenever we write C in this proof, we denote a positive constant, large enough depending only on n, p, γ and $\|u\|_{L^\infty(Q_4)}$, which may vary from line to line. Recall that we have already chosen L_2 above depending on $\|u\|_{L^\infty}$ only.

Note that $|q| = L_1|\phi'(\delta)|$. By choosing L_1 large enough, δ will be small, $|\phi'(\delta)|$ will thus be large, and $|q| \gg L_2$. In particular,

$$\frac{|q|}{2} \le |q + L_2 x| \le 2|q| \quad \text{and} \quad \frac{|q|}{2} \le |q - L_2 y| \le 2|q|. \tag{2.5}$$

From (2.3) and the fact that $\phi''(\delta) < 0$, we have

$$\begin{aligned} X = \nabla^2 u(t,x) &\le L_1 \frac{\phi'(\delta)}{\delta}(I - \hat{a}\otimes\hat{a}) + L_2 I, \\ -Y = -\nabla^2 u(t,y) &\le L_1 \frac{\phi'(\delta)}{\delta}(I - \hat{a}\otimes\hat{a}) + L_2 I. \end{aligned} \tag{2.6}$$

Making use of (2.4), (2.5) and (2.6), we have

$$\begin{aligned} \operatorname{Tr}(A(q + L_2 x)X) &= (|q + L_2 x|^2 + \varepsilon^2)^{-\frac{\gamma}{2}} L_2 t + \Big(\frac{|q - L_2 y|^2 + \varepsilon^2}{|q + L_2 x|^2 + \varepsilon^2}\Big)^{\frac{\gamma}{2}} \operatorname{Tr}(A(q - L_2 y)Y) \\ &\ge -C\Big(|q|^{-\gamma} + L_1 \frac{\phi'(\delta)}{\delta} + 1\Big). \end{aligned}$$

Therefore, it follows from (2.6) and the ellipticity of A that

$$|X| \le C\Big(|q|^{-\gamma} + L_1 \frac{\phi'(\delta)}{\delta} + 1\Big). \tag{2.7}$$

Similarly,

$$|Y| \le C\Big(|q|^{-\gamma} + L_1 \frac{\phi'(\delta)}{\delta} + 1\Big).$$

Let

$$B(z) = (|z|^2 + \varepsilon)^{\gamma} A(z).$$

We get from (2.4) and (2.2) the following inequality:

$$-C \le \operatorname{Tr}[B(q + L_2 x)X] - \operatorname{Tr}[B(q - L_2 y)Y] \le T_1 + T_2, \tag{2.8}$$

where

$$T_1 = \operatorname{Tr}[B(q - L_2 y)(X - Y)] \quad \text{and} \quad T_2 = |X|\big|B(q + L_2 x) - B(q - L_2 y)\big|.$$

We first estimate T_2. Using successively (2.2), (2.5), (2.7) and the mean value theorem, we get

$$\begin{aligned} T_2 &\le C|X||q|^{\gamma-1}|x + y| \\ &\le C|X||q|^{\gamma-1} \\ &\le C\Big(|q|^{-\gamma} + \frac{L_1\phi'(\delta)}{\delta} + 1\Big)|q|^{\gamma-1} \\ &\le C\Big(|q|^{-1} + \frac{|q|^{\gamma}}{\delta} + |q|^{\gamma-1}\Big). \end{aligned} \tag{2.9}$$

We now turn to T_1. On the one hand, evaluating (2.3) with respect to a vector of the form (ξ,ξ), for all $\xi \in \mathbb{R}^d$ we have

$$(X - Y)\xi\cdot\xi \le 2L_2|\xi|^2. \tag{2.10}$$

On the other hand, when we evaluate (2.3) with respect to $(\hat{a},\hat{a})$, we get

$$(X - Y)\hat{a}\cdot\hat{a} \le 4L_1\phi''(\delta) + 2L_2. \tag{2.11}$$

Inequality (2.10) tells us that all eigenvalues of $(X - Y)$ are bounded above by a constant C. Inequality (2.11) tells us that there is at least one eigenvalue that is less than the negative number $4L_1\phi''(\delta) + 2L_2$. Because of the uniform ellipticity of A, we obtain

$$T_1 \leq C|q|^\gamma(L_1\phi''(\delta) + 1).$$

In view of the estimates for T_1 and T_2, we finally get from (2.8) that

$$-L_1\phi''(\delta)|q|^\gamma \leq C\Big(|q|^\gamma + |q|^{-1} + \frac{|q|^\gamma}{\delta} + |q|^{\gamma-1} + 1\Big),$$

or equivalently

$$-L_1\phi''(\delta) \leq C\Big(1 + |q|^{-1-\gamma} + \frac{1}{\delta} + |q|^{-1} + |q|^{-\gamma}\Big). \tag{2.12}$$

Our purpose is to choose L_1 large in order to get a contradiction in (2.12).

Recall that we have the estimate $\delta \leq C/L_1$. From our choice of ϕ, we obtain $\phi'(\delta) > 1$ for δ small and $-\phi''(\delta) = \frac{1}{\delta} \geq cL_1$.

For L_1 sufficiently large, since $\gamma > -2$,

$$C(1 + |q|^{-1-\gamma} + |q|^{-1} + |q|^{-\gamma}) \leq C(1 + L_1^{-1-\gamma} + L_1^{-1} + L_1^{-\gamma}) \leq \frac{c}{2}L_1^2 \leq -\frac{1}{2}L_1\phi''(\delta).$$

The remaining term is handled because of the special form of the function ϕ. We have

$$-L_1\phi''(\delta) = \frac{L_1}{\delta} > \frac{2C}{\delta}$$

for L_1 sufficiently large.

Therefore, we reached a contradiction. The proof of this lemma is thereby completed. □

By letting $t = t_0$ and $y = x_0$ in Lemma 2.1 and since (x_0, t_0) is arbitrary, we have the following corollary.

Corollary 2.2. *Let u be a smooth solution of* (1.3) *in Q_4 with $\gamma > -2$ and $\varepsilon \in (0, 1)$. Then there exists a positive constant C depending only on n, γ, p and $\|u\|_{L^\infty(Q_4)}$ such that for every $(t, x), (t, y) \in Q_3$ and $|x - y| < \frac{1}{2}$, we have*

$$|u(t, x) - u(t, y)| \leq C|x - y|\big|\log|x - y|\big|.$$

We shall make use of the above log-Lipschitz estimate and the Ishii–Lions method [12] again to prove the following Lipschitz estimate.

Lemma 2.3 (Lipschitz estimate). *Let u be a smooth solution of* (1.3) *in Q_4 with $\gamma > -2$ and $\varepsilon \in (0, 1)$. Then there exist two positive constants L_1 and L_2 depending only on n, p, γ and $\|u\|_{L^\infty(Q_4)}$ such that for every $(t_0, x_0) \in Q_1$ we have*

$$u(t, x) - u(t, y) \leq L_1|x - y| + \frac{L_2}{2}|x - x_0|^2 + \frac{L_2}{2}|y - x_0|^2 + \frac{L_2}{2}(t - t_0)^2$$

for all $t \in [t_0 - 1, t_0]$ and $x, y \in B_{1/4}(x_0)$.

Proof. The proof of this lemma follows the same computations as that of Lemma 2.1, but we make use of the conclusion of Corollary 2.2 in order to improve our estimate.

Without loss of generality, we assume $x_0 = 0$ and $t_0 = 0$. As before, we define

$$M := \max_{-1 \leq t \leq 0,\, x, y \in B_1} \Big\{u(t, x) - u(t, y) - L_1\phi(|x - y|) - \frac{L_2}{2}|x|^2 - \frac{L_2}{2}|y|^2 - \frac{L_2}{2}t^2\Big\}$$

and prove that it is non-positive, where

$$\phi(r) = \begin{cases} r - \dfrac{1}{2 - \gamma_0}r^{2-\gamma_0} & \text{for } r \in [0, 1], \\ 1 - \dfrac{1}{2 - \gamma_0} & \text{for } r \geq 1 \end{cases}$$

for some $\gamma_0 \in (\frac{1}{2}, 1)$.

We assume this is not true in order to obtain a contradiction. In the remaining of the proof of the lemma, $t \in [-1, 0]$ and $x, y \in \overline{B}_{1/4}$ denote the points realizing the maximum defining M.

For the same reasons as in the proof of Lemma 2.1, inequalities (2.1) and (2.2) also apply in this case. Thus, we can use the same choice of L_2 depending on $\|u\|_{L^\infty}$ only that ensures $t \in (-1, 0]$ and $x, y \in B_1$.

From Corollary 2.2, we already know that $u(t,x) - u(t,y) \le C|x-y||\log|x-y||$. Since $M \ge 0$,

$$L_1\phi(|x-y|) + \frac{L_2}{2}(|x|^2 + |y|^2 + t^2) \le C|x-y||\log|x-y||.$$

In particular, we obtain an improvement of (2.2):

$$|t| + |x| + |y| \le C\sqrt{\frac{\delta|\log\delta|}{L_2}}.$$

This gives us an upper bound for $|x+y|$ that we can use to improve (2.9):

$$T_2 \le C|X||q|^{\gamma-1}|x+y| \le C\Big(|q|^{-1} + \frac{|q|^\gamma}{\delta} + |q|^{\gamma-1}\Big)\sqrt{\delta|\log\delta|}.$$

The estimate for T_1 stays unchanged. Hence, (2.12) becomes

$$-L_1\phi''(\delta) \le C\Big(1 + \sqrt{\delta|\log\delta|}\Big(|q|^{-1} + |q|^{-1-\gamma} + \frac{1}{\delta} + |q|^{-\gamma}\Big)\Big).$$

Recall that $|q| = L_1\phi'(\delta) \ge L_1/2$ and $\phi''(\delta) = (\gamma_0 - 1)\delta^{-\gamma_0}$. Then

$$L_1\delta^{-\gamma_0} \le C\Big(1 + \sqrt{\delta|\log\delta|}(1 + L_1^{-1} + L_1^{-1-\gamma} + \delta^{-1} + L_1^{-\gamma})\Big).$$

The term $+1$ inside the innermost parenthesis is there just to ensure that the inequality holds both for $\gamma < 0$ and $\gamma > 0$. Recalling that $\delta < C/L_1$, we obtain an inequality in terms of L_1 only:

$$L_1^{1+\gamma_0} \le C\Big(1 + L_1^{-\frac{1}{2}}\sqrt{\log L_1}(1 + L_1^{-1} + L_1^{-1-\gamma} + L_1 + L_1^{-\gamma})\Big)$$

Choosing L_1 large, we arrive at a contradiction given that $1 + \gamma_0 > \max(\frac{1}{2}, -\frac{1}{2} - \gamma)$ since $\gamma_0 > \frac{1}{2}$ and $\gamma > -2$. □

Again, by letting $t = t_0$ and $y = x_0$ in Lemma 2.3 and since (x_0, t_0) is arbitrary, we have the following corollary.

Corollary 2.4. *Let u be a smooth solution of* (1.3) *in Q_4 with $\gamma > -2$ and $\varepsilon \in (0,1)$. Then there exists a positive constant C depending only on n, γ, p and $\|u\|_{L^\infty(Q_4)}$ such that for every $(t,x), (t,y) \in Q_3$ and $|x-y| < 1$,*

$$|u(t,x) - u(t,y)| \le C|x-y|.$$

3 Hölder estimates in the time variable

Using the Lipschitz continuity in x and a simple comparison argument, we show that the solution of (1.3) is Hölder continuous in t.

Lemma 3.1. *Let u be a smooth solution of* (1.3) *in Q_4 with $\gamma > -1$ and $\varepsilon \in (0,1)$. Then there holds*

$$\sup_{t\neq s,\,(t,x),(s,x)\in Q_1} \frac{|u(t,x) - u(s,x)|}{|t-s|^{1/2}} \le C,$$

where C is a positive constant depending only on n, p, γ and $\|u\|_{L^\infty(Q_4)}$.

Remark 3.2. Deriving estimates in the time variable for estimates in the space variable by maximum principle techniques is classical. As far as viscosity solutions are concerned, the reader is referred to [2, Lemma 9.1, p. 317] for instance.

Proof. Let $\beta = \max(2, (2+\gamma)/(1+\gamma))$. We claim that for all $t_0 \in [-1, 0)$ and $\eta > 0$ there exist $L_1 > 0$ and $L_2 > 0$ such that

$$u(t,x) - u(t_0, 0) \le \eta + L_1(t - t_0) + L_2|x|^\beta =: \varphi(t,x) \quad \text{for all } (t,x) \in [t_0, 0] \times \overline{B}_1. \tag{3.1}$$

We first choose $L_2 \ge 2\|u\|_{L^\infty(Q_3)}$ such that (3.1) holds true for $x \in \partial B_1$. We will next choose L_2 such that (3.1) holds true for $t = t_0$. In this step, we shall use Corollary 2.4 to find that u is Lipschitz continuous with respect to the spatial variables. From Corollary 2.4, $\|\nabla u\|_{L^\infty(Q_3)}$ is bounded depending on $\|u\|_{L^\infty(Q_4)}$ only. It is enough to choose

$$\|\nabla u\|_{L^\infty(Q_3)}|x| \le \eta + L_2|x|^\beta,$$

which holds true if

$$L_2 \ge \frac{\|\nabla u\|^\beta_{L^\infty(Q_3)}}{\eta^{\beta-1}}.$$

We finally choose L_1 such that the function $\varphi(t,x)$ is a supersolution of an equation which u is a solution of. Inequality (3.1) thus follows from the comparison principle. We use a slightly different equation depending on whether $\gamma \le 0$ or $\gamma > 0$.

Let us start with the case $\gamma \le 0$. In this case, we will prove that φ is a supersolution of the nonlinear equation (1.3). That is,

$$\varphi_t - (\varepsilon^2 + |\nabla\varphi|^2)^{\frac{\gamma}{2}}\Big(\delta_{ij} + (p-2)\frac{\varphi_i\varphi_j}{\varepsilon^2 + |\nabla\varphi|^2}\Big)\varphi_{ij} > 0. \tag{3.2}$$

In order to ensure this inequality, we choose L_1 so that

$$L_1 > (p-1)|\nabla\varphi|^\gamma|D^2\varphi| \ge (\varepsilon^2 + |\nabla\varphi|^2)^{\frac{\gamma}{2}}\Big(\delta_{ij} + (p-2)\frac{\varphi_i\varphi_j}{\varepsilon^2 + |\nabla\varphi|^2}\Big)\varphi_{ij}.$$

We chose the exponent β so that when $\gamma \le 0$, $|\nabla\varphi|^\gamma|D^2\varphi| = CL_1^{1+\gamma}$ for some constant C depending on n and γ. Thus, we must choose $L_1 = CL_2^{1+\gamma}$ in order to ensure (3.2).

Therefore, still for the case $\gamma \le 0$, $\beta = (2+\gamma)/(1+\gamma)$ and for any choice of $\eta > 0$, using the comparison principle, we have

$$\begin{aligned} u(t,0) - u(t_0,0) &\le \eta + C(\eta^{(1-\beta)}\|\nabla u\|^\beta_{L^\infty(Q_3)} + 2\|u\|_{L^\infty(Q_3)} + \varepsilon)^{\gamma+1}(t - t_0) \\ &\le \eta + C\eta^{-1}\|\nabla u\|^{\gamma+2}_{L^\infty(Q_3)}|t - t_0| + C(\|u\|_{L^\infty(Q_3)} + \varepsilon)^{\gamma+1}|t - t_0|. \end{aligned}$$

By choosing $\eta = \|\nabla u\|^{\gamma/2+1}_{L^\infty(Q_3)}|t - t_0|^{1/2}$, it follows that for $t \in (t_0, 0]$,

$$u(t,0) - u(t_0,0) \le C(\|\nabla u\|_{L^\infty(Q_3)})^{\frac{\gamma+2}{2}}|t - t_0|^{\frac{1}{2}} + C(\|u\|_{L^\infty(Q_3)} + \varepsilon)^{\gamma+1}|t - t_0|.$$

The lemma is then concluded in the case $\gamma \le 0$.

Let us now analyze the case $\gamma > 0$. In this case, we prove that φ is a supersolution to a linear parabolic equation whose coefficients depend on u. That is,

$$\varphi_t - (\varepsilon^2 + |\nabla u|^2)^{\frac{\gamma}{2}}\Big(\delta_{ij} + (p-2)\frac{u_iu_j}{\varepsilon^2 + |\nabla u|^2}\Big)\varphi_{ij} > 0.$$

Since $\gamma > 0$ and ∇u is known to be bounded after Corollary 2.4, we can rewrite the equation assumption as

$$\varphi_t - a_{ij}(t,x)\varphi_{ij} > 0, \tag{3.3}$$

where the coefficients $a_{ij}(t,x)$ are bounded by

$$|a_{ij}(t,x)| \le C(\varepsilon + \|\nabla u\|_{L^\infty(Q_3)})^\gamma.$$

Since $\gamma > 0$, we pick $\beta = 2$ and $D^2\varphi$ is a constant multiple of L_2. In particular, we ensure that (3.3) holds if

$$L_1 > C(\varepsilon + \|\nabla u\|_{L^\infty(Q_3)})^\gamma L_2.$$

Therefore, for the case $\gamma > 0$, $\beta = 2$ and for any choice of $\eta > 0$, by using the comparison principle,

$$u(t,0) - u(t_0,0) \le \eta + C(\varepsilon + \|\nabla u\|_{L^\infty(Q_3)})^\gamma (\eta^{-1}\|\nabla u\|^2_{L^\infty(Q_3)} + \|u\|_{L^\infty(Q_3)})(t - t_0).$$

Choosing

$$\eta = (\varepsilon + \|\nabla u\|_{L^\infty(Q_3)})^{\frac{\gamma}{2}+1}(t-t_0)^{\frac{1}{2}},$$

we obtain,

$$u(t,0) - u(t_0,0) \le C(\varepsilon + \|\nabla u\|_{L^\infty(Q_3)})^{\frac{\gamma}{2}+1}(t-t_0)^{\frac{1}{2}} + C(\varepsilon + \|\nabla u\|_{L^\infty(Q_3)})^\gamma \|u\|_{L^\infty(Q_3)}(t-t_0).$$

This finishes the proof for $\gamma > 0$ as well. □

4 Hölder estimates for the spatial gradients

In this section, we assume that $\gamma > -1$, so that Corollary 2.4 and Lemma 3.1 hold, that is, the solution of (1.3) in Q_2 has uniform interior Lipschitz estimates in x and uniform interior Hölder estimates in t, both of which are independent of $\varepsilon \in (0,1)$. For $\rho, r > 0$, we denote

$$Q_r = B_r \times (-r^2, 0], \quad Q_r^\rho = B_r \times (-\rho^{-\gamma} r^2, 0].$$

This same family of cylinders Q_r^ρ was used in [8]. They are the natural ones that correspond to the two-parameter family of scaling of the equation. Indeed, if u solves (1.3) in Q_r^ρ and we let $v(x,t) = \frac{1}{r\rho} u(rx, r^2\rho^{-\gamma}t)$, then

$$v_t(t,x) = (|\nabla v|^2 + \varepsilon^2\rho^{-2})^{\frac{\gamma}{2}}\Big(\Delta v + (p-2)\frac{v_i v_j}{|\nabla v|^2 + \varepsilon^2\rho^{-2}} v_{ij}\Big) \quad \text{in } Q_1.$$

If we choose $\rho \ge \|\nabla u\|_{L^\infty(Q_1)} + 1$, we may assume that the solution of (1.3) satisfies $|\nabla u| \le 1$ in Q_1.

We are going to show that ∇u is Hölder continuous in space-time at the point $(0,0)$. The idea of the proof in this step is similar to that in [14]. First we show that if the projection of ∇u onto the direction $e \in \mathbb{S}^{n-1}$ is away from 1 in a positive portion of Q_1, then $\nabla u \cdot e$ has improved oscillation in a smaller cylinder.

Lemma 4.1. *Let u be a smooth solution of (1.3) with $\varepsilon \in (0,1)$ such that $|\nabla u| \le 1$ in Q_1. For every $\frac{1}{2} < \ell < 1$ and $\mu > 0$, there exists $\tau_1 \in (0, \frac{1}{4})$ depending only on μ, n and there exist $\tau, \delta > 0$ depending only on n, p, γ, μ and ℓ such that for arbitrary $e \in \mathbb{S}^{n-1}$ if*

$$\big|\{(x,t) \in Q_1 : \nabla u \cdot e \le \ell\}\big| > \mu |Q_1|,$$

then

$$\nabla u \cdot e < 1 - \delta \quad \text{in } Q_\tau^{1-\delta}$$

and $Q_\tau^{1-\delta} \subset Q_{\tau_1}$.

Proof. Let

$$a_{ij}(q) = (|q|^2 + \varepsilon^2)^{\frac{\gamma}{2}}\Big(\delta_{ij} + (p-2)\frac{q_i q_j}{|q|^2 + \varepsilon^2}\Big), \quad q \in \mathbb{R}^n, \tag{4.1}$$

and denote

$$a_{ij,m} = \frac{\partial a_{ij}}{\partial q_m}.$$

Differentiating (1.3) in x_k, we have

$$(u_k)_t = a_{ij}(u_k)_{ij} + a_{ij,m} u_{ij}(u_k)_m.$$

Then

$$(\nabla u \cdot e - \ell)_t = a_{ij}(\nabla u \cdot e - \ell)_{ij} + a_{ij,m} u_{ij}(\nabla u \cdot e - \ell)_m,$$

and for

$$v = |\nabla u|^2$$

we have

$$v_t = a_{ij}v_{ij} + a_{ij,m}u_{ij}v_m - 2a_{ij}u_{ki}u_{kj}.$$

For $\rho = \frac{\ell}{4}$, let

$$w = (\nabla u \cdot e - \ell + \rho|\nabla u|^2)^+.$$

Then in the region $\Omega_+ = \{(x,t) \in Q_1 : w > 0\}$ we have

$$w_t = a_{ij}w_{ij} + a_{ij,m}u_{ij}w_m - 2\rho a_{ij}u_{ki}u_{kj}.$$

Since $|\nabla u| > \frac{\ell}{2}$ in Ω_+, we have

$$|a_{ij,m}| \leq \begin{cases} c(p,n,\gamma)\ell^{-1} & \text{if } \gamma \geq 0, \\ c(p,n,\gamma)\ell^{\gamma-1} & \text{if } \gamma < 0, \end{cases}$$

in Ω_+, where $c(p,n,\gamma)$ is a positive constant depending only on p, n and γ. By the Cauchy–Schwarz inequality, it follows that

$$w_t \leq a_{ij}w_{ij} + c_1(\ell)|\nabla w|^2 \quad \text{in } \Omega_+,$$

where

$$c_1(\ell) = \begin{cases} c_0\ell^{-\gamma-3} & \text{if } \gamma \geq 0, \\ c_0\ell^{2\gamma-3} & \text{if } \gamma < 0 \end{cases}$$

for some constant $c_0 > 0$ depending only on p, γ and n. Therefore, it satisfies in the viscosity sense that

$$w_t \leq \tilde{a}_{ij}w_{ij} + c_1(\ell)|\nabla w|^2 \quad \text{in } Q_1,$$

where

$$\tilde{a}_{ij}(x) = \begin{cases} a_{ij}(\nabla u(x)) & \text{if } x \in \Omega_+, \\ \delta_{ij} & \text{elsewhere.} \end{cases}$$

Notice that since $\ell \in (\frac{1}{2}, 1)$, the coefficient $\tilde{a}_{ij}$ is uniformly elliptic with ellipticity constants depending only on p and γ. We can choose $c_2(\ell) > 0$ depending only on p, γ, n and ℓ such that if we let

$$W = 1 - \ell + \rho$$

and

$$\overline{w} = \frac{1}{c_2}(1 - e^{c_2(w-W)}),$$

then we have

$$\overline{w}_t \geq \tilde{a}_{ij}\overline{w}_{ij} \quad \text{in } Q_1$$

in the viscosity sense. Since $W \geq \sup_{Q_1} w$, we obtain $\overline{w} \geq 0$ in Q_1.

If $\nabla u \cdot e \leq \ell$, then $\overline{w} \geq (1 - e^{c_2(\ell-1)})/c_2$. Therefore, it follows from the assumption that

$$\left|\left\{(x,t) \in Q_1 : \overline{w} \geq \frac{(1 - e^{c_2(\ell-1)})}{c_2}\right\}\right| > \mu|Q_1|.$$

By [14, Proposition 2.3], there exist $\tau_1 > 0$ depending only on μ and n, and $\nu > 0$ depending only on μ, ℓ, n, γ and p such that

$$\overline{w} \geq \nu \quad \text{in } Q_{\tau_1}.$$

Meanwhile, we have

$$\overline{w} \leq W - w.$$

This implies that

$$W - w \geq \nu \quad \text{in } Q_{\tau_1}.$$

Therefore, we have

$$\nabla u \cdot e + \rho |\nabla u|^2 \le 1 + \rho - \nu \quad \text{in } Q_{\tau_1}.$$

Since $|\nabla u \cdot e| \le |\nabla u|$, we have

$$\nabla u \cdot e + \rho (\nabla u \cdot e)^2 \le 1 + \rho - \nu \quad \text{in } Q_{\tau_1}.$$

Therefore, remarking that $\nu \le 1 + \rho$, we have

$$\nabla u \cdot e \le \frac{-1 + \sqrt{1 + 4\rho(1 + \rho - \nu)}}{2\rho} \le 1 - \delta \quad \text{in } Q_{\tau_1}$$

for some $\delta > 0$ depending only on p, γ, μ, ℓ and n. Finally, we can choose $\tau = \tau_1$ if $\gamma < 0$ and $\tau = \tau_1(1-\delta)^{\gamma/2}$ if $\gamma \ge 0$ such that $Q_\tau^{1-\delta} \subset Q_{\tau_1}$. □

Note that our choice of τ and δ above implies that

$$\tau < (1-\delta)^{\frac{\gamma}{2}} \quad \text{when } \gamma \ge 0.$$

In the rest of the paper, we will choose τ even smaller such that

$$\tau < (1-\delta)^{1+\gamma} \quad \text{for all } \gamma > -1. \tag{4.2}$$

This fact will be used in the proof of Theorem 4.8.

In case we can assume that Lemma 4.1 holds in all directions $e \in \partial B_1$, then it effectively implies a reduction in the oscillation of ∇u in a smaller parabolic cylinder. If such an improvement of oscillation takes place at all scales, it leads to the Hölder continuity of ∇u at $(0,0)$ by iteration and scaling. The following corollary describes this favorable case in which the assumption of the previous lemma holds in all directions.

Corollary 4.2. *Let u be a smooth solution of* (1.3) *with $\varepsilon \in (0,1)$ such that $|\nabla u| \le 1$ in Q_1. For every $0 < \ell < 1$ and $\mu > 0$, there exist $\tau \in (0, \frac{1}{4})$ depending only on μ and n, and $\delta > 0$ depending only on n, p, γ, μ and ℓ, such that for every nonnegative integer $k \le \log \varepsilon / \log(1-\delta)$ if*

$$\left|\{(x,t) \in Q_{\tau^i}^{(1-\delta)^i} : \nabla u \cdot e \le \ell(1-\delta)^i\}\right| > \mu \left|Q_{\tau^i}^{(1-\delta)^i}\right| \quad \textit{for all } e \in \mathbb{S}^{n-1} \textit{ and } i = 0, \dots, k, \tag{4.3}$$

then

$$|\nabla u| < (1-\delta)^{i+1} \quad \textit{in } Q_{\tau^{i+1}}^{(1-\delta)^{i+1}} \textit{ for all } i = 0, \dots, k.$$

Remark 4.3. Note that we can further impose on δ that $\delta < \frac{1}{2}$ and $\delta < 1 - \tau$.

Proof. When $i = 0$, it follows from Lemma 4.1 that $\nabla u \cdot e < 1 - \delta$ in Q_τ for all $e \in \mathbb{S}^{n-1}$. This implies that $|\nabla u| < 1 - \delta$ in $Q_\tau^{1-\delta}$.

Suppose this corollary holds for $i = 0, \dots, k-1$. We are going prove it for $i = k$. Let

$$v(x,t) = \frac{1}{\tau^k (1-\delta)^k} u(\tau^k x, \tau^{2k}(1-\delta)^{-k\gamma} t).$$

Then v satisfies

$$v_t = \left(|\nabla v|^2 + \frac{\varepsilon^2}{(1-\delta)^{2k}}\right)^{\frac{\gamma}{2}} \left(\Delta v + (p-2)\frac{v_i v_j}{|\nabla v|^2 + \varepsilon^2 (1-\delta)^{-2k}} v_{ij}\right) \quad \text{in } Q_1.$$

By the induction hypothesis, we also know that $|\nabla v| \le 1$ in Q_1, and

$$\left|\{(x,t) \in Q_1 : \nabla v \cdot e \le \ell\}\right| > \mu |Q_1| \quad \text{for all } e \in \mathbb{S}^{n-1}.$$

Notice that $\varepsilon \le (1-\delta)^k$. Therefore, by Lemma 4.1 we have

$$\nabla v \cdot e \le 1 - \delta \quad \text{in } Q_\tau^{1-\delta} \text{ for all } e \in \mathbb{S}^{n-1}.$$

Hence, $|\nabla v| \le 1 - \delta$ in $Q_\tau^{1-\delta}$. Consequently,

$$|\nabla u| < (1-\delta)^{k+1} \quad \text{in } Q_{\tau^{k+1}}^{(1-\delta)^{k+1}}.$$ □

Unless $\nabla u(0,0) = 0$, the above iteration will inevitably stop at some step. There will be a first value of k where the assumptions of Corollary 4.2 do not hold in some direction $e \in \mathbb{S}^{n-1}$. This means that ∇u is close to some fixed vector in a large portion of $Q^{(1-\delta)^k}_{\tau^k}$. We then prove that u is close to some linear function, from which the Hölder continuity of ∇u will follow by applying a result from [25].

Having ∇u close to a vector e for most points tells us that for every fixed time t the function $u(x,t)$ will be approximately linear. However, it does not say anything about how u varies with respect to time. We must use the equation in order to prove that the function $u(x,t)$ will be close to some linear function uniformly in t. This is the main purpose of the following set of lemmas.

Lemma 4.4. *Let $u \in C(\overline{Q}_1)$ be a smooth solution of (1.3) with $\gamma > -1$ and $\varepsilon \in (0,1)$ such that $|\nabla u| \le M$ in Q_1. Let A be a positive constant. Assume that for all $t \in [-1,0]$ we have*

$$\operatorname{osc}_{B_1} u(\cdot,t) \le A.$$

Then

$$\operatorname{osc}_{Q_1} u \le \begin{cases} CA & \text{if } \gamma \ge 0,\\ C(A + A^{1+\gamma}) & \text{if } -1 < \gamma < 0,\end{cases}$$

where C is a positive constant depending only on M, γ, p and the dimension n.

Proof. When $\gamma \ge 0$, for the a_{ij} in (4.1) we have $|a_{ij}| \le \Lambda := (M^2+1)^{\gamma/2}\max(p-1,1)$, and therefore the conclusion follows from the same proof of [14, Lemma 4.3].

When $\gamma \in (-1,0)$, we choose different comparison functions from [14]. Let

$$\overline{w}(x,t) = \overline{a} + \Lambda A^{1+\gamma} t + 2A|x|^\beta,$$
$$\underline{w}(x,t) = \underline{a} - \Lambda A^{1+\gamma} t - 2A|x|^\beta,$$

where $\beta = (2+\gamma)/(1+\gamma)$ and Λ is to be fixed later. As far as $\overline{a}$ and $\underline{a}$ are concerned, $\overline{a}$ is chosen so that $\overline{w}(\cdot,-1) \ge u(\cdot,-1)$ in B_1 and $\overline{w}(\bar{x},-1) = u(\bar{x},-1)$ for some $\bar{x} \in \overline{B}_1$, and $\underline{a}$ is chosen so that $\underline{w}(\cdot,-1) \le u(\cdot,-1)$ in B_1 and $\underline{w}(\underline{x},-1) = u(\underline{x},-1)$ for some $\underline{x} \in \overline{B}_1$. This implies that

$$\overline{a} - \underline{a} = u(\bar{x},-1) - u(\underline{x},-1) + 2\Lambda A^{1+\gamma} - 2A|\bar{x}|^2 - 2A|\underline{x}|^2 \le A + 2\Lambda A^{1+\gamma}.$$

Notice that $\beta > 2$ since $\gamma \in (-1,0)$. We now remark that if Λ is chosen as $\Lambda = (2\beta)^{\gamma+1}(\beta-1)pn^2 + 1$, then the first inequality

$$\Lambda A^{1+\gamma} \le ((2A\beta|x|^{\beta-1})^2 + \varepsilon^2)^{\frac{\gamma}{2}} \cdot pn^2 \cdot 2A\beta(\beta-1)|x|^{\beta-2} \le (2\beta)^{\gamma+1}(\beta-1)pn^2A^{1+\gamma}$$

(we used that $\gamma < 0$) cannot hold true for $x \in B_1$. This implies that $\overline{w}$ is a strict supersolution of the equation satisfied by u. Similarly, $\underline{w}$ is a strict subsolution.

We claim that

$$\overline{w} \ge u \quad \text{in } Q_1 \qquad \text{and} \qquad \underline{w} \le u \quad \text{in } Q_1.$$

We only justify the first inequality since we can proceed similarly to get the second one. Suppose that the first inequality is false. Let $m = -\inf_{Q_1}(\overline{w} - u) > 0$ and $(x_0,t_0) \in \overline{Q}_1$ be such that $m = u(x_0,t_0) - \overline{w}(x_0,t_0)$. Then $\overline{w} + m \ge u$ in Q_1 and $\overline{w}(x_0,t_0) + m = u(x_0,t_0)$. By the choice of $\bar{a}$, we know that $t_0 > -1$. If $x_0 \in \partial B_1$, then

$$2A = (\overline{w}(x_0,t_0) + m) - (\overline{w}(0,t_0) + m) \le u(x_0,t_0) - u(0,t_0) \le \operatorname{osc}_{B_1} u(\cdot,t_0) \le A,$$

which is impossible. Therefore, $x_0 \in B_1$. But this is not possible since $\overline{w}$ is a strict supersolution of the equation satisfied by u. This proves the claim.

Therefore, we have

$$\operatorname{osc}_{Q_1} u \le \sup_{Q_1} \overline{w} - \inf_{Q_1} \underline{w} \le \bar{a} - \underline{a} + 4A = 2\Lambda A^{\gamma+1} + 5A. \qquad \square$$

Lemma 4.5. *Let $u \in C(\overline{Q}_1)$ be a smooth solution of (1.3) with $\gamma \in \mathbb{R}$ and $\varepsilon \in (0,1)$. Let $e \in \mathbb{S}^{n-1}$ and $0 < \delta < \frac{1}{8}$. Assume that for all $t \in [-1, 0]$ we have*

$$\operatorname{osc}_{x \in B_1}(u(x,t) - x \cdot e) \leq \delta.$$

Then

$$\operatorname{osc}_{(x,t) \in Q_1}(u(x,t) - x \cdot e) \leq C\delta,$$

where C is a positive constant depending only on γ, p and the dimension n.

Proof. Let

$$\overline{w}(x,t) = \overline{a} + x \cdot e + \Lambda\delta t + 2\delta|x|^2,$$
$$\underline{w}(x,t) = \underline{a} + x \cdot e - \Lambda\delta t - 2\delta|x|^2,$$

where $\Lambda > 0$ will be fixed later, $\overline{a}$ is chosen so that $\overline{w}(\cdot,-1) \geq u(\cdot,-1)$ in B_1 and $\overline{w}(\bar{x},-1) = u(\bar{x},-1)$ for some $\bar{x} \in \overline{B}_1$, and $\underline{a}$ is chosen so that $\underline{w}(\cdot,-1) \leq u(\cdot,-1)$ in B_1 and $\underline{w}(\underline{x},-1) = u(\underline{x},-1)$ for some $\underline{x} \in \overline{B}_1$. This implies that

$$\overline{a} - \underline{a} = u(\bar{x},-1) - \bar{x} \cdot e - (u(\underline{x},-1) - \underline{x} \cdot e) + 2\Lambda\delta - 2\delta|\bar{x}|^2 - 2\delta|\underline{x}|^2 \leq (2\Lambda + 1)\delta.$$

For every $x \in \overline{B}_1$ and $t \in [-1,0]$, since $\delta < \frac{1}{8}$, we have

$$|\nabla\overline{w}(x,t)| \geq |e| - 4\delta|x| \geq \frac{1}{2}, \qquad |\nabla\underline{w}(x,t)| \geq |e| - 4\delta|x| \geq \frac{1}{2}.$$

Similarly, $|\nabla\overline{w}(x,t)| \leq \frac{3}{2}$ and $|\nabla\underline{w}(x,t)| \leq \frac{3}{2}$. Therefore, using the notation from (4.1), there is a constant A_0 (depending on p and γ) so that

$$a_{ij}(\nabla\overline{w}(x,t)) \leq A_0\mathrm{I} \qquad \text{and} \qquad a_{ij}(\nabla\underline{w}(x,t)) \leq A_0\mathrm{I}.$$

We choose $\Lambda = 5nA_0$. We claim that

$$\overline{w} \geq u \quad \text{in } Q_1 \qquad \text{and} \qquad \underline{w} \leq u \quad \text{in } Q_1.$$

We only justify the first inequality since we can proceed similarly to get the second one. Suppose that the first inequality is false. Let $m = -\inf_{Q_1}(\overline{w} - u) > 0$ and $(x_0,t_0) \in \overline{Q}_1$ be such that $m = u(x_0,t_0) - \overline{w}(x_0,t_0)$. Then $\overline{w} + m \geq u$ in Q_1 and $\overline{w}(x_0,t_0) + m = u(x_0,t_0)$. By the choice of $\bar{a}$, we know that $t_0 > -1$. If $x_0 \in \partial B_1$, then

$$\begin{aligned} 2\delta &= (\overline{w}(x_0,t_0) + m) - x_0 \cdot e - (\overline{w}(0,t_0) + m) \\ &\leq u(x_0,t_0) - x_0 \cdot e - u(0,t_0) \\ &\leq \operatorname{osc}_{x \in B_1}(u(x,t_0) - x \cdot e) \\ &\leq \delta, \end{aligned}$$

which is impossible. Hence, $x_0 \in B_1$. Therefore, we have the classical relations

$$\begin{aligned} u(x_0,t_0) &= \overline{w}(x_0,t_0) + m, \\ \nabla u(x_0,t_0) &= \nabla\overline{w}(x_0,t_0) \in \overline{B}_{3/2} \setminus B_{1/2}, \\ D^2u(x_0,t_0) &\leq D^2\overline{w}(x_0,t_0) = 4\delta\mathrm{I}, \\ \partial_t u(x_0,t_0) &\geq \partial_t\overline{w}(x_0,t_0) = \Lambda\delta. \end{aligned}$$

It follows that

$$u_t(x_0,t_0) - a_{ij}(\nabla u(x_0,t_0))\partial_{ij}u(x_0,t_0) \geq \overline{w}_t(x_0,t_0) - a_{ij}(\nabla\overline{w}(x_0,t_0))\partial_{ij}\overline{w}(x_0,t_0) > 0,$$

which is a contradiction. This proves the claim.

Therefore, we have

$$\operatorname{osc}_{(x,t) \in Q_1}(u(x,t) - x \cdot e) \leq \sup_{Q_1}(\overline{w} - x \cdot e) - \inf_{Q_1}(\underline{w} - x \cdot e) \leq \bar{a} - \underline{a} + 4\delta = (2\Lambda + 5)A. \qquad \square$$

Lemma 4.6. *Let η be a positive constant and let u be a smooth solution of* (1.3) *with $\gamma > -1$ and $\varepsilon \in (0, 1)$ such that $|\nabla u| \le 1$ in Q_1. Assume*

$$\big|\{(x, t) \in Q_1 : |\nabla u - e| > \varepsilon_0\}\big| \le \varepsilon_1$$

for some $e \in \mathbb{S}^{n-1}$ and two positive constants ε_0, ε_1. Then if ε_0 and ε_1 are sufficiently small, there exists a constant $a \in \mathbb{R}$ such that

$$|u(x, t) - a - e \cdot x| \le \eta \quad \text{for all } (x, t) \in Q_{1/2}.$$

Here, both ε_0 and ε_2 depend only on n, p, γ and η.

Proof. Let

$$f(t) := \big|\{x \in B_1 : |\nabla u(x, t) - e| > \varepsilon_0\}\big|.$$

By the assumptions and Fubini's theorem, we have that $\int_{-1}^{0} f(t)\,dt \le \varepsilon_1$. For $E := \{t \in (-1, 0) : f(t) \ge \sqrt{\varepsilon_1}\}$, we obtain

$$|E| \le \frac{1}{\sqrt{\varepsilon_1}} \int_E f(t)\,dt \le \frac{1}{\sqrt{\varepsilon_1}} \int_{-1}^{0} f(t)\,dt \le \sqrt{\varepsilon_1}.$$

Therefore, for all $t \in (-1, 0] \setminus E$ with $|E| \le \sqrt{\varepsilon_1}$ we have

$$\big|\{x \in B_1 : |\nabla u(x, t) - e| > \varepsilon_0\}\big| \le \sqrt{\varepsilon_1}. \tag{4.4}$$

It follows from (4.4) and Morrey's inequality that for all $t \in (-1, 0] \setminus E$ we have

$$\operatorname{osc}_{B_{1/2}}(u(\cdot, t) - e \cdot x) \le C(n)\|\nabla u - e\|_{L^{2n}(B_1)} \le C(n)(\varepsilon_0 + \varepsilon_1^{\frac{1}{4n}}), \tag{4.5}$$

where $C(n) > 0$ depends only on n.

Meanwhile, since $|\nabla u| \le 1$ in Q_1, we have that $\operatorname{osc}_{B_1} u(\cdot, t) \le 2$ for all $t \in (-1, 0]$. Therefore, applying Lemma 4.4, we have that $\operatorname{osc}_{Q_1} u \le C$ for some constant C. Note that $u(t, x) - u(0, 0)$ also satisfies (1.3) and

$$\|u(t, x) - u(0, 0)\|_{L^\infty(Q_1)} \le \operatorname{osc}_{Q_1} u \le C.$$

By applying Lemma 3.1 to $u(t, x) - u(0, 0)$, we have

$$\sup_{t \ne s,\,(t,x),(s,x) \in Q_1} \frac{|u(t, x) - u(s, x)|}{|t - s|^{1/2}} \le C.$$

Therefore, by (4.5) and the fact that $|E| \le \sqrt{\varepsilon_1}$ we obtain

$$\operatorname{osc}_{B_{1/2}}(u(\cdot, t) - e \cdot x) \le C(\varepsilon_0 + \varepsilon_1^{\frac{1}{4n}} + \varepsilon_1^{\frac{1}{4}})$$

for all $t \in (-\frac{1}{4}, 0]$ (that is, including $t \in E$). If ε_0 and ε_1 are sufficiently small, we obtain from Lemma 4.5 that

$$\operatorname{osc}_{Q_{1/2}}(u - e \cdot x) \le C(\varepsilon_0 + \varepsilon_1^{\frac{1}{4n}} + \varepsilon_1^{\frac{1}{4}}).$$

Hence, if ε_0 and ε_1 are sufficiently small, there exists a constant $a \in \mathbb{R}$ such that

$$|u(t, x) - a - e \cdot x| \le \eta \quad \text{for all } (x, t) \in Q_{1/2}.$$ □

Theorem 4.7 (Regularity of small perturbation solutions). *Let u be a smooth solution of* (1.3) *in Q_1. For each $\beta \in (0, 1)$, there exist two positive constants η (small) and C (large), both of which depend only on β, n, γ and p, such that if $|u(x, t) - L(x)| \le \eta$ in Q_1 for some linear function L of x satisfying $\frac{1}{2} \le |\nabla L| \le 2$, then*

$$\|u - L\|_{C^{2,\beta}(Q_{1/2})} \le C.$$

Proof. Since L is a solution of (1.3), the conclusion follows from [25, Corollary 1.2]. □

Now we are ready to prove the following Hölder gradient estimate.

Theorem 4.8. *Let u be a smooth solution of (1.3) with $\varepsilon \in (0,1)$ and $\gamma > -1$ such that $|\nabla u| \le 1$ in Q_1. Then there exist two positive constants α and C depending only on n, γ and p such that*

$$|\nabla u(x,t) - \nabla u(y,s)| \le C\big(|x-y|^\alpha + |t-s|^{\frac{\alpha}{2-\alpha\gamma}}\big)$$

for all $(x,t),(y,s) \in Q_{1/2}$. Also, there holds

$$|u(x,t) - u(x,s)| \le C|t-s|^{\frac{1+\alpha}{2-\alpha\gamma}}$$

for all $(x,t),(x,s) \in Q_{1/2}$.

Proof. We first show the Hölder estimate of ∇u at $(0,0)$ and the Hölder estimate in t at $(0,0)$.

Let η be the one from Theorem 4.7 with $\beta = \frac{1}{2}$, and for this η let $\varepsilon_0, \varepsilon_1$ be two sufficiently small positive constants so that the conclusion of Lemma 4.6 holds. For $\ell = 1 - \varepsilon_0^2/2$ and $\mu = \varepsilon_1/|Q_1|$ if

$$\big|\{(x,t) \in Q_1 : \nabla u \cdot e \le \ell\}\big| \le \mu|Q_1| \quad \text{for any } e \in \mathbb{S}^{n-1},$$

then

$$\big|\{(x,t) \in Q_1 : |\nabla u - e| > \varepsilon_0\}\big| \le \varepsilon_1.$$

This is because if $|\nabla u(x,t) - e| > \varepsilon_0$ for some $(x,t) \in Q_1$, then

$$|\nabla u|^2 - 2\nabla u \cdot e + 1 \ge \varepsilon_0^2.$$

Since $|\nabla u| \le 1$, we have

$$\nabla u \cdot e \le 1 - \frac{\varepsilon_0^2}{2}.$$

Therefore, if $\ell = 1 - \varepsilon_0^2/2$ and $\mu = \varepsilon_1/|Q_1|$, then

$$\{(x,t) \in Q_1 : |\nabla u - e| > \varepsilon_0\} \subset \{(x,t) \in Q_1 : \nabla u \cdot e \le \ell\}, \tag{4.6}$$

from which it follows that

$$\big|\{(x,t) \in Q_1 : |\nabla u - e| > \varepsilon_0\}\big| \le \big|\{(x,t) \in Q_1 : \nabla u \cdot e \le \ell\}\big| \le \mu|Q_1| \le \varepsilon_1.$$

Let τ and δ be the constants from Corollary 4.2. We denote by $[\log\varepsilon/\log(1-\delta)]$ the integer part of $\log\varepsilon/\log(1-\delta)$. Let k be either $[\log\varepsilon/\log(1-\delta)]$ or the minimum nonnegative integer such that condition (4.3) does not hold, whichever is smaller. Then it follows from Corollary 4.2 that for all $\ell = 0, 1, \dots, k$ we have

$$|\nabla u(x,t)| \le (1-\delta)^\ell \quad \text{in } Q_{\tau^\ell}^{(1-\delta)^\ell}.$$

Then for

$$(x,t) \in Q_{\tau^\ell}^{(1-\delta)^\ell} \setminus Q_{\tau^{\ell+1}}^{(1-\delta)^{\ell+1}}$$

we obtain

$$|\nabla u(x,t)| \le (1-\delta)^\ell \le C\big(|x|^\alpha + |t|^{\frac{\alpha}{2-\alpha\gamma}}\big),$$

where

$$C = \frac{1}{1-\delta} \quad \text{and} \quad \alpha = \frac{\log(1-\delta)}{\log\tau}.$$

Thus,

$$|\nabla u(x,t) - q| \le C\big(|x|^\alpha + |t|^{\frac{\alpha}{2-\alpha\gamma}}\big) \quad \text{in } Q_1 \setminus Q_{\tau^{k+1}}^{(1-\delta)^{k+1}} \tag{4.7}$$

for every $q \in \mathbb{R}^n$ such that $|q| \le (1-\delta)^k$. Note that when $\gamma \ge 0$, it follows from (4.2) that

$$2 - \alpha\gamma > 0 \quad \text{and} \quad \frac{\alpha}{2-\alpha\gamma} < \frac{1}{2}.$$

For $\ell = 0, 1, \dots, k$, let

$$u_\ell(x,t) = \frac{1}{\tau^\ell(1-\delta)^\ell} u(\tau^\ell x, \tau^{2\ell}(1-\delta)^{-\ell\gamma} t). \tag{4.8}$$

Then $|\nabla u_\ell(x,t)| \le 1$ in Q_1, and

$$\partial_t u_\ell = (|\nabla u_\ell|^2 + \varepsilon^2(1-\delta)^{-2\ell})^{\frac{\gamma}{2}} \Big(\delta_{ij} + (p-2)\frac{\partial_i u_\ell \partial_j u_\ell}{|\nabla u_\ell|^2 + \varepsilon^2(1-\delta)^{-2\ell}}\Big)\partial_{ij} u_\ell \quad \text{in } Q_1. \tag{4.9}$$

Notice that $\varepsilon^2(1-\delta)^{-2\ell} \le \varepsilon^2(1-\delta)^{-2k} \le 1$. By Lemma 4.4, we have

$$\operatorname{osc}_{Q_1} u_\ell \le C,$$

and thus

$$\operatorname{osc}_{Q_{\tau^\ell}^{(1-\delta)^\ell}} u \le C\tau^\ell(1-\delta)^\ell. \tag{4.10}$$

Let $v = u_k$.

Case 1: $k = [\log\varepsilon/\log(1-\delta)]$. Then we have $(1-\delta)^{k+1} < \varepsilon \le (1-\delta)^k$, and thus $\frac{1}{2} < 1-\delta < \varepsilon(1-\delta)^{-k} \le 1$. Therefore, when $\ell = k$, equation (4.9) is a uniformly parabolic quasilinear equation with smooth and bounded coefficients. By the standard quasilinear parabolic equation theory (see, e.g., [17, Theorem 4.4, p. 560]) and Schauder estimates, there exists $b \in \mathbb{R}^n$, $|b| \le 1$, such that

$$|\nabla v(x,t) - b| \le C(|x| + |t|^{\frac{1}{2}}) \le C\big(|x|^\alpha + |t|^{\frac{\alpha}{2-\alpha\gamma}}\big) \quad \text{in } Q_\tau^{1-\delta} \subset Q_{1/4}$$

and

$$|\partial_t v| \le C \quad \text{in } Q_\tau^{1-\delta} \subset Q_{1/4},$$

where $C > 0$ depends only on γ, p and n, and we used that $\frac{\alpha}{2-\alpha\gamma} \le \frac{1}{2}$. Rescaling back, we have

$$|\nabla u(x,t) - (1-\delta)^k b| \le C\big(|x|^\alpha + |t|^{\frac{\alpha}{2-\alpha\gamma}}\big) \quad \text{in } Q_{\tau^{k+1}}^{(1-\delta)^{k+1}} \tag{4.11}$$

and

$$|u(x,t) - u(x,0)| \le C\tau^{-k}(1-\delta)^{k(\gamma+1)}|t| \quad \text{in } Q_{\tau^{k+1}}^{(1-\delta)^{k+1}}. \tag{4.12}$$

Then we can conclude from (4.7) and (4.11) that

$$|\nabla u(x,t) - q| \le C\big(|x|^\alpha + |t|^{\frac{\alpha}{2-\alpha\gamma}}\big) \quad \text{in } Q_{1/2},$$

where $C > 0$ depends only on γ, p and n. From (4.12) we obtain that for $|t| \le \tau^{2m}(1-\delta)^{-m\gamma}$ with $m \ge k+1$,

$$|u(0,t) - u(0,0)| \le C\tau^{-k}(1-\delta)^{k(\gamma+1)}\tau^{2m}(1-\delta)^{-m\gamma} \le C\tau^m(1-\delta)^m, \tag{4.13}$$

where in the last inequality we have used (4.2). From (4.10) and (4.13) we have

$$|u(0,t) - u(0,0)| \le C|t|^\beta$$

for all $t \in (-\frac{1}{4}, 0]$, where β is chosen such that

$$\tau(1-\delta) = (\tau^2(1-\delta)^{-\gamma})^\beta.$$

That is,

$$\beta = \frac{1+\alpha}{2-\alpha\gamma}. \tag{4.14}$$

Note that $\beta > \frac{1}{2}$ if $\gamma > -2$.

Case 2: $k < [\log \varepsilon / \log(1-\delta)]$. Then

$$\big|\{(x,t) \in Q_{\tau^k}^{(1-\delta)^k} : \nabla u \cdot e \le \ell(1-\delta)^k\}\big| \le \mu \big|Q_{\tau^k}^{(1-\delta)^k}\big| \quad \text{for some } e \in \mathbb{S}^{n-1}.$$

Also,

$$|\nabla u| < (1-\delta)^\ell \quad \text{in } Q_{\tau^\ell}^{(1-\delta)^\ell} \text{ for all } \ell = 0, 1, \ldots, k.$$

Recall $v = u_k$ as defined in (4.8), which satisfies (4.9) with $\ell = k$. Then $|\nabla v| \le 1$ in Q_1, and

$$\big|\{(x,t) \in Q_1 : \nabla v \cdot e \le \ell\}\big| \le \mu |Q_1| \quad \text{for some } e \in \mathbb{S}^{n-1}.$$

Consequently, using (4.6), we get

$$\big|\{(x,t) \in Q_1 : |\nabla v - e| > \varepsilon_0\}\big| \le \varepsilon_1.$$

It follows from Lemma 4.6 that there exists $a \in \mathbb{R}$ such that

$$|v(x,t) - a - e \cdot x| \le \eta \quad \text{for all } (x,t) \in Q_{1/2}.$$

By Theorem 4.7, there exists $b \in \mathbb{R}^n$ such that

$$|\nabla v - b| \le C(|x| + \sqrt{|t|}) \quad \text{for all } (x,t) \in Q_\tau^{1-\delta} \subset Q_{1/4}$$

and

$$|\partial_t v| \le C \quad \text{in } Q_\tau^{1-\delta} \subset Q_{1/4}.$$

Rescaling back, we have

$$|\nabla u(x,t) - (1-\delta)^k b| \le C\big(|x|^\alpha + |t|^{\frac{\alpha}{2-\alpha\gamma}}\big) \quad \text{in } Q_{\tau^{k+1}}^{(1-\delta)^{k+1}}$$

and

$$|u(x,t) - u(x,0)| \le C\tau^{-k}(1-\delta)^{k(\gamma+1)}|t| \quad \text{in } Q_{\tau^{k+1}}^{(1-\delta)^{k+1}}.$$

Together with (4.7) and (4.10), we can conclude as in case 1 that

$$|\nabla u(x,t) - q| \le C\big(|x|^\alpha + |t|^{\frac{\alpha}{2-\alpha\gamma}}\big) \quad \text{in } Q_{1/2}$$

and

$$|u(0,t) - u(0,0)| \le C|t|^\beta$$

for all $t \in (-\frac{1}{4}, 0]$, where $C > 0$ depends only on γ, p and n.

In conclusion, we have proved that there exist $q \in \mathbb{R}^n$ with $|q| \le 1$, and two positive constants α and C depending only on γ, p and n such that

$$|\nabla u(x,t) - q| \le C\big(|x|^\alpha + |t|^{\frac{\alpha}{2-\alpha\gamma}}\big) \quad \text{for all } (x,t) \in Q_{1/2}$$

and

$$|u(0,t) - u(0,0)| \le C|t|^\beta \quad \text{for } t \in (-\frac{1}{4}, 0],$$

where β is given in (4.14). Then the conclusion follows from standard translation arguments. □

5 Approximation

As mentioned in the introduction, the viscosity solutions to

$$u_t = |\nabla u|^\gamma\big(\Delta u + (p-2)|\nabla u|^{-2} u_i u_j u_{ij}\big) \quad \text{in } Q_1 \tag{5.1}$$

with $\gamma > -1$ and $p > 1$ fall into the general framework studied by Ohnuma and Sato in [21], which is an extension of the work of Barles and Georgelin [6] and Ishii and Souganidis [13] on the viscosity solutions of singular/degenerate parabolic equations. Let us recall from [21] the definition of viscosity solutions to (5.1).

We denote

$$F(\nabla u, \nabla^2 u) = |\nabla u|^\gamma(\Delta u + (p-2)|\nabla u|^{-2} u_i u_j u_{ij}).$$

Let $\mathcal{F}$ be the set of functions $f \in C^2([0,\infty))$ satisfying

$$f(0) = f'(0) = f''(0) = 0, \qquad f''(r) > 0 \quad \text{for all } r > 0$$

and

$$\lim_{|x|\to 0,\, x\neq 0} F(\nabla g(x), \nabla^2 g(x)) = \lim_{|x|\to 0,\, x\neq 0} F(-\nabla g(x), -\nabla^2 g(x)) = 0, \quad \text{where } g(x) = f(|x|).$$

This set $\mathcal{F}$ is not empty when $\gamma > -1$ and $p > 1$ since $f(r) = r^\beta \in \mathcal{F}$ for any $\beta > \max((\gamma+2)/(\gamma+1), 2)$. Moreover, if $f \in \mathcal{F}$, then $\lambda f \in \mathcal{F}$ for all $\lambda > 0$.

Because equation (5.1) may be singular or degenerate, one needs to choose the test functions properly when defining viscosity solutions. A function $\varphi \in C^2(Q_1)$ is admissible, which is denoted as $\varphi \in \mathcal{A}$, if for every $\hat{z} = (\hat{x}, \hat{t}) \in Q_1$ such that $\nabla\varphi(\hat{z}) = 0$ there exist $\delta > 0$, $f \in \mathcal{F}$ and $\omega \in C([0,\infty))$ satisfying $\omega \geq 0$ and $\lim_{r\to 0} \frac{\omega(r)}{r} = 0$ such that for all $z = (x,t) \in Q_1$, $|z - \hat{z}| < \delta$, we have

$$|\varphi(z) - \varphi(\hat{z}) - \varphi_t(\hat{z})(t - \hat{t})| \leq f(|x - \hat{x}|) + \omega(|t - \hat{t}|).$$

Definition 5.1. An upper (resp. lower) semi-continuous function u in Q_1 is called a viscosity subsolution (resp. supersolution) of (5.1) if for every admissible $\varphi \in C^2(Q_1)$ such that $u - \varphi$ has a local maximum (resp. minimum) at $(x_0, t_0) \in Q_1$, the following conditions hold:

$$\varphi_t \leq (\text{resp. } \geq)\ |\nabla\varphi|^\gamma(\Delta\varphi + (p-2)|\nabla\varphi|^{-2}\varphi_i\varphi_j\varphi_{ij}) \quad \text{at } (x_0, t_0) \text{ when } \nabla\varphi(x_0, t_0) \neq 0$$

and

$$\varphi_t \leq (\text{resp. } \geq)\ 0 \quad \text{at } (x_0, t_0) \text{ when } \nabla\varphi(x_0, t_0) = 0.$$

A function $u \in C(Q_1)$ is called a viscosity solution of (1.1), if it is both a viscosity subsolution and a viscosity supersolution.

We shall use two properties about the viscosity solutions defined above. The first one is the comparison principle for (5.1), which is [21, Theorem 3.1].

Theorem 5.2 (Comparison principle). *Let u and v be a viscosity subsolution and a viscosity supersolution of* (5.1) *in Q_1, respectively. If $u \leq v$ on $\partial_p Q_1$, then $u \leq v$ in $\overline{Q}_1$.*

The second one is the stability of viscosity solutions of (5.1), which is an application of [21, Theorem 6.1]. Its application to equation (5.1) with $\gamma = 0$ and $1 < p \leq 2$ is given in [21, Proposition 6.2] with detailed proof. It is elementary to check that it applies to (5.1) for all $\gamma > -1$ and all $p > 1$ (which was also pointed out in [21]).

Theorem 5.3 (Stability). *Let $\{u_k\}$ be a sequence of bounded viscosity subsolutions of* (1.3) *in Q_1 with $\varepsilon_k \geq 0$ such that $\varepsilon_k \to 0$ and u_k converges locally uniformly to u in Q_1. Then u is a viscosity subsolution of* (5.1) *in Q_1.*

Now we shall use the solution of (1.3) to approximate the solution of (5.1). Since $p > 1$, the following lemma ensues from classical quasilinear equations theory (see, e.g., [17, Theorem 4.4, p. 560]) and the Schauder estimates.

Lemma 5.4. *Let $g \in C(\partial_p Q_1)$. For $\varepsilon > 0$, there exists a unique solution $u^\varepsilon \in C^\infty(Q_1) \cap C(\overline{Q}_1)$ of* (1.3) *with $p > 1$ and $\gamma \in \mathbb{R}$ such that $u^\varepsilon = g$ on $\partial_p Q_1$.*

The last ingredient we need in the proof of Theorem 1.1 is the following continuity estimate up to the boundary for the solutions of (1.3). Its proof is given in Appendix A. For two real numbers a and b, we denote $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$.

Theorem 5.5 (Boundary estimates). *Let $u \in C(\overline{Q}_1) \cap C^\infty(Q_1)$ be a solution of (1.3) with $\gamma > -1$ and $\varepsilon \in (0,1)$. Let $\varphi := u|_{\partial_p Q_1}$ and let ρ be a modulus of continuity of φ. Then there exists another modulus of continuity ρ^* depending only on n, γ, p, ρ and $\|\varphi\|_{L^\infty(\partial_p Q_1)}$ such that*

$$|u(x,t) - u(y,s)| \le \rho^*(|x-y| \vee \sqrt{|t-s|})$$

for all $(x,t),(y,s) \in \overline{Q}_1$.

Proof of Theorem 1.1. Given Theorem 4.8, Theorem 5.2, Theorem 5.3, Lemma 5.4 and Theorem 5.5, the proof of Theorem 1.1 is identical to that of [14, Theorem 1]. □

A Proof of Theorem 5.5

We will adapt some arguments in [7] to prove Theorem 5.5. In the following, c denotes some positive constant depending only on n, γ and p, which may vary from line to line. Denote

$$F_\varepsilon(\nabla u, \nabla^2 u) = (|\nabla u|^2 + \varepsilon^2)^{\frac{\gamma}{2}}\Big(\delta_{ij} + (p-2)\frac{u_i u_j}{|\nabla u|^2 + \varepsilon^2}\Big)u_{ij}.$$

Lemma A.1. *For every $z \in \partial B_1$, there exists a function $W_z \in C(\overline{B}_1)$ such that $W_z(z) = 0$ and $W_z > 0$ in $\overline{B}_1 \setminus \{z\}$, and*

$$F_\varepsilon(\nabla W_z, \nabla^2 W_z) \le -1 \quad \text{in } B_1.$$

Proof. Let $z \in \partial B_1$. Let $f(r) = \sqrt{(r-1)^+}$ and $w_z(x) = f(|x-2z|)$. Then for $x \in B_1$ we have

$$F_\varepsilon(\nabla w_z, \nabla^2 w_z) = (f'^2 + \varepsilon^2)^{\frac{\gamma}{2}}\Big(\Big(1 + (p-2)\frac{f'^2}{f'^2+\varepsilon^2}\Big)f'' + \frac{n-1}{|x-2z|}f'\Big).$$

Then there exists $\delta > 0$ depending only on n, γ and p such that for $x \in B_1 \cap B_{1+\delta}(2z)$ we have

$$F_\varepsilon(\nabla w_z, \nabla^2 w_z) \le -1.$$

For

$$\sigma = \frac{2n}{\min(p-1,1)} + 2 \quad \text{and} \quad a > 0,$$

let

$$G_z(x) = a\Big(2^\sigma - \frac{1}{|x-2z|^\sigma}\Big).$$

Then $G_z(x) \ge a(2^\sigma - 1)$ in B_1. Also, for $r = |x-2z|$ and $x \in B_1$ we have

$$\begin{aligned} F_\varepsilon(\nabla G_z, \nabla^2 G_z) &= a(\sigma^2 r^{-2\sigma-2} + \varepsilon^2)^{\frac{\gamma}{2}}\Big(\Big(1 + \frac{(p-2)\sigma^2}{\sigma^2 + \varepsilon^2 r^{2\sigma+2}}\Big)\sigma(-\sigma-1)r^{-\sigma-2} + (n-1)\sigma r^{-\sigma-2}\Big) \\ &\le -\frac{a}{2}\sigma r^{-\sigma-2}(\sigma^2 r^{-2\sigma-2} + \varepsilon^2)^{\frac{\gamma}{2}} \\ &\le \begin{cases} -\frac{a}{2}3^{-\sigma-2-\gamma(\sigma+1)}\sigma^{1+\gamma} & \text{if } \gamma \ge 0, \\ -\frac{a}{2}3^{-\sigma-2}(\sigma^2+1)^{\frac{\gamma}{2}}\sigma & \text{if } \gamma < 0, \end{cases} \end{aligned}$$

where in the first inequality we used the choice of σ. Then we choose a such that

$$a\Big(2^\sigma - \frac{1}{|1+\delta|^\sigma}\Big) = \sqrt{\frac{\delta}{2}}.$$

Since $w_z(z) = 0$ and $G_z(z) > 0$, the function

$$W_z(x) = \begin{cases} G_z(x) & \text{for } x \in \overline{B}_1,\ |x-2z| \ge 1+\delta, \\ \min(G_z(x), w_z(x)) & \text{for } x \in \overline{B}_1,\ |x-2z| \le 1+\delta, \end{cases}$$

agrees with w_z in a neighborhood of z (relative to $\overline{B}_1$). Also, because of the choice of a, the function W_z agrees with G_z when $x \in \overline{B}_1$ and $|x-2z| \geq 1+\tilde{\delta}$ for some $\tilde{\delta} \in (0,\delta)$. Moreover,

$$F_\varepsilon(\nabla W_z, \nabla^2 W_z) \leq -\kappa$$

for some constant $\kappa > 0$ depending only on n, γ and p. By multiplying a large positive constant to W_z, we finish the proof of this lemma. □

Lemma A.2. *For every $(z,\tau) \in \partial_p Q_1$, there exists $W_{z,\tau} \in C(\overline{Q}_1)$ such that $W_{z,\tau}(z,\tau) = 0$, $W_{z,\tau} > 0$ in $\overline{Q}_1 \setminus \{(z,\tau)\}$ and*

$$\partial_t W_{z,\tau} - F_\varepsilon(\nabla W_{z,\tau}, \nabla^2 W_{z,\tau}) \geq 1 \quad \text{in } Q_1.$$

Proof. For $\tau > -1$ and $x \in \partial B_1$,

$$W_{z,\tau}(x,t) = \frac{(t-\tau)^2}{2} + 2W_z$$

is a desired function, where W_z is the one from Lemma A.1. For $\tau = -1$ and $x \in B_1$, we let

$$W_{z,\tau}(x,t) = A(t+1) + |x-z|^\beta,$$

where

$$\beta = \max\Big(\frac{\gamma+2}{\gamma+1}, 2\Big).$$

If we choose $A > 0$ large, which depends only on n, γ and p, then $W_{z,\tau}$ will be a desired function. □

For two real numbers a and b, we denote $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.

Theorem A.3. *Let $u \in C(\overline{Q}_1) \cap C^\infty(Q_1)$ be a solution of (1.3) with $\gamma > -1$ and $\varepsilon \in (0,1)$. Let $\varphi := u|_{\partial_p Q_1}$ and let ρ be a modulus of continuity of φ. Then there exists another modulus of continuity ρ^* depending only on n, γ, p and ρ such that*

$$|u(x,t) - u(y,s)| \leq \tilde{\rho}(|x-y| \vee |t-s|)$$

for all $(x,t) \in \overline{Q}_1$ and $(y,s) \in \partial_p Q_1$.

Proof. For every $\kappa > 0$ and $(z,\tau) \in \partial_p Q_1$, let

$$W_{\kappa,z,\tau}(x,t) = \varphi(z,\tau) + \kappa + M_\kappa W_{z,\tau}(x,t),$$

where $M_\kappa > 0$ is chosen so that

$$\varphi(z,\tau) + \kappa + M_\kappa W_{z,\tau}(y,s) \geq \varphi(y,s) \quad \text{for all } (y,s) \in \partial_p Q_1.$$

Indeed,

$$M_k = \inf_{(y,s)\in\partial_p Q_1,\,(y,s)\neq(z,\tau)} \frac{(\rho(|z-y| \vee |\tau-s|) - \kappa)^+}{W_{z,\tau}(y,s)}$$

would suffice, and is independent of the choice of (z,τ). Finally, let

$$W(x,t) = \inf_{\kappa>0,\,(z,\tau)\in\partial_p Q_1} W_{\kappa,z,\tau}(x,t).$$

Note that for every $\kappa > 0$ and $(z,\tau) \in \partial_p Q_1$,

$$\begin{aligned} W(x,t) - \varphi(z,\tau) &\leq W_{\kappa,z,\tau}(x,t) - \varphi(z,\tau) \\ &\leq \kappa + M_\kappa W_{z,\tau}(x,t) \\ &\leq \kappa + M_\kappa (W_{z,\tau}(x,t) - W_{z,\tau}(z,\tau)) \\ &\leq \kappa + M_\kappa \omega(|z-x| \vee |\tau-t|), \end{aligned}$$

where ω is the modulus of continuity for $W_{z,\tau}$, which is evidently independent of (z,τ). Let

$$\tilde{\rho}(r) = \inf_{\kappa>0}(\kappa + M_\kappa \omega(r))$$

for all $r \geq 0$. Then $\tilde{\rho}$ is a modulus of continuity, and

$$W(x,t) - \varphi(z,\tau) \leq \tilde{\rho}(|z-x| \vee |\tau - t|) \quad \text{for all } (x,t) \in \overline{Q}_1,\ (z,\tau) \in \partial_p Q_1.$$

By Lemma A.2, $W_{\kappa,z,\tau}$ is a supersolution of (1.3) for every $\kappa > 0$ and $(z,\tau) \in \partial_p Q_1$, and therefore W is also a supersolution of (1.3). By the comparison principle,

$$u(x,t) - \varphi(z,\tau) \leq W(x,t) - \varphi(z,\tau) \leq \tilde{\rho}(|z-x| \vee |\tau - t|)$$

for all $(x,t) \in \overline{Q}_1$ and $(z,\tau) \in \partial_p Q_1$.

Similarly, one can show that $u(x,t) - \varphi(z,\tau) \geq -\tilde{\rho}(|z-x| \vee |\tau - t|)$ for all $(x,t) \in \overline{Q}_1$ and $(z,\tau) \in \partial_p Q_1$. This finishes the proof of this theorem. □

Proof of Theorem 5.5. By the maximum principle, we have that

$$M := \|u\|_{L^\infty(Q_1)} = \|\varphi\|_{L^\infty(\partial_p Q_1)}.$$

Let $(x,t), (y,s) \in Q_1$, and assume that $t \geq s$. Let x_0 be such that $|x - x_0| = 1 - |x| = r$. Let $\tilde{\rho}$ be the one from the conclusion of Theorem A.3. Without loss of generality, we may assume that $2M + 2 \geq \tilde{\rho}(r) \geq r$ for all $r \in [0,2]$ (e.g., replacing $\tilde{\rho}(r)$ by $\tilde{\rho}(r) + r$), and $\tilde{\rho}(r) \leq 2M + 2$ for all $r \geq 2$.

In the following, if $\gamma \in (-1,0)$, then we will assume first that

$$r^{1+\gamma}(2M+2)^{-\gamma} \leq 1,$$

and will deal with the other situation in the end of this proof. Under the above assumption, we have that $r^{2+\gamma}(\tilde{\rho}(2r))^{-\gamma} \leq r^{2+\gamma}(2M+2)^{-\gamma} \leq r$ when $\gamma < 0$, and $r^{2+\gamma}(\tilde{\rho}(2r))^{-\gamma} \leq r^{2+\gamma}(\tilde{\rho}(r))^{-\gamma} \leq r^2 \leq r$ when $\gamma \geq 0$. Thus, for all $\gamma > -1$ we have

$$r^{2+\gamma}(\tilde{\rho}(2r))^{-\gamma} \leq r.$$

We will deal with the situation that $\gamma \in (-1,0)$ and $r^{1+\gamma}(2M+2)^{-\gamma} \geq 1$ at the very end of the proof.

Case 1: $r^{2+\gamma}(\tilde{\rho}(2r))^{-\gamma} \leq 1 + t$. If $|y - x| \leq \frac{r}{2}$ and $|s - t| \leq r^{2+\gamma}(\tilde{\rho}(2r))^{-\gamma}/4$, then we do a scaling:

$$v(z,\tau) = \frac{u\big(rz + x, r^{2+\gamma}(\tilde{\rho}(2r))^{-\gamma}\tau + t\big) - u(x_0,t)}{\tilde{\rho}(2r)}.$$

Then

$$v_\tau = (|\nabla v|^2 + \varepsilon^2 r^2 \tilde{\rho}(2r)^{-2})^{\frac{\gamma}{2}}\Big(\delta_{ij} + (p-2)\frac{v_i v_j}{|\nabla u|^2 + \varepsilon^2 r^2 \tilde{\rho}(2r)^{-2}}\Big)u_{ij} \quad \text{in } Q_1.$$

Notice that $\varepsilon r/\tilde{\rho}(2r) \leq \varepsilon r/\tilde{\rho}(r) \leq \varepsilon < 1$ and $r^{2+\gamma}(\tilde{\rho}(2r))^{-\gamma} \leq r$. Thus, $|v(z,\tau)| \leq 1$ for $(z,\tau) \in Q_1$. By applying Corollary 2.4 and Lemma 3.1 to v and rescaling to u, there exists $\alpha > 0$ depending only on γ such that v is C^α in (x,t), and there exists $C > 0$ depending only on n, γ and p such that

$$|u(y,s) - u(x,s)| \leq C\tilde{\rho}(2r)\frac{|x-y|^\alpha}{r^\alpha}$$

and

$$|u(x,t) - u(x,s)| \leq C\tilde{\rho}(2r)^{1+\alpha\gamma}\frac{|t-s|^\alpha}{r^{\alpha(2+\gamma)}},$$

Therefore,

$$|u(y,s) - u(x,t)| \leq C\tilde{\rho}(2r)\frac{|x-y|^\alpha}{r^\alpha} + C\tilde{\rho}(2r)^{1+\alpha\gamma}\frac{|t-s|^\alpha}{r^{\alpha(2+\gamma)}}.$$

Since $|y - x| \leq \frac{r}{2}$ and $|s - t| \leq r^{2+\gamma}(\tilde{\rho}(2r))^{-\gamma}/4 \leq \frac{r}{4}$, we have $2^{-m-1}r < |x-y| \vee |t-s| \leq 2^{-m}r$ for some integer $m \geq 1$. Then

$$\begin{aligned}|u(y,s) - u(x,t)| &\leq C\frac{\tilde{\rho}(2^{m+2}(|x-y| \vee |t-s|))}{2^{m\alpha}} + C\frac{\tilde{\rho}(2^{m+2}(|x-y| \vee |t-s|))^{1+\alpha\gamma}}{2^{m\alpha}r^{\alpha(1+\gamma)}}\\ &\leq C\frac{\tilde{\rho}(2^{m+2}(|x-y| \vee |t-s|)) + \tilde{\rho}(2^{m+2}(|x-y| \vee |t-s|))^{1+\alpha\gamma}}{2^{m\alpha}}.\end{aligned}$$

Notice that

$$\sup_{m\geq 1}\frac{\tilde\rho(2^{m+2}r)+\tilde\rho(2^{m+2}r)^{1+\alpha\gamma}}{2^{m\alpha}}\to 0\quad\text{as } r\to 0.$$

Therefore, we can choose a modulus of continuity ρ_1 such that

$$\rho_1(r)\geq C\sup_{m\geq 1}\frac{\tilde\rho(2^{m+2}r)+\tilde\rho(2^{m+2}r)^{1+\alpha\gamma}}{2^{m\alpha}}\quad\text{for all } r\geq 0,$$

and we have

$$|u(y,s)-u(x,t)|\leq\rho_1(|x-y|\vee|t-s|).$$

If $|y-x|\geq\frac{r}{2}$, then

$$\begin{aligned}|u(x,t)-u(y,s)|&\leq|u(x,t)-u(x_0,t)|+|u(x_0,t)-u(y,s)|\\&\leq\tilde\rho(r)+\tilde\rho(|x_0-y|\vee|t-s|)\\&\leq\tilde\rho(2(|x-y|\vee|t-s|))+\tilde\rho((|x-y|+r)\vee|t-s|)\\&\leq\tilde\rho(2(|x-y|\vee|t-s|))+\tilde\rho(3(|x-y|\vee|t-s|))\\&\leq 2\tilde\rho(3(|x-y|\vee|t-s|)).\end{aligned}$$

If $|x-y|\leq\frac{r}{2}$ and $|s-t|\geq r^{2+\gamma}(\tilde\rho(2r))^{-\gamma}/4$, then

$$r\leq 4^{\frac{1}{2+\gamma}}(2M+2)^{\frac{\gamma}{2+\gamma}}|s-t|^{\frac{1}{2+\gamma}}$$

when $\gamma\geq 0$, and $r\leq 2|s-t|^{1/2}$ when $\gamma\leq 0$. Then one can show similar to the above that

$$\begin{aligned}|u(x,t)-u(y,s)|&\leq 2\tilde\rho(c(|x-y|\vee|t-s|^{\frac12}\vee|s-t|^{\frac{1}{2+\gamma}}))\\&\leq\rho_2(|x-y|\vee|t-s|),\end{aligned}$$

where $\rho_2(r)=2\tilde\rho(cr^{1/2})$ or $\rho_2(r)=2\tilde\rho(cr^{1/(2+\gamma)})$ depending on whether $\gamma\geq 0$ or $\gamma\leq 0$ is a modulus of continuity, c is a positive constant depending only on M and γ.

This finishes the proof in this first case.

Case 2: $r^{2+\gamma}(\tilde\rho(2r))^{-\gamma}\geq 1+t$. Then let $\lambda=\sqrt{|t+1|}$ when $\gamma\geq 0$, and $\lambda=(2M+2)^{\gamma/(2+\gamma)}|t+1|^{1/(2+\gamma)}$ when $\gamma\in(-1,0)$. Then one can check that $\lambda\leq r$.

If $|y-x|\leq\frac{\lambda}{2}$ and $|s-t|\leq\lambda^{2+\gamma}(\tilde\rho(2\lambda))^{-\gamma}/4$, let

$$v(z,\tau)=\frac{u(\lambda z+x,\lambda^{2+\gamma}(\tilde\rho(2\lambda))^{-\gamma}\tau+t)-u(x_0,t)}{\tilde\rho(2\lambda)}\quad\text{for }(z,\tau)\in Q_1.$$

Then

$$v_\tau=(|\nabla v|^2+\varepsilon^2r^2\tilde\rho(2\lambda)^{-2})^{\frac{\gamma}{2}}\Big(\delta_{ij}+(p-2)\frac{v_iv_j}{|\nabla u|^2+\varepsilon^2\lambda^2\tilde\rho(2\lambda)^{-2}}\Big)u_{ij}\quad\text{in } Q_1.$$

Notice that $\lambda^{2+\gamma}(\tilde\rho(2\lambda))^{-\gamma}\leq\lambda^2\leq\lambda$ when $\gamma\geq 0$, and $\lambda^{2+\gamma}(\tilde\rho(2\lambda))^{-\gamma}\leq\lambda r^{1+\gamma}(\tilde\rho(2r))^{-\gamma}\leq\lambda$ when $\gamma\in(-1,0)$. Thus, $|v(z,\tau)|\leq 1$ for $(z,\tau)\in Q_1$. Also, $\varepsilon\lambda/\tilde\rho(2\lambda)\leq\varepsilon\lambda/\tilde\rho(\lambda)\leq\varepsilon<1$. Then, by arguments similar to the ones in case 1, we have

$$|u(y,s)-u(x,t)|\leq\rho_1(|x-y|\vee|t-s|).$$

If $|y-x|\geq\frac{\lambda}{2}$, then $|t+1|\leq c(|x-y|^2\vee|x-y|^{2+\gamma})\leq c|x-y|$ for some $c>0$ depending only on M and γ. Therefore,

$$\begin{aligned}|u(x,t)-u(y,s)|&\leq|u(x,t)-u(x,-1)|+|u(x,-1)-u(y,s)|\\&\leq\tilde\rho(|t+1|)+\tilde\rho(|x-y|\vee|1+s|)\\&\leq\tilde\rho(c|x-y|)+\tilde\rho((|x-y|)\vee|1+t|)\\&\leq\tilde\rho(c(|x-y|\vee|t-s|))+\tilde\rho(c|x-y|\vee|t-s|)\\&=2\tilde\rho(c(|x-y|\vee|t-s|))\\&\leq\rho_2(|x-y|\vee|t-s|).\end{aligned}$$

If $|x-y| \le \frac{\lambda}{2}$ and $|s-t| \ge \lambda^{2+\gamma}(\tilde{\rho}(2\lambda))^{-\gamma}/4$, then

$$\lambda \le 4^{\frac{1}{2+\gamma}}(2M+2)^{\frac{\gamma}{2+\gamma}}|s-t|^{\frac{1}{2+\gamma}}$$

when $\gamma \ge 0$, and $\lambda \le 2|s-t|^{1/2}$ when $\gamma \le 0$. Then one can show similar to the above that

$$\begin{aligned}|u(x,t)-u(y,s)| &\le |u(x,t)-u(x,-1)| + |u(x,-1)-u(y,s)|\\ &\le \tilde{\rho}(|t+1|) + \tilde{\rho}(|x-y| \vee |1+s|)\\ &\le \tilde{\rho}\big(c\big(|s-t|^{\frac{2}{2+\gamma}} \vee |s-t|^{\frac{2+\gamma}{2}}\big)\big) + \tilde{\rho}((|x-y|) \vee |1+t|)\\ &\le \tilde{\rho}\big(c\big(|s-t|^{\frac{1}{2+\gamma}} \vee |s-t|^{\frac{1}{2}}\big)\big) + \tilde{\rho}\big(c\big(|s-t|^{\frac{1}{2+\gamma}} \vee |s-t|^{\frac{1}{2}}\big)\big)\\ &\le \rho_2(|x-y| \vee |t-s|).\end{aligned}$$

This finishes the proof in this second case.

In the end, we deal with the situation that $\gamma \in (-1,0)$ and $r^{1+\gamma}(2M+2)^{-\gamma} \ge 1$. Then we have $r \ge c$ for $c = (2M+2)^{\gamma/(1+\gamma)}$. Let

$$\lambda = (2M+2)^{\frac{\gamma}{2+\gamma}}|t+1|^{\frac{1}{2+\gamma}}.$$

There exists $\mu > 0$ depending only on M and γ such that if $|t+1| \le \mu$, then $\lambda \le c$, $c^{2+\gamma}(\tilde{\rho}(2c))^{-\gamma} \ge 1+t$ and $\lambda^{1+\gamma}(2M+2)^{-\gamma} \le 1$. Then, for $t \le -1+\mu$, the same arguments as in case 2 work without any change.

Now the final case left is that $(x,t) \in \overline{B}_{1-c} \times [-1+\mu, 0]$. Then we only need to consider that

$$(y,s) \in B_{1-c/2} \times \Big[-1+\frac{\mu}{2}, 0\Big].$$

It follows from Corollary 2.4 and Lemma 3.1 that there exists a modulus of continuity $\bar{\rho}$ depending only on n, γ, p and M such that

$$|u(x,t)-u(y,s)| \le \overline{\rho}(|x-y| \vee |t-s|).$$

This finishes the final situation.

Then $\rho^*(r) := \rho_1(r) + \rho_2(r) + \bar{\rho}(r)$ is a desired modulus of continuity. The proof of this theorem is thereby completed. □

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