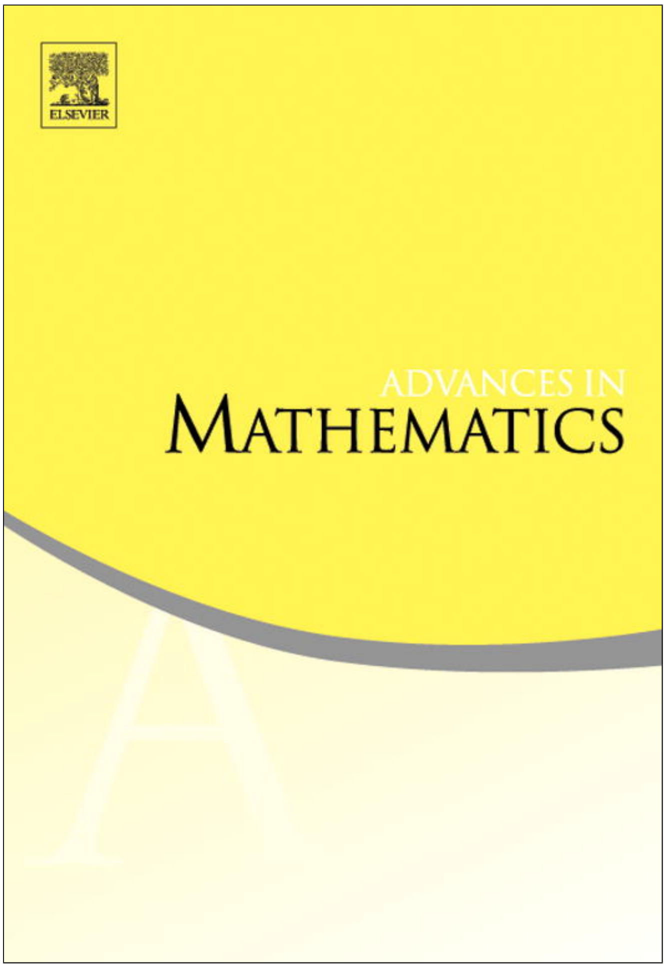
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$C^{1,\alpha}$ regularity of solutions of some degenerate fully non-linear elliptic equations

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Abstract

In the present paper, a class of fully non-linear elliptic equations are considered, which are degenerate as the gradient becomes small. Hölder estimates obtained by the first author (2011) are combined with new Lipschitz estimates obtained through the Ishii–Lions method in order to get $C^{1,\alpha}$ estimates for solutions of these equations.

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1. Introduction

This paper is concerned with the study of the regularity of solutions of the following non-linear elliptic equation

$$|\nabla u|^{\gamma} F(D^2 u) = f \quad \text{in } B_1 \tag{1}$$

where B_1 is the unit ball of $\mathbb{R}^d$ and $\gamma > 0$, F is uniformly elliptic, $F(0) = 0$ and f is bounded.

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Singular/degenerate fully non-linear elliptic equations. Eq. (1) makes part of a class of non-linear elliptic equations studied in a series of papers by Birindelli and Demengel, starting with [3]. The specificity of these equations is that they are not uniformly elliptic; they are either singular or degenerate (in a way to be made precise).

Birindelli and Demengel proved many important results in the singular case such as comparison principles and Liouville type results [3], regularity and uniqueness of the first eigenfunction [4] *etc.* In the degenerate case, the set of results [5,6] is less complete and in particular, there was no $C^{1,\alpha}$ estimate in the non-radial case (see [7] for the radial case).

Alexandrov–Bakelman–Pucci (ABP) estimates were obtained for such equations independently in [9,11]. It was used to derive the Harnack inequality in the singular case in [10] and in both cases in [11]. From the Harnack inequality, it is classical to derive Hölder estimates ([10] in the singular case, [11] in both cases).

Main result. The main result of this paper is the following.

Theorem 1. *Assume that $\gamma \geq 0$, F is uniformly elliptic, $F(0) = 0$, and f is bounded in B_1. There exist $\alpha > 0$ and $C > 0$ only depending on γ, the ellipticity constants of F and dimension d, such that any viscosity solution u of (1) is $C^{1,\alpha}$ and*

$$[u]_{1+\alpha,B_{1/2}} \leq C\left(\|u\|_{L^\infty} + \|f\|_{L^\infty}^{\frac{1}{1+\gamma}}\right).$$

Comments. Getting $C^{1,\alpha}$ estimates consists in proving that the graph of the function u can be approximated by planes with an error bounded by $Cr^{1+\alpha}$ in balls of radius r. The proof is based on an iterative argument, in which we show that the graph of u gets *flatter* (meaning better approximated by planes) in smaller balls. The iterative step, after a rescaling, amounts to show that if $p \cdot x + u$ satisfies (1) in B_1 with $\operatorname{osc} u \leq 1$, then the oscillation of u, up to a linear function $p' \cdot x$, is smaller in a smaller ball. This is proved by compactness. In order to make such an argument work, the modulus of continuity of u has to be controlled independently of the slopes p and p' which can vary from one scale to the other. There is a difficulty since $u - p \cdot x$ does not satisfy any PDE independently of p. The main originality of this paper is to combine the method introduced by Ishii and Lions [12] to get Lipschitz estimate in the case of large slopes and the Harnack inequality approach of Caffarelli and Cabré [8] adapted in [11] to the present framework for small slopes.

An alternative approach to find a modulus of continuity for solutions of the rescaled equation (see (6) below) for large slopes could be to apply the Harnack inequality from [14] to get a uniform Hölder modulus of continuity for $|p|$ large enough instead of the Ishii–Lions method to get a uniform Lipschitz estimate. We chose the latter approach because of its simplicity.

The following example shows that solutions u of (1) cannot be more regular than $C^{1,\alpha}$, even if f is Hölder continuous.

Example 1. The function $u(x) = |x|^{1+\alpha}$ satisfies

$$|Du|^\gamma \Delta u = C|x|^{(1+\alpha)(\gamma+1)-(\gamma+2)}$$

where $C = (1+\alpha)^{1+\gamma}(d+\alpha-1)$. In particular, if we choose $\alpha = 1/(1+\gamma)$ the right hand side is simply constant. This example shows that even for a constant right hand side and $F(D^2u) = \Delta u$, we cannot expect in general the solution to be more regular than $C^{1,\alpha}$ with $\alpha < 1$.

As far as the authors know, the result of Theorem 1 is new even for the simple equation $|\nabla u|^\gamma \Delta u = f(x)$. For this case we expect the optimal α to be in fact equal to $1/(1+\gamma)$ although we did not work on that issue. For general fully nonlinear equations $F(D^2u)$ the value of α can get arbitrarily small even in the case $\gamma = 0$ (see [13] for an example).

The paper is organized as follows. In Section 2 we specify the notation to be used in the paper and we review a few well known definitions and results for fully nonlinear elliptic equations. In Section 3 we restate Theorem 1 in a simplified form simply by rescaling. In Section 3, we also show how the iteration of the *improvement of flatness* lemma implies the main theorem. The methods of Section 3 are more or less standard for proving $C^{1,\alpha}$ regularity for elliptic equations. In Section 4 we find a uniform modulus of continuity for the difference between the solution and a plane appropriately rescaled. Based on this continuity estimates we prove the improvement of oscillation lemma by a compactness argument. In the last section we show a technical lemma that says that viscosity solutions to $|\nabla u|^\gamma F(D^2u) = 0$ are also viscosity solutions to $F(D^2u) = 0$. This lemma is used to characterize the limits in the compactness argument for the proof of the improvement of flatness lemma in Section 4.

2. Preliminaries

2.1. Notation

For $r > 0$, $B_r(x)$ denotes the open ball of radius r centred at x. B_r denotes $B_r(0)$. $\mathbb{S}_d$ denotes the set of symmetric $d \times d$ real matrices. I denotes the identity matrix.

For $\alpha \in (0, 1]$ and $Q \subset \mathbb{R}^d$, we consider

$$[u]_{\alpha,Q} = \sup_{x,y\in Q, x\neq y} \frac{u(x)-u(y)}{|x-y|^\alpha},$$

$$[u]_{1+\alpha,Q} = \sup_{\rho>0, x\in Q} \inf_{p\in\mathbb{R}^d, c\in\mathbb{R}} \sup_{z\in B_\rho(x)\cap Q} \rho^{-1-\alpha}|u(z) - p\cdot z - c|.$$

2.2. Uniform ellipticity

We recall the definition of uniform ellipticity (see [8] for more details). We say that a function F defined on the set of real symmetric matrices and taking real values is uniformly elliptic if there exist two positive constants λ and Λ such that for any two symmetric matrices X and Y, with $Y \geq 0$ we have

$$\lambda \operatorname{tr} Y \leq F(X) - F(X+Y) \leq \Lambda \operatorname{tr} Y.$$

The constants λ and Λ are called the *ellipticity constants*. Under this definition $F(X) = -\operatorname{tr}(X)$ is uniformly elliptic with ellipticity constants $\lambda = \Lambda = 1$, and $F(D^2u) = -\Delta u = f(x)$ is a uniformly elliptic equation.

The maximum and minimum of all the uniformly elliptic functions F such that $F(0) = 0$, are called the *Pucci operators*. We write them P^+ and P^-. Recall that P^- has the closed form

$$P^-(X) = -\Lambda \operatorname{tr} X^+ - \lambda \operatorname{tr} X^-,$$

where $\operatorname{tr} X^+$ is the sum of all positive eigenvalues of X and $\operatorname{tr} X^-$ is the sum of all negative eigenvalues of X. With the definition of P^+ and P^- at hand, it is equivalent that F is uniformly elliptic with the inequality

$$P^-(Y) \leq F(X+Y) - F(X) \leq P^+(Y),$$

for any two symmetric matrices X and Y.

2.3. Two observations

The uniform ellipticity hypothesis on F implies that there exist $\alpha_0 \in (0,1)$ and $C > 0$ such that viscosity solutions of $F(D^2u) = 0$ in B_1 are C^{1,α_0} in the interior of B_1 and

$$[u]_{1+\alpha_0, B_{1/2}} \leq C\|u\|_{L^\infty(B_1)}.$$

The constants α_0 and C depend on the ellipticity constants and dimension only.

Note that for any constant $a > 0$, the function $a^{-1}F(aX)$ has the same ellipticity constants as F. This will be important when rescaling the equation.

3. Reduction of the problem

In this section, we first show that a simple rescaling reduces the proof of the problem to the case that $\|u\|_{L^\infty} \leq 1/2$ and $\|f\|_{L^\infty} \leq \varepsilon_0$ for some small constant ε_0 which will be chosen later. We then further reduce the proof to an *improvement of flatness* lemma.

3.1. Rescaling

We work with the arbitrary normalization $\|u\|_{L^\infty} \leq 1/2$ because that implies that $\operatorname{osc} u \leq 1$ and that will be a good starting point for our iterative proof of $C^{1,\alpha}$ regularity.

Proposition 1. *In order to prove Theorem 1, it is enough to prove that*

$$[u]_{1+\alpha, B_{1/2}} \leq C$$

assuming $\|u\|_{L^\infty(B_1)} \leq 1/2$ and $\|f\|_{L^\infty(B_1)} \leq \varepsilon_0$ for some $\varepsilon_0 > 0$ which only depends on the ellipticity constants, dimension and γ.

Proof. Given any function u under the assumptions of Theorem 1, we can take $\kappa = \big(2\|u\|_{L^\infty} + (\|f\|_{L^\infty}/\varepsilon_0)^{1/(1+\gamma)}\big)^{-1}$ and consider the scaled function $\tilde{u}(x) = \kappa u(x)$ solving the equation

$$|\nabla \tilde{u}|^\gamma \kappa F(\kappa^{-1} D^2 \tilde{u}) = \kappa^{1+\gamma} f(x).$$

We previously made the observation that the function $\kappa F(\kappa^{-1}X)$ has the same ellipticity constants as $F(X)$. But now $\|\tilde{u}\|_{L^\infty} \leq 1/2$ and $\|\tilde{f}\|_{L^\infty} \leq \varepsilon_0$. Therefore, if

$$[\tilde{u}]_{1+\alpha, B_{1/2}} \leq C,$$

by scaling back to u, we get

$$[u]_{1+\alpha, B_{1/2}} \leq C\kappa^{-1} \leq C\big(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty}^{1/(1+\gamma)}\big)$$

which concludes the proof. $\square$

It is enough to prove that the solution u of (1) is $C^{1,\alpha}$ at 0 that is to say that there exist $C > 0$ and α(only depending on the ellipticity constants, dimension and γ) such that for all $r \in (0,1)$, there exists $p \in \mathbb{R}^d$ such that

$$\operatorname*{osc}_{B_r}(u - p \cdot x) \le Cr^{1+\alpha}. \tag{2}$$

If we start with a function u such that $\operatorname{osc}_{B_1} u \le 1$, we already have the inequality for $r = 1$ with $C = 1$. In order to get such a result for all $r \in (0, 1)$, it is enough to find $\rho, \alpha \in (0, 1)$ such that for all $k \in \mathbb{N}$ there exists $p_k \in \mathbb{R}^d$ such that

$$\operatorname*{osc}_{B_{\rho^k}}(u - p_k \cdot x) \le \rho^{k(1+\alpha)}.$$

The inequality (2) follows with $C = \rho^{-(1+\alpha)}$.

This is the reason why we consider $r_k = \rho^k$ and we aim at proving by induction on $k \in \mathbb{N}$ the following.

Lemma 1. *There exist $\rho, \alpha \in (0, 1)$ and $\varepsilon_0 \in [0, 1]$ only depending on γ, ellipticity constants and dimension such that, as soon as a viscosity solution u of (1) with $\|f\|_{L^\infty} \le \varepsilon_0$ satisfies $\operatorname{osc}_{B_1} u \le 1$, then for all $k \in \mathbb{N}$, there exists $p_k \in \mathbb{R}^d$ such that*

$$\operatorname*{osc}_{B_{r_k}}(u - p_k \cdot x) \le r_k^{1+\alpha} \tag{3}$$

where $r_k = \rho^k$.

The choice of ρ depends on the C^{1,α_0} estimates for $F(D^2u) = 0$. Precisely, since we assume that any viscosity solution u of $F(D^2u) = 0$ in B_1 is C^{1,α_0}, it is in particular C^{1,α_0} at 0, that is to say there exists $C_0 > 0$ such that for all $r \in (0, 1)$, there exists $p \in \mathbb{R}^d$ such that

$$\operatorname*{osc}_{B_r}(u - p \cdot x) \le C_0 r^{1+\alpha_0}.$$

We then pick $\rho \in (0, 2^{-\gamma-1})$ such that

$$C_0 \rho^{\alpha_0} \le \frac{1}{4}. \tag{4}$$

Given a solution u of $F = 0$ in B_1, we also pick $p_\rho = p_\rho(u)$ such that

$$\operatorname*{osc}_{B_\rho}(u - p_\rho \cdot x) \le \frac{1}{4}\rho. \tag{5}$$

3.2. Reduction to the improvement of flatness lemma

In order to prove Lemma 1, we prove an *improvement of flatness* lemma; it is the core of the paper. It basically says that if $p \cdot x + u$ solves (1) in B_1 and the oscillation of u in B_1 is less than 1, say, then the function u can be approximated by a linear function in a smaller ball with an error that is less than the radius of the ball. We make this statement rigorous and quantitative now.

Lemma 2 (*Improvement of Flatness Lemma*)**.** *There exist $\varepsilon_0 \in [0, 1]$ and $\rho \in (0, 1)$ only depending on γ, ellipticity constants and dimension such that, for any $p \in \mathbb{R}^d$ and any viscosity solution u of*

$$|p + \nabla u|^\gamma F(D^2u) = f \quad \text{in } B_1 \tag{6}$$

such that $\operatorname{osc}_{B_1} u \leq 1$ *and* $\|f\|_{L^\infty(B_1)} \leq \varepsilon_0$, *there exists* $p' \in \mathbb{R}^d$ *such that*

$$\operatorname*{osc}_{B_\rho}(u - p' \cdot x) \leq \frac{1}{2}\rho.$$

It is important to remark that the choice of ρ and ε_0 works for all vectors p in the previous lemma. No constant depends on p.

We now explain how to derive Lemma 1 from Lemma 2.

Proof of Lemma 1. For $k = 0$, we simply choose $p_0 = 0$ and (3) is guaranteed by the assumption $\operatorname{osc} u \leq 1$.

We choose $\alpha > 0$ small such that $\rho^\alpha > 1/2$.

We assume now that $k \geq 0$ and that we constructed already $p_k \in \mathbb{R}^d$ such that (3) holds true. We then consider for $x \in B_1$,

$$u_k(x) = r_k^{-1-\alpha}[u(r_k x) - p_k \cdot (r_k x)].$$

The vector p_k is such that $\operatorname{osc}_{B_1} u_k \leq 1$. Moreover, u_k satisfies

$$|r_k^{-\alpha} p_k + Du_k|^\gamma \, r_k^{1-\alpha} F(r_k^{\alpha-1} D^2 u_k) = f_k(x)$$

with $f_k(x) = r_k^{1-\alpha(1+\gamma)} f(r_k x)$. In particular, $\|f_k\|_{L^\infty(B_1)} \leq \varepsilon_0$ as long as $\alpha \leq 1/(1+\gamma)$.

Notice that the function $r_k^{1-\alpha} F(r_k^{\alpha-1} X)$ has the same ellipticity constants as $F(X)$, therefore the C^{1,α_0} estimates are conserved by this scaling.

Now we apply Lemma 2 and get q_{k+1} such that

$$\operatorname*{osc}_{B_\rho}(u_k - q_{k+1} \cdot x) \leq \frac{1}{2}\rho.$$

Because of our choice of α, we then obtain p_{k+1} such that

$$\operatorname*{osc}_{B_{r_{k+1}}}(u - p_{k+1} \cdot x) \leq r_k^{1+\alpha} \frac{1}{2}\rho \leq r_{k+1}^{1+\alpha}.$$

The proof is now complete. □

4. Equi-continuity of rescaled solutions

The proof of Lemma 2 relies on the following lemma in which the modulus of continuity of solutions of (6) is controlled.

Lemma 3 (*Modulus of Continuity Independent of p*)**.** *For all* $r > 0$, *there exist* $\beta \in (0, 1)$ *and* $C > 0$ *only depending on ellipticity constants, dimension,* γ *and* r *and such that for all viscosity solutions* u *of* (6) *with* $\operatorname{osc}_{B_1} u \leq 1$ *and* $\|f\|_{L^\infty(B_1)} \leq \varepsilon_0 < 1$ *satisfies*

$$[u]_{\beta, B_r} \leq C. \tag{7}$$

In particular, the modulus of continuity of u *is controlled independently of* p.

4.1. Proof of Lemma 3

This lemma is a consequence of the two following ones.

Lemma 4 (*Lipschitz Estimate for Large p's*)**.** *Assume u solves (6) with $\operatorname{osc}_{B_1} u \le 1$ and $\|f\|_{L^\infty(B_1)} \le \varepsilon_0 < 1$. If $|p| \ge 1/a_0$, with $a_0 = a_0(\lambda, \Lambda, d, \gamma, r)$, then any viscosity solution u of (6) is Lipschitz continuous in B_r and*

$$[u]_{1,B_r} \le C \tag{8}$$

where $C = C(\lambda, \Lambda, \gamma, d, r)$.

Lemma 5 (*Hölder Estimate for Small p's*)**.** *Assume u solves (6) with $\operatorname{osc}_{B_1} u \le 1$ and $\|f\|_{L^\infty(B_1)} \le \varepsilon_0 < 1$. If $|p| \le 1/a_0$, then u is β-Hölder continuous in B_r and*

$$[u]_{\beta,B_r} \le C$$

where $\beta = \beta(\lambda, \Lambda, d, r, a_0)$ and $C = C(\lambda, \Lambda, d, r, a_0)$.

We now turn to the proof of these two lemmas.

Proof of Lemma 4. We rewrite (6) as

$$|e + aDu|^\gamma F(D^2 u) = \tilde{f}$$

where $e = p/|p|$ and $a = 1/|p| \in [0, a_0]$ and

$$\tilde{f} = |p|^{-\gamma} f.$$

Remark that

$$\|\tilde{f}\|_{L^\infty(B_1)} \le a_0^\gamma \varepsilon_0.$$

We use viscosity solution techniques first introduced in [12]. For all $x_0 \in B_{r/2}$, we look for $L_1 > 0$ and $L_2 > 0$ such that

$$M = \sup_{x,y \in B_r} u(x) - u(y) - L_1 \omega(|x-y|) - L_2|x-x_0|^2 - L_2|y-x_0|^2 \le 0$$

where $\omega(s) = s - \omega_0 s^{\frac{3}{2}}$ if $s \le s_0 := (2/3\omega_0)^2$ and $\omega(s) = \omega(s_0)$ if $s \ge s_0$. We choose ω_0 such that $s_0 \ge 1$. We notice that if we proved such an inequality, the Lipschitz constant is bounded from above by any $L > L_1$.

We argue by contradiction by assuming that $M > 0$. If $(x, y) \in \bar{B}_r \times \bar{B}_r$ denotes a point where the maximum is reached (recall that u is continuous and its oscillation is bounded), we conclude that

$$L_1 \omega(|x-y|) + L_2|x-x_0|^2 + L_2|y-x_0|^2 \le \operatorname*{osc}_{B_1} u \le 1.$$

We choose $L_2 = (4/r)^2$, so that $|x - x_0| \le \frac{r}{4}$ and $|y - x_0| \le \frac{r}{4}$. With this choice, we force the points x and y where the supremum is achieved to be in B_r. Remark also that the supremum cannot be reached at (x, y) with $x = y$, otherwise $M \le 0$. Hence, we can write two viscosity inequalities.

Before doing so, we compute the gradient of the test-function for u with respect to x and y at (x, y)

$$q_x = q + 2L_2(x - x_0) \quad \text{and} \quad q_y = q - 2L_2(y - x_0)$$

where $q = L_1\omega'(|\delta|)\hat{\delta}$, $\delta = x - y$ and $\hat{\delta} = \delta/|\delta|$. To get appropriate viscosity inequalities, we shall use Jensen–Ishii's Lemma in order to construct a limiting sub-jet (q_x, X) of u at x and a limiting super-jet (q_y, Y) of u at y such that the following $2n \times 2n$ matrix inequality holds for all $\iota > 0$ small enough (depending on the norm of Z):

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + (2L_2 + \iota)I$$

where $Z = L_1 D^2(\omega(|\cdot|))(x-y)$. We refer the reader to [2,1] for details. Applying the previous matrix inequality as a quadratic form inequality to vectors of the form (v, v) we obtain

$$\langle (X - Y)v, v\rangle \leq (4L_2 + \iota)|v|^2. \tag{9}$$

Therefore $X - Y \leq (4L_2 + \iota)I$, or equivalently, all eigenvalues of $X - Y$ are less than $4L_2 + \iota$. On the other hand, applying now the particular vector $(\hat{\delta}, -\hat{\delta})$, we obtain

$$\langle (X - Y)\hat{\delta}, \hat{\delta}\rangle \leq (4L_2 + \iota - 6\omega_0 L_1|x-y|^{-1/2})|\hat{\delta}|^2 \leq (4L_2 + \iota - 3\sqrt{2}\omega_0 L_1)|\hat{\delta}|^2. \tag{10}$$

Thus, at least one eigenvalue of $X - Y$ is less than $(4L_2 + \iota - 3\omega_0\sqrt{2}L_1)$ (which will be a negative number). We next consider the minimal Pucci operator P^-. We recall that $-P^-(A)$ equals λ times the sum of all negative eigenvalues of A plus Λ times the sum of all positive eigenvalues. Therefore, from (9) and (10), we obtain

$$\begin{aligned} P^-(X - Y) &\geq -\lambda(4L_2 + \iota - 3\sqrt{2}\omega_0 L_1) - \Lambda(d-1)(4L_2 + \iota) \\ &\geq -(\lambda + (d-1)\Lambda)(4L_2 + \iota) + 3\sqrt{2}\omega_0\lambda L_1. \end{aligned}$$

We now write the two viscosity inequalities and we combine them in order to get a contradiction.

$$\begin{aligned} &|e + aq_x|^\gamma F(X) \leq \tilde{f}(x) \\ &|e + aq_y|^\gamma F(Y) \geq \tilde{f}(y). \end{aligned}$$

We will choose a_0 small enough depending on L_1 and L_2 so that $|aq_x| \leq 1/2$ and $|aq_y| \leq 1/2$. The constant L_1 will be chosen later and its value does not depend on this choice of a_0. In particular, we have

$$\frac{1}{2} \leq \min(|e + aq_x|, |e + aq_y|).$$

We now use that F is uniformly elliptic to write

$$F(X) \geq F(Y) + P^-(X - Y).$$

Combining the previous displayed inequalities and recalling $\|f\|_{L^\infty} \leq \varepsilon_0$ yields

$$3\sqrt{2}\omega_0\lambda L_1 \leq (\lambda + \Lambda(d-1))(4L_2 + \iota) + 2^{\gamma+1}\varepsilon_0.$$

Choosing L_1 large enough depending on λ, Λ, d, γ, and the previous choice of L_2 (which depends on r only), we obtain a contradiction.

Note that this choice of L_1 does not depend on the previous choice of a_0, so we should first choose L_1 large and then a_0 small. The proof of the lemma is now complete. □

Proof of Lemma 5. The equation can be written as $G(Du, D^2u) = f$ with

$$G(q, X) = |p + q|^\gamma F(X).$$

In particular, if $|q| \geq 2a_0^{-1}$ then $|p+q|^\gamma \geq a_0^{-\gamma}$. In particular,

$$\left.\begin{matrix} G(q,X) = f \\ |q| \geq 2a_0^{-1} \end{matrix}\right\} \Rightarrow \begin{cases} P^+(D^2u) + a_0^\gamma |f| \geq 0 \\ P^-(D^2u) - a_0^\gamma |f| \leq 0 \end{cases}$$

where $P^\pm$ denote the extremal Pucci operators associated with the ellipticity constants of F. We know from [11] that there exist $\beta_1 \in (0,1)$ and C_1 only depending on r, dimension and ellipticity constants of F such that

$$\begin{aligned} [u]_{\beta_1, B_r} &\leq C_1 \left(\operatorname*{osc}_{B_1} u + \max(2a_0^{-1}, \|f\|_{L^n(B_1)}) \right) \\ &\leq C_1 (1 + \max(2a_0^{-1}, \varepsilon_0)). \end{aligned}$$

The proof of the lemma is now complete. □

4.2. *Proof of the improvement of flatness lemma*

With Lemma 3 in hand, we can now turn to the proof of Lemma 2.

Proof of Lemma 2. We argue by contradiction and we assume that there exist sequences $\varepsilon_n \to 0$, $p_n \in \mathbb{R}^d$, f_n such that $\|f_n\|_{L^\infty(B_1)} \leq \varepsilon_n$, and u_n satisfying (6) with $(p,f) = (p_n, f_n)$ such that for all $p' \in \mathbb{R}^d$,

$$\operatorname*{osc}_{B_\rho}(u_n - p' \cdot x) > \frac{1}{2}\rho.$$

Remark that $f_n \to 0$ as $n \to \infty$.

Thanks to Lemma 3, we can extract a subsequence of $(u_n - u_n(0))_n$ converging locally uniformly in B_1 to a continuous function u_∞. Remark that we have in particular for all $p' \in \mathbb{R}^d$,

$$\operatorname*{osc}_{B_\rho}(u_\infty - p' \cdot x) > \frac{1}{2}\rho. \tag{11}$$

We are going to prove that u_∞ satisfies $F(D^2 u_\infty) = 0$ in B_1. This will imply that there exists a vector p_ρ such that (5) holds true. This is the desired contradiction with (11).

To prove that $F(D^2 u_\infty) = 0$ in B_1, we now distinguish two cases.

If we can extract a converging subsequence of p_n, then we also do it for u_n and we get at the limit

$$|p_\infty + \nabla u_\infty|^\gamma F(D^2 u_\infty) = 0 \quad \text{in } B_1.$$

In particular, we have $F(D^2 u_\infty) = 0$ in B_1 (see Lemma 6 in the next subsection).

If now we cannot extract a converging subsequence of p_n, then $|p_n| \to \infty$ and in this case, we extract a converging subsequence from $e_n = p_n/|p_n|$ and dividing the equation by $|p_n|$ we get at the limit

$$|e_\infty + 0\nabla u_\infty|^\gamma F(D^2 u_\infty) = 0 \quad \text{in } B_1$$

for $e_\infty \neq 0$ so that we also have in this case $F(D^2 u_\infty) = 0$ in B_1. The proof of the lemma is now complete. □

5. Viscosity solutions of $|\nabla u|^\gamma F(D^2 u) = 0$

In the previous subsection, we used the following lemma.

Lemma 6. *Assume that u is a viscosity solution of*

$$|p + \nabla u|^\gamma F(D^2 u) = 0 \quad \textit{in } B_1.$$

Then u is a viscosity solution of $F(D^2 u) = 0$ in B_1.

Proof. We reduce the problem to $p = 0$ as follows. The function $v = u + p \cdot x$ satisfies $|\nabla v|^\gamma F(D^2 v) = 0$ in B_1. If we proved the result for $p = 0$, we conclude that $F(D^2 u) = F(D^2 v) = 0$ in B_1.

We now assume that $p = 0$. We only prove the super-solution property since the sub-solution property is very similar.

Consider a test-function ϕ touching u strictly from below at $x \in B_1$. We assume for simplicity that $x = 0$. Hence, we have, $\phi(0) = u(0) = 0$ and $\phi < u$ in $B_r \setminus \{0\}$ for some $r > 0$. We can assume without loss of generality that ϕ is quadratic: $\phi(x) = \frac{1}{2} Ax \cdot x + b \cdot x$. If $b \neq 0$, then we get the desired inequality: $F(A) \geq 0$.

If $b = 0$, we argue by contradiction by assuming that $F(A) < 0$. Since F is uniformly elliptic, this implies that A has at least one positive eigenvalue. Let S be the direct sum of eigensubspace corresponding to non-negative eigenvalues. Let P_S denote the orthogonal projection on S. We then consider the following test function

$$\psi(x) = \phi(x) + \varepsilon |P_S x|.$$

Since $\phi < u$ in B_r, then $u - \psi$ reaches its minimum at x_0 in $\bar{B}_r$ in the interior of the ball for ε small enough.

We claim first that $P_S x_0 \neq 0$. Indeed, if this is not true, we use the fact that

$$|P_S x| = \max_{|e|=1} e \cdot P_S x$$

and we deduce that for all $e \in \mathbb{R}^d$ such that $|e| = 1$, the test-function $\phi(x) + \varepsilon e \cdot P_S x$ touches u from below at x_0 and we thus have for all such e's

$$|Ax_0 + \varepsilon P_S e|^\gamma F(A) \geq 0.$$

On the other hand, we can choose e such that $D\phi(x_0) + \varepsilon P_S e \neq 0$ and get the contradiction $F(A) \geq 0$.

Since $P_S x_0 \neq 0$, ψ is smooth in a neighbourhood of x_0 and we get the following viscosity inequality

$$|Ax_0 + \varepsilon e_0|^\gamma F(A + \varepsilon B) \geq 0$$

where $e_0 = P_S x_0 / |P_S x_0|$ and $B \geq 0$ since $x \mapsto |P_S x|$ is convex. Remark next that

$$(Ax_0 + \varepsilon e_0) \cdot P_S x_0 = P_S A x_0 \cdot x_0 + \varepsilon |P_S x_0| \geq \varepsilon |P_S x_0| > 0.$$

Hence $Ax_0 + \varepsilon e_0 \neq 0$ and we get the following contradiction

$$F(A) \geq F(A + \varepsilon B) \geq 0.$$

The proof is now complete. $\square$

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