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Convexity of solutions and $C^{1,1}$ estimates for fully nonlinear elliptic equations

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Abstract

The starting point of this work is a paper by Alvarez, Lasry and Lions (1997) concerning the convexity and the partial convexity of solutions of fully nonlinear degenerate elliptic equations. We extend their results in two directions. First, we deal with possibly sublinear (but epi-pointed) solutions instead of 1-coercive ones; secondly, the partial convexity of C^2 solutions is extended to the class of continuous viscosity solutions. A third contribution of this paper concerns $C^{1,1}$ estimates for convex viscosity solutions of strictly elliptic nonlinear equations. To finish with, all the tools and techniques introduced here permit us to give a new proof of the Alexandroff estimate obtained by Trudinger (1988) and Caffarelli (1989).

Résumé

Le point de départ de cet article est un résultat d'Alvarez, Lasry et Lions (1997) sur la convexité et la convexité partielle de solutions d'équations elliptiques dégénérées complètement non linéaires. Nous étendons leurs résultats de deux façons. D'une part, nous pouvons traiter le cas de solutions non nécessairement 1-coercitives, éventuellement sous linéaires (mais épi-pointées) ; d'autre part, la convexité partielle de solutions C^2 est étendue aux solutions de viscosité simplement continues. Une troisième contribution de cet article sont des estimations $C^{1,1}$ pour les solutions de viscosité convexes des équations non linéaires strictement elliptiques ; pour finir, les outils et techniques de cet article nous permettent de donner une nouvelle démonstration de l'estimation d'Alexandroff, adaptée par Trudinger (1988) et Caffarelli (1989) aux équations elliptiques complètement non linéaires.

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0. Introduction

In [1], a new method for establishing the convexity of a (viscosity) solution of a second order equation of the form,

$$F\big(x, u(x), \nabla u(x), D^2 u(x)\big) = 0 \quad \text{in } \Omega, \tag{1}$$

is developed under the assumption that F is degenerate elliptic, i.e.,

$$F(x, r, p, A) \leqslant F(x, r, p, B) \quad \text{as soon as } A \geqslant B;$$

that Ω is an open convex subset of $\mathbb{R}^n$ and that

$$(x, r, A) \mapsto F\big(x, r, p, A^{-1}\big) \text{ is concave on } \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n_{++}. \tag{2}$$

Their work follows the ones of Korevaar [13] and Kennington [12]. The idea of [1] is to prove that the convex envelope of a viscosity supersolution u of (1) is still a viscosity supersolution. In order to obtain such a result, they assume that u is 1-coercive, i.e.,

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|} = +\infty.$$

Moreover, Condition (2) implies that one must work with nondegenerate semijets of the convex envelope of a solution.

In this paper, we explain that Condition (2) can be understood in the following sense: for any $p \in \mathbb{R}^n$, for any $(x, r, A), (y, s, B) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{S}^n$ and $\lambda \in]0, 1[$, such that $\frac{1}{\lambda} A + \frac{1}{1-\lambda} B \geqslant 0$,

$$\lambda F(x, r, p, A) + (1-\lambda) F(y, s, p, B) \leqslant F\big(\lambda x + (1-\lambda) y, \lambda r + (1-\lambda) s, p, \lambda \odot A \,\square\, (1-\lambda) \odot B\big), \tag{3}$$

where $\square$ denotes the inf-convolution of matrices, seen as quadratic functions, and $\lambda \odot A = \lambda^{-1} A$. Indeed, if A and B is definite positive, then $\lambda \odot A \,\square\, (1-\lambda) \odot B = (\lambda A^{-1} + (1-\lambda) B^{-1})^{-1}$. We will see through the paper that thinking of $(\lambda A^{-1} + (1-\lambda) B^{-1})^{-1}$ as a certain "convex combination" of A and B helps understanding the proofs and the structure conditions imposed to the equations, especially when studying partial convexity. Another way to say this is that it is natural to redefine the product of a real number and a matrix in our context. Roughly speaking, we can say that the natural sum on matrices is replaced with $\square$ and that the natural product of a matrix A with a scalar α is replaced with $\alpha \odot A$. See Section 1 for details. Remark that there is no more restriction about the degeneracy of the matrices.

We extend the results of [1] in two directions. First, we deal with solutions that are not necessarily 1-coercive; in particular, sublinear convex functions can be considered. However, the behaviour of u at infinity is restricted; precisely, the function must satisfy:

$$\liminf_{|x| \to \infty} \frac{u(x) - \langle s, x \rangle}{|x|} > 0,$$

for some $s \in \mathbb{R}^n$. Such functions are said to be *epi-pointed*. This is equivalent to impose that the Legendre–Fenchel transform of u has a nonempty interior (see [2]). This is of interest when dealing with equations for which no comparison principle in the class of subquadratic functions is available. Indeed, imposing 1-coercivity to solutions u of an equation forces to be able to prove a comparison principle for solutions with superlinear growth at infinity and it is not an easy task. For instance, in [1, Theorem 3], the authors use results of [8] to establish the convexity of a solution and consequently, equations depending on x, for instance, cannot be treated. Working with this bigger class of functions when considering solutions of (1) leads to be able to treat "points at infinity" when proving that the convex envelope of a supersolution of (1) is still a supersolution.

Secondly, a structure condition analogous to (2) that ensures the partial convexity of a C^2 solution u is exhibited in [1]. They explain that they cannot treat the case of general viscosity solutions because they need to ensure that some "partial subjets" of u are nonnegative. The new point of view permits to overcome this difficulty. The price to pay is quite a lot of technicalities in the proofs.

The third contribution of this paper concerns $C^{1,1}$ estimates of convex viscosity solutions in $\mathbb{R}^n$. We prove that if the equation is strictly elliptic (in the usual fully nonlinear sense), then any convex viscosity solution of this equation is $C^{1,1}$ and we give an estimate of the second derivatives. Using the same kind of ideas, we also give a new proof of the Alexandroff estimate obtained by Trudinger [16] and Caffarelli [4] (in the second paper, the Hamiltonian is independent of u and ∇u). See also [3]. As explained for instance in [5], the main difference between the geometric proof of [4] and the classical one (see [9]) is the way the Hessian matrix of a convex envelope is bounded from above on the so-called contact set. In [4], a barrier argument is used; instead, we directly give an estimate of these second derivatives by using the techniques developed previously. Let us point out further differences: we treat general nonlinear equations, depending on the solution, its gradient and the space variable (like in [16,3]); we do not assume that the supersolution is continuous but merely lsc; eventually, our estimate is true even if the Hamiltonian is strictly elliptic but not uniformly elliptic.

In the first section, we list a number of equations that satisfy the structure Condition (3). The second order operator that appears in the Monge–Ampère equation satisfies it. However, we would like to point out that Condition (3) is especially relevant for convex solutions that are *not* strictly convex. Moreover, it is explained in [1] that the relevant boundary conditions when proving, via these techniques, that the solution of an equation on a bounded domain is convex are of state constraints type [15]. Eventually, let us point out that we do not know yet if these techniques can be useful in the context of other geometrical equations such as the prescribed Gauss curvature equation.

The paper is organised as follows. In Section 1, we describe some properties of the inf-convolution of matrices seen as quadratic functions and we redefine the external law of the vector space $\mathbb{S}^n$ and we give various examples of equations that satisfy (3). In the first subsection of Section 2, we mainly revisit and refine the results of the paper [1]. We first describe the subjet of the convex envelope of functions that are not necessarily 1-coercive; the proper assumption of the behaviour at infinity is to assume that the function is *epi-pointed*; next, we prove that the convex envelope of a viscosity supersolution of a nonlinear elliptic equation is still a viscosity supersolution under the Structure Condition (3) on the Hamiltonian; eventually, we give an example of application of such a result to prove that a viscosity solution of a nonlinear equation is convex. In Section 2.3, we generalize the results of the previous subsection to the case of the partial convex envelope and the partial convexity of a solution of a nonlinear alliptic equation. In Section 3, we first prove a general $C^{1,1}$ estimate for viscosity solutions of strictly elliptic nonlinear equations and give a new proof of the Alexandroff estimate adapted by Caffarelli [4] to fully nonlinear elliptic equations.

Notations. The inner product of $x, y \in \mathbb{R}^n$ is simply denoted $x \cdot y$. The transpose matrix of a $n \times n$ matrix A is denoted A^*. The vector space of symmetric real $n \times n$ matrices is denoted $\mathbb{S}^n$. The subset of $\mathbb{S}^n$ made of the symmetric matrices with nonnegative eigenvalues is denoted $\mathbb{S}^n_+$. The space $\mathbb{S}^n$ is endowed with its usual partial order: $A \leqslant B$ if $B - A \in \mathbb{S}^n_+$. The set of nondegenerate matrices of $\mathbb{S}^n_+$ is denoted $\mathbb{S}^n_{++}$. Operations $\Box$, $\boxminus$ and $\odot$ and the matrices A^ε and A_ε are introduced in Section 1. $B_r(x)$ refers to the open ball centered at x and of radius r; B_r stands for $B_r(0)$.

1. Preliminaries

In this section, we give several results related with nonlinear convolutions of matrices, seen as pure quadratic functions. We first describe the infimum convolution of two matrices. Precisely, given $A, B \in \mathbb{S}^n$ such that $A + B \geqslant 0$, we consider the inf-convolution of two (pure) quadratic functions:

$$A \Box B(x) = \inf_{y \in \mathbb{R}^n} \bigl\{ A(x-y) \cdot (x-y) + By \cdot y \bigr\}.$$

Analogously, sup-convolutions of two matrices A and B can be considered if $A \leqslant B$:

$$A \boxminus B(x) = \sup_{y \in \mathbb{R}^n} \bigl\{ A(x-y) \cdot (x-y) - By \cdot y \bigr\}.$$

In the following, we essentially study $\Box$ and analogous properties can be obtained by remarking that $A \boxminus B = -[(-A) \Box B]$. Here are elementary properties of the $\Box$ operation.

Proposition 1.

- *The operation $\Box$ is associative and commutative and continuous.*
- *Given $A, B \in \mathbb{S}^n$ such that $A + B \geqslant 0$ and $\alpha, \beta, \varepsilon > 0$,*

$$A \Box B \leqslant A \text{ and } A \Box B \leqslant B, \quad A \Box 0 = 0; \tag{4}$$

$$\text{if } A \text{ and } B \text{ are not degenerate}, \quad A \Box B = \bigl(A^{-1} + B^{-1}\bigr)^{-1}; \tag{5}$$

$$\textit{if } A+B \textit{ is not degenerate}, \quad A \,\square\, B = A(A+B)^{-1}B = B(A+B)^{-1}A; \tag{6}$$

$$\textit{for any } A\in\mathbb{S}^n \textit{ and } \varepsilon \textit{ s.t. } I+\varepsilon A \textit{ definite positive}, \quad A_\varepsilon := A \,\square\, \varepsilon^{-1}I = (I+\varepsilon A)^{-1}A; \tag{7}$$

$$\textit{for any } A\in\mathbb{S}^n_+, \quad \alpha A \,\square\, \beta A = \left(\alpha^{-1}+\beta^{-1}\right)^{-1}A, \quad \underbrace{A \,\square\, \cdots \,\square\, A}_{n \text{ times}} = \frac{1}{n}A. \tag{8}$$

The previous properties exhibit the fact that it can be useful to redefine the product of a real number and a matrix when dealing with nonlinear convolutions. Namely, it is convenient to define $\lambda \odot A = \lambda^{-1}A$. Hence, (8) is equivalent to:

$$\alpha \odot A \,\square\, \beta \odot A = (\alpha+\beta)\odot A, \quad \underbrace{A \,\square\, \cdots \,\square\, A}_{n \text{ times}} = n \odot A.$$

We can obtain analogous elementary properties for $A \boxminus B$. Let us state the ones used in the following:

Proposition 2.

- *The operation $\boxminus$ is associative and continuous.*
- *For any $A, B \in \mathbb{S}^n$ such that $A \leqslant B$,*

$$\textit{if } A-B \textit{ is not degenerate}, \quad A \boxminus B = A(B-A)^{-1}B = B(B-A)^{-1}A; \tag{9}$$

$$\textit{for any } A\in\mathbb{S}^n \textit{ and } \varepsilon \textit{ s.t. } I-\varepsilon A \textit{ definite positive}, \quad A^\varepsilon := A \boxminus \varepsilon \odot I = (I-\varepsilon A)^{-1}A; \tag{10}$$

$$\left(A^{2\varepsilon}\right)_\varepsilon = A^\varepsilon \quad \textit{and} \quad (A_{2\varepsilon})^\varepsilon = A_\varepsilon; \tag{11}$$

$$\textit{for } \beta > \alpha \textit{ and } A \geqslant 0, \quad \beta\odot A \boxminus \alpha \odot A = (\beta-\alpha)\odot A. \tag{12}$$

The next natural step is to redefine concavity of a function $F:\mathbb{S}^n \to \mathbb{R}$ with $\square$ and $\odot$. Namely, we say that F is $\square$-concave if for any $\lambda \in]0,1[$ and $A, B \in \mathbb{S}^n_+$:

$$\lambda F(A) + (1-\lambda)F(B) \leqslant F\left(\lambda \odot A \,\square\, (1-\lambda)\odot B\right)$$

and Condition (3) only says that F is $\square$-concave. Let us give several examples of $\square$-concave function.

(1) *Concave equations*. If $A \mapsto F(A)$ is concave on $\mathbb{S}^n_+$ in the classical sense and degenerate elliptic, it is $\square$-concave; this is a consequence of the degenerate ellipticity and of the following inequality: for any $A, B \in \mathbb{S}^n$, $\lambda \in]0,1[$ such that $\lambda A + (1-\lambda)B \geqslant 0$: $\lambda \odot A \,\square\, (1-\lambda)\odot B \leqslant \lambda A + (1-\lambda)B$. To see this, use the fact that $A \mapsto A^{-1}$ is convex and nonincreasing on $\mathbb{S}^n_{++}$. An important special case is the following Pucci extremal operator:

$$\mathcal{P}^-(u) = -\Lambda \operatorname{Tr}\left(D^2u\right)^+ + \lambda \operatorname{Tr}\left(D^2u\right)^- = \inf_{A\in\mathbb{S}^n,\ \lambda I \leqslant A \leqslant \Lambda I} \left\{-\operatorname{Tr}(AM)\right\}.$$

This corresponds to $-\mathcal{M}^+$ in [5].

(2) *Convex equations*. If $A \mapsto F(A)$ is convex on $-\mathbb{S}^n_+$ and degenerate elliptic, then one can consider the change of unknown function $v(x) = -u(x)$ and if u solves (in the viscosity sense) $F(D^2u) = 0$, then $G(D^2v) = 0$ with $G(A) = -F(-A)$ degenerate elliptic and concave on $\mathbb{S}^n_+$. The second Pucci extremal operator is an important example: for given $\lambda, \Lambda > 0$, consider:

$$\mathcal{P}^+(u) = -\lambda \operatorname{Tr}\left(D^2u\right)^+ + \Lambda \operatorname{Tr}\left(D^2u\right)^- = \sup_{A\in\mathbb{S}^n,\ \lambda I \leqslant A \leqslant \Lambda I} \left\{-\operatorname{Tr}(AM)\right\}.$$

This corresponds to $-\mathcal{M}^-$ in [5].

(3) $\square$-*linear equations*. A natural question to be addressed is: what are the $\square$-linear functions? The answer is: the functions $\mathcal{L}_A : M \in \mathbb{S}^n_{++} \mapsto \operatorname{Tr}(AM^{-1})$ extended by $-\infty$ elsewhere with $A \in \mathbb{S}^n$; and $-A \in \mathbb{S}^n_+$ if one imposes degenerate ellipticity.

(4) *Monge–Ampère equations*. The Monge–Ampère equation writes $\det(D^2u) = f(x)$ with $f > 0$. If one considers strictly convex viscosity solutions, this is equivalent to solving $-\ln\det D^2u = -\ln(f(x))$. We claim that $A \mapsto -\ln\det A$ is $\square$-concave. See below for a proof of it. The prescribed Gauss curvature equation involves the same

nonlinear second order operator; precisely, it can be written under the following form: $\det D^2u = H(x,u,\nabla u)$ with $H > 0$.

(5) *Perturbed equations*. One can consider $F(D^2u) + H(x,u,\nabla u) = 0$ with F $\square$-concave (for instance of the four previous forms) and H such that $(x,u) \mapsto H(x,u,p)$ is concave for any $p \in \mathbb{R}^n$. Convex Hamiltonians H can also be treated by making the change of variables $v(x) = -u(-x)$.

(6) *Quasilinear equations*. Any quasilinear equations of the form $-\mathrm{Tr}(A(\nabla u)D^2u) = f(x)$ is $\square$-concave as soon as $A \geqslant 0$ and f is convex. An important special case is the minimal surface equation for which $A(p) = I - \frac{p\otimes p}{\varepsilon^2+|p|^2}$ with $\varepsilon = 0, 1$.

(7) *Special Bellmann equations*. If L_A denotes the function that maps $M \in \mathbb{S}^n$ to $\mathrm{Tr}(AM)$, then one can consider $\inf_{\alpha\in\mathcal{A}}\{-L_{A_\alpha}(D^2u) + b_\alpha \cdot \nabla u + c_\alpha u + d_\alpha(x)\} = 0$ with $d_\alpha(\cdot)$ concave.

In order to prove the claim about Monge–Ampère equations, for $M \in \mathbb{S}^n_{++}$, use the fact that $\ln(\sigma) = \inf_{\gamma>0}\{\gamma\sigma - \ln(\gamma) - 1\}$ and write:

$$\begin{aligned}
-\ln\det(M) = \sum_{i=1}^n \ln\lambda_i\bigl(M^{-1}\bigr) &= \inf_{\gamma_1,\dots,\gamma_n>0}\sum_{i=1}^n\bigl(\gamma_i\lambda_i\bigl(M^{-1}\bigr) - \ln(\gamma_i) - 1\bigr)\\
&= \inf_{\gamma_1,\dots,\gamma_n>0}\bigl\{\mathrm{Tr}\bigl(\mathrm{Diag}(\gamma_i)M^{-1}\bigr) - \ln\det\bigl(\mathrm{Diag}(\gamma_i)\bigr) - n\bigr\}\\
&= \inf_{\gamma_1,\dots,\gamma_n>0}\bigl\{\mathcal{L}_{\mathrm{Diag}(\gamma_i)}(M) - \ln\det\bigl(\mathrm{Diag}(\gamma_i)\bigr) - n\bigr\},
\end{aligned}$$

where $\lambda_i(M^{-1})$ are the eigenvalues of M^{-1} and $\mathrm{Diag}(\gamma_i)$ is a diagonal matrix in the spectral base of M^{-1} with γ_i as diagonal entries. Notice that $A \mapsto -(\det A)^{1/n}$ is convex on $\mathbb{S}^n_+$, and not on $-\mathbb{S}^n_+$; this can be seen by using Formula (5.19) from [10, p. 54]. But this cannot help us since we cannot make the change of variables explained above.

2. Convexity of viscosity solutions

2.1. Subjets of convex envelopes of noncoercive functions

We first describe the subjets of convex envelopes of functions that are not necessarily 1-coercive and we next explain how to use it to get results analogous to these from [1].

Let us show how to treat points at infinity when dealing with the convex envelope of noncoercive functions. It is possible to do so by working in the class of epi-pointed functions. See the introduction for a definition. The following proposition must be compared with Proposition 1 of [1].

Proposition 3. *Let Ω be a convex open set and $u : \overline{\Omega} \to \mathbb{R}$ be lsc and epi-pointed. For $x \in \overline{\Omega}$, consider $(p,A) \in D^{2,-}_{\Omega}u^{**}(x)$. Consider the points x_i and x_j^∞ and the real numbers λ_i such that* (33) *hold true. Then for every $\varepsilon > 0$, there are $A_i, B_j \in \mathbb{S}^n$, $i = 1,\dots,p$, $j = 1,\dots,q$, such that*

$$\begin{cases}(p,A_i) \in \overline{D}^{2,-}u(x_i), \quad (p,B_j) \in \overline{D}^{2,-}u_\infty(x_j^\infty),\\ A_\varepsilon \leqslant (\square_{i=1}^p \lambda_i \odot A_i) \,\square\, (\square_{j=1}^q B_j).\end{cases} \tag{13}$$

Remark 1.

- The two main differences with Proposition 1 of [1] is that u is only assumed epi-pointed and A is not supposed to be nonnegative. Both improvements will be crucial when applying this proposition. See Theorems 1 and 8.
- Even if A is not necessarily nonnegative, a consequence of (13) is that $\sum_{i=1}^p \lambda_i \odot A_i + \sum_{j=1}^q B_j \geqslant 0$ and, because of (4), $A_\varepsilon \leqslant \lambda_i \odot A_i$ and $A_\varepsilon \leqslant B_j$ for any i, j.
- Following the remark made in [1] (Eq. (14), p. 274), we observe that for $(p,A) \in D^{2,-}u(x)$, $A \geqslant 0$, we have:

$$Ah\cdot h = 0 \text{ for every } h \in \mathrm{span}\bigl(x_1 - x,\dots,x_p - x, x_1^\infty,\dots,x_q^\infty\bigr).$$

This is a consequence of the fact that u^{**} is affine on the closed convex polyhedron

$$P = \mathrm{co}\{x_1, \ldots, x_p\} + \mathbb{R}^+ x_1^\infty + \cdots + \mathbb{R}^+ x_q^\infty$$

(see [2] for a proof of this assertion).

Proof of Proposition 3. We proceed as in [1]. Since $(p, A) \in D^{2,-}_{\Omega} u^{**}(x)$, there exists a C^2 function ϕ such that $u^{**} - \phi$ attains a local minimum at x. Then for any $y_i, y_j^\infty \in \mathbb{R}^n$, we obtain:

$$\begin{aligned}
&\sum_{i=1}^p \lambda_i u(y_i) + \sum_{j=1}^q u_\infty(y_j^\infty) - \phi\left(\sum_{i=1}^p \lambda_i y_i + \sum_{j=1}^q y_j^\infty\right) \\
&\quad \geqslant u^{**}\left(\sum_{i=1}^p \lambda_i y_i\right) + \sum_{j=1}^q (u^{**})_\infty(y_j^\infty) - \phi\left(\sum_{i=1}^p \lambda_i y_i + \sum_{j=1}^q y_j^\infty\right) \geqslant (u^{**} - \phi)\left(\sum_{i=1}^p \lambda_i y_i + \sum_{j=1}^q y_j^\infty\right) \\
&\quad \geqslant (u^{**} - \phi)(x) = \sum_{i=1}^p \lambda_i u(x_i) + \sum_{j=1}^q u_\infty(x_j^\infty) - \phi\left(\sum_{i=1}^p \lambda_i x_i + \sum_{j=1}^q x_j^\infty\right).
\end{aligned} \tag{14}$$

We successively used the fact that $u^{**} \leqslant u$, that u^{**} is convex, that taking the recession function preserves order and that relation (31) holds true. We then apply Ishii's lemma with $p + q$ functions and we obtain that for any $\varepsilon > 0$, there exist $p + q$ matrices $A_i, B_j \in \mathbb{S}^n_+$ such that

$$(p, A_i) \in \overline{D}^{2,-} u(x_i), \qquad (p, B_j) \in \overline{D}^{2,-} u(x_j^\infty)$$

and such that

$$\begin{bmatrix}
\lambda_1^2 A & \ldots & \lambda_1\lambda_p A & \lambda_1 A & \ldots & \lambda_1 A \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\lambda_p\lambda_1 A & \cdots & \lambda_p^2 A & \lambda_p A & \ldots & \lambda_p A \\
\lambda_1 A & \ldots & \lambda_p A & A & \ldots & A \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\lambda_1 A & \ldots & \lambda_p A & A & \cdots & A
\end{bmatrix}_\varepsilon \leqslant
\begin{bmatrix}
\lambda_1 A_1 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \lambda_p A_p & 0 & \ldots & 0 \\
0 & \cdots & 0 & B_1 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & B_q
\end{bmatrix}.$$

Applying the previous matrix inequality to vectors $\zeta_i/\lambda_i, \xi_j \in \mathbb{R}^n$ yields:

$$A_{n\varepsilon}\zeta \cdot \zeta \leqslant \sum_{i=1}^p (\lambda_i \odot A_i)\zeta_i \cdot \zeta_i + \sum_{j=1}^q B_j \xi_j \cdot \xi_j,$$

with $\zeta = \sum_{i=1}^p \zeta_i + \sum_{j=1}^q \xi_j$ which exactly means (13) with $n\varepsilon$ instead of ε. □

Remark 2. The proof can be seen from the following point of view. First, the convex envelope u^{**} equals $(\square_{i=1}^p u_{\lambda_i}) \square (\square_{i=1}^q u_\infty)$ where $u_{\lambda_i}(x) = \lambda_i u(x/\lambda_i)$; secondly, at the point $x = \sum_{i=1}^p (\lambda_i x_i) + \sum_{j=1}^q x_j^\infty$, we have $(\square_{i=1}^p u_{\lambda_i}) \square (\square_{i=1}^q u_\infty)(x) = \sum_{i=1}^p u_{\lambda_i}(\lambda x_i) + \sum_{j=1}^q u_\infty(x_j^\infty)$. Hence, Proposition 3 can be seen as a consequence of the following proposition adapted from [1]:

Proposition 4. (See [1].) *Consider k lsc functions $v_1, \ldots, v_k$ defined on $\overline{\Omega}$ and consider $v(x) = \square_{i=1}^k v_i$. Suppose it is finite at x and that the infimum is attained at x_i:*

$$v(x) = \sum_{i=1}^k v_i(x_i).$$

Then for any $(p, A) \in D^{2,-} v(x)$ and any $\varepsilon > 0$ small enough, there exists $A_i \in \mathbb{S}^n$ such that

$$(p, A_i) \in \overline{D}^{2,-} v_i(x_i) \quad \text{and} \quad A_\varepsilon \leqslant \square_{i=1}^k A_i.$$

This must be related to Proposition 10 in Appendix B, that is more precise in the case of a quadratic or a C^2 function. This point of view will be useful when dealing with partial convexity.

2.2. Convexity

Let us now use this proposition to show that the convex envelope of a supersolution of (1) is still a supersolution under appropriate assumptions.

Proposition 5. *Let Ω be a convex open subset of $\mathbb{R}^n$. Let $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ be continuous and degenerate elliptic and satisfy* (3). *Let $u : \overline{\Omega} \to \mathbb{R}$ be lsc and epi-pointed and be a supersolution of* (1) *in $\overline{\Omega}$. Then so is u^{**}.*

Remark 3.

- The difference with Proposition 3 from [1] is that we assume u epi-pointed instead of 1-coercive (see the Introduction for definitions).
- We remark that a viscosity solution u of $F = 0$ is such that $v(x) = -u(-x)$ is a viscosity solution of $G = 0$ with $G(x,u,p,A) = -F(-x,-u,p,A)$. Hence if F is convex instead of concave (in the sense of (3)), we can conclude that u is concave.

Proof. We argue as in [1]; let us first state and prove two lemmata. The first one is quite classical; it uses the notion of relaxed semi-limits first introduced by Barles and Perthame.

Lemma 1. *Let u be a solution of* (1). *Then u_∞ is a solution of*:

$$F^\infty\big(x, u_\infty, \nabla u_\infty, D^2 u_\infty\big) = 0 \quad \textit{in } \Omega_\infty,$$

with $\Omega_\infty = \{y \in \mathbb{R}^n\colon\ \exists (t_n, x_n) \in (0;+\infty) \times \Omega,\ t_n \to 0,\ t_n x_n \to y\}$ and

$$F^\infty(y,v,p,B) = \limsup_{\substack{(y',v',B') \to (y,v,B)\\ t \to 0^+}} tF\left(\frac{y'}{t}, \frac{v'}{t}, p, \frac{1}{t} \odot B'\right).$$

Proof. Let us see u_∞ as the relaxed lower limit of the family of functions $\{u_t\}_{t>0}$ with $u_t(x) = tu(x/t)$. Next, since u_t solves: $F_t = 0$ with $F_t(y,v,p,B) = tF(\frac{y}{t}, \frac{v}{t}, p, \frac{1}{t} \odot B)$, we use the classical discontinuous stability result [7] and get that u_∞ solves $F^* = 0$ where F^* is the relaxed upper limit of the family $\{F_t\}_{t>0}$. And by definition, $F^\infty = F^*$. □

The function F^∞ is the opposite of the recession function of $-F$ but since we redefined the addition and the external law on matrices and because the proofs are very simple, we state and prove the properties of F^∞ that we need later. The reader may compare them with the usual properties of recession functions listed in Appendix A.

Lemma 2. *The function F^∞ satisfies*:

$$F^\infty(ty, tv, p, t \odot B) = tF^\infty(y,v,p,B) \quad \textit{for } t > 0,$$
$$F^\infty(y,v,p,B) + F(x,u,p,A) \leqslant F(x+y, u+v, p, A \,\square\, B),$$
$$F^\infty(y,v,p,B) + F^\infty(z,w,q,C) \leqslant F^\infty(y+z, v+w, p, B \,\square\, C).$$

Proof. Looking at the definition of F^∞, the first equality is clear. In order to prove the second one, consider sequences $\{t_n\}_n, \{y_n\}_n, \{v_n\}_n, \{B_n\}_n$ realizing the lim sup defining F^∞:

$$F^\infty(y,v,p,B) = \lim_{n \to +\infty} t_n F\left(\frac{y_n}{t_n}, \frac{v_n}{t_n}, p, \frac{1}{t_n} \odot B_n\right)$$

and define $d_n = \frac{y_n}{t_n}$, $w_n = \frac{v_n}{t_n}$, and $C_n = \frac{1}{t_n} \odot B_n$. Now use the fact that F is continuous and get:

$$\begin{aligned}
F(x+y,u+v,p,A\,\Box\,B) &= \lim_{n\to+\infty} F\big((1-t_n)x+t_n d_n,(1-t_n)u+t_n w_n,p,(1-t_n)\odot A\,\Box\,t_n\odot C_n\big)\\
&\geqslant \lim_{n\to+\infty}\left\{(1-t_n)F(x,u,p,A)+t_nF\left(\frac{y_n}{t_n},\frac{v_n}{t_n},p,\frac{1}{t_n}\odot B_n\right)\right\}\\
&= F(x,u,p,A)+F^\infty(y,v,p,B)
\end{aligned}$$

and the first inequality is proved. Combining the two first properties gives the third one. □

Proposition 5 can now be easily proved by using these two lemmata. Consider $(p,A)\in D^{2,-}u^{**}(x)$ for $x\in\overline{\Omega}$. By Proposition 3, there exist $p+q$ points x_i, x_j^∞ and $p+q$ matrices A_i, B_j such that (13) holds true. Since u is a supersolution of $F=0$ at x_i and u_∞ is a supersolution of F^∞ at x_j^∞ (Lemma 1), we have:

$$F\big(x_i,u(x_i),p,A_i\big)\geqslant 0 \quad\text{and}\quad F\big(x_j^\infty,u_\infty\big(x_j^\infty\big),p,B_j\big)\geqslant 0.$$

Using Lemma 2, (33) and (3), we therefore obtain,

$$\begin{aligned}
F\big(x,u^{**}(x),p,A_\varepsilon\big) &\geqslant F\big(x,u^{**}(x),p,\big(\Box_{i=1}^p\lambda_i\odot A_i\big)\,\Box\,\big(\Box_{j=1}^q B_j\big)\big)\\
&\geqslant F\left(\sum_{i=1}^p\lambda_i x_i,\sum_{i=1}^p\lambda_i u(x_i),p,\Box_{i=1}^p\lambda_i\odot A_i\right)+F^\infty\left(\sum_{j=1}^q x_j^\infty,\sum_{j=1}^q u_\infty(x_j^\infty),p,\Box_{j=1}^q B_j\right)\\
&\geqslant \sum_{i=1}^p\lambda_i F\big(x_i,u(x_i),p,A_i\big)+\sum_{j=1}^q F^\infty\big(x_j^\infty,u_\infty\big(x_j^\infty\big),p,B_j\big)\geqslant 0.
\end{aligned}$$

Letting $\varepsilon\to 0$ achieves the proof. □

We now use Proposition 5 to establish the convexity of a viscosity solution of a fully nonlinear second order equation whose Hamiltonian depends explicitly on x. Precisely, we consider:

$$F\big(x,u,\nabla u,D^2u\big)=0 \quad \text{for } x\in\mathbb{R}^n. \tag{15}$$

In order that a comparison principle be satisfied, we make the following (very) classical assumptions:

(A1) $F:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n\to\mathbb{R}$ is continuous;
(A2) There exists $\gamma>0$ such that for any $x,p\in\mathbb{R}^n$ and $A\in\mathbb{S}^n$,

$$F(x,u,p,A)-F(x,v,p,A)\geqslant\gamma(u-v);$$

(A3) There exists a modulus of continuity $m(\cdot)$ such that for any $x,y\in\mathbb{R}^n$ and any $X\in\mathbb{S}^n$,

$$F\left(y,u,\frac{x-y}{\varepsilon},X^{\varepsilon/3}\right)-F\left(x,u,\frac{x-y}{\varepsilon},X\right)\leqslant m\left(\frac{|x-y|^2}{\varepsilon}+|x-y|\right); \tag{16}$$

(A4) $F(x,u,p,X)$ is uniformly continuous in (p,X), uniformly in (x,u);
(A5) $F(x,0,0,0)$ is uniformly continuous.

We recall that (16) implies that F is degenerate elliptic. The following theorem must be compared with [1, Theorem 3].

Theorem 1. *Let F satisfy Assumptions* (A1)–(A5) *and* (3). *Then the unique uniformly continuous viscosity solution of* (15) *is convex.*

Proof. For $\delta>0$, consider u_δ the unique uniformly continuous viscosity solution of

$$F\big(x,u,\nabla u,D^2u\big)=\gamma\delta|x| \quad \text{in } \mathbb{R}^n. \tag{17}$$

Let $|\cdot|_\varepsilon$ denote the inf-convolution of $|\cdot|$ and $\varepsilon^{-1}|\cdot|^2$. For C_ε large enough, the function $u_-^\delta(x)=\delta|x|_\varepsilon-C_\varepsilon$ is a subsolution of (17) with sublinear growth and by the comparison principle, we conclude that $u^\delta\geqslant u_-^\delta$ so that u^δ is epi-pointed. The function $F_\delta(x,u,p,A)=F(x,u,p,A)-\gamma\delta|x|$ still satisfies (3). Hence Proposition 5 implies that the convex envelope of u^δ is a supersolution of (17) and by the comparison principle we conclude that u^δ is convex. By stability, we know that $u=\lim_{\delta\to 0}u^\delta$ so that u is also convex. □

2.3. Partial convexity

We next extend the results of [1] about partial convexity from regular solutions of nonlinear elliptic equations to viscosity solutions. We use the framework and the notations of [1]. Let us recall them now. We consider two integers n' and n'' and $n = n' + n''$, an open subset $\Omega'' \subset \mathbb{R}^{n''}$ and we define $\Omega = \mathbb{R}^{n'} \times \Omega''$. We establish conditions under which a viscosity solution of (1) is convex with respect to the first variable x', i.e., such that $x' \mapsto u(x', x'')$ is convex for any $x'' \in \overline{\Omega''}$. In order to do so, we study its partial convex envelope, namely $u^{\star\star} = (u^\star)^\star$ where $\star$ denotes the Legendre–Fenchel transform with respect to x':

$$f^\star(q', x'') = \sup_{x' \in \mathbb{R}^n} \{q' \cdot x' - f(x', x'')\}.$$

We also consider partial recession functions, i.e., recession functions with respect to x':

$$f_{\underline{\infty}}(x', x'') = \liminf_{t \to 0^+,\ y' \to x'} tf\left(\frac{y'}{t}, x''\right).$$

We will consider inf-convolution with respect to x'; precisely, we define $\Box'$ by:

$$u \,\Box'\, v(x', x'') = \inf_{y' \in \mathbb{R}^{n'}} \{u(y', x'') + v(x' - y', x'')\}.$$

Next, any matrix $A \in \mathbb{S}^n$ is decomposed in four blocks denoted as follows:

$$A = \begin{bmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{bmatrix}.$$

Eventually, it will be useful to redefine $\odot$ in the following way:

$$\alpha \odot' A = \begin{bmatrix} \alpha^{-1} a_1 & a_2 \\ a_2^* & \alpha a_3 \end{bmatrix}.$$

We consider a function u that is epi-pointed with respect to x' uniformly in x''; this means that there exists $s \in \mathbb{R}^{n'}$, $\sigma > 0$ and $r \in \mathbb{R}$ such that

$$u(x', x'') \geqslant \langle s, x' \rangle + \sigma |x'| - r \quad \text{for any } (x', x'') \in \overline{\Omega}. \tag{18}$$

We want to obtain a proposition equivalent to Proposition 3. Let $x \in \overline{\Omega}$ and consider $(p, A) \in D^{2,-}_{\overline{\Omega}} u^{\star\star}(x)$. Consider the points x_i and x_j^∞ and the real numbers λ_i such that

$$\begin{cases} x' = \sum_{i=1}^p \lambda_i x_i' + \sum_{j=1}^q x_j'^\infty, \\ u^{\star\star}(x', x'') = \sum_{i=1}^p \lambda_i u(x_i', x'') + \sum_{j=1}^q u_{\underline{\infty}}(x_j'^\infty, x''). \end{cases} \tag{19}$$

Using Remark 2, we notice that $u^{\star\star}$ equals $(\Box')_{i=1}^p u_{\lambda_i} \,\Box'\, (\Box')_{i=1}^q u_{\underline{\infty}}$ where $u_{\lambda_i}(x', x'') = \lambda_i u(x'/\lambda_i, x'')$. We thus need to prove a result analogous to Proposition 4.

Lemma 3. *Consider k lsc functions $v_1, \ldots, v_k$ defined on $\mathbb{R}^{n'} \times \overline{\Omega''}$ and consider $v = (\Box')_{i=1}^k v_i$. Suppose it is finite at $x = (x', x'')$ and that the infimum is attained at (x_i', x''):*

$$v(x', x'') = \sum_{i=1}^k v_i(x_i', x'').$$

Then for any $((p', p''), A) \in D^{2,-} v(x)$ and any $\varepsilon > 0$ small enough, there exist $k+1$ vectors $p''_{i,\varepsilon} \in \mathbb{R}^{n''}$, $p'_\varepsilon \in \mathbb{R}^{n'}$ and a vector $p''_\varepsilon \in \mathbb{R}^{n''}$ and k matrices $A_i \in \mathbb{S}^n$ and $B \in \mathbb{S}^{2n}$ such that

$$\big((p'_\varepsilon, p''_{i,\varepsilon}), A_i\big) \in \overline{D}^{2,-} v_i(x'_{i,\varepsilon}, x''_{i,\varepsilon}) \quad \text{and} \quad p''_\varepsilon = \sum_{i=1}^k p''_{i,\varepsilon}, \quad B_\varepsilon \leqslant (\Box')_{i=1}^k A_i \quad \text{and}$$
$$\big(p'_\varepsilon, p''_\varepsilon, B, x'_{i,\varepsilon}, x''_{i,\varepsilon}, v_i(x'_{i,\varepsilon}, x''_{i,\varepsilon})\big) \to \big(p', p'', A, x_i', x'', v_i(x_i', x'')\big) \quad \text{as } \varepsilon \to 0.$$

Proof. We only do the proof for two functions, denoted u and v. By a classical property of subjets, we know that there exists a C^2 function ϕ such that $u \oplus v - \phi$ attains a local strict minimum at (x', x'') and $(p', p'') = (\nabla_{x'}\phi(x', x''), \nabla_{x''}\phi(x', x''))$ and $A = D^2\phi(x', x'')$. This implies that the function $(y_1', y_2', y'') \mapsto u(y_1', y'') + v(y_2', y'') - \phi(y_1' + y_2', y'')$ attains a local strict minimum at (x_1', x_2', x''). We next dedouble the variable y'' and we consider the infimum of the perturbed function $(y_1', y_1'', y_2', y_2'') \mapsto u(y_1', y_1'') + v(y_2', y_2'') - \phi(y_1' + y_2', \frac{y_1''+y_2''}{2}) - \frac{|y_1''-y_2''|^2}{2\varepsilon}$. We know that the infimum is attained at a point $(x'_{1,\varepsilon}, x'_{2,\varepsilon}, x''_{1,\varepsilon}, x''_{2,\varepsilon}) \to (x_1', x_2', x'', x'')$ as $\varepsilon \to 0$. Let us compute the derivative of the new test function $\psi(y_1', y_1'', y_2', y_2'') = \phi(y_1' + y_2', \frac{y_1''+y_2''}{2}) + \frac{|y_1''-y_2''|^2}{2\varepsilon}$; we do not specify the variables in ϕ and B denotes $D^2\phi(x'_{1,\varepsilon} + x'_{2,\varepsilon}, x''_{1,\varepsilon})$ and $B = \left[\begin{smallmatrix} b_1 & b_2 \\ b_2^* & b_3 \end{smallmatrix}\right]$:

$$\nabla\psi = \left(\nabla_{y'}\phi, \frac{1}{2}\nabla_{y''}\phi + \frac{y_1''-y_2''}{\varepsilon}, \nabla_{y'}\phi, \frac{1}{2}\nabla_{y''}\phi - \frac{y_1''-y_2''}{\varepsilon}\right),$$

$$D^2\psi = \begin{bmatrix} b_1 & b_2/2 & b_1 & b_2/2 \\ b_2^*/2 & b_3/4 + \varepsilon^{-1}I & b_2^*/2 & b_3/4 - \varepsilon^{-1}I \\ b_1 & b_2/2 & b_1 & b_2/2 \\ b_2^*/2 & b_3/4 - \varepsilon^{-1}I & b_2^*/2 & +b_3/4 + \varepsilon^{-1}I \end{bmatrix} = \begin{bmatrix} \tilde B + C & \tilde B - C \\ \tilde B^* - C & \tilde B + C \end{bmatrix},$$

with $\tilde B = \left[\begin{smallmatrix} b_1 & b_2/2 \\ b_2^*/2 & b_3/4 \end{smallmatrix}\right]$ and $C = \left[\begin{smallmatrix} 0 & 0 \\ 0 & \varepsilon^{-1}I \end{smallmatrix}\right]$. By Ishii's lemma, for any $\nu > 0$, there exists $X, Y \in \mathbb{S}^n$ such that: $C_\nu \leqslant \left[\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right]$. Consider $\zeta = (\zeta', \zeta'')$, apply this inequality to $(\zeta_1', \zeta'', \zeta' - \zeta_1', \zeta'')$, minimize with respect to ζ_1' and get:

$$\begin{aligned}
(X \,\square'\, Y)\zeta\cdot\zeta &\geqslant \inf_{\zeta_1', \xi_1', \xi_1'', \xi_2', \xi_2''} \Bigg\{ \tilde B \begin{pmatrix} \zeta' - (\xi_1' + \xi_2') \\ 2\zeta'' - (\xi_1'' + \xi_2'') \end{pmatrix} \cdot \begin{pmatrix} \zeta' - (\xi_1' + \xi_2') \\ 2\zeta'' - (\xi_1'' + \xi_2'') \end{pmatrix} \\
&\quad + \frac{1}{\varepsilon}|\xi_1'' - \xi_2''|^2 + \frac{1}{2\nu}|\xi_1'|^2 + \frac{1}{2\nu}|\xi_1''|^2 + \frac{1}{2\nu}|\xi_2'|^2 + \frac{1}{2\nu}|\xi_2''|^2 \Bigg\} \\
&= \inf_{\xi_1', \xi_1'', \xi_2', \xi_2''} \Bigg\{ B \begin{pmatrix} \zeta' - (\xi_1' + \xi_2') \\ \zeta'' - \frac{\xi_1''+\xi_2''}{2} \end{pmatrix} \cdot \begin{pmatrix} \zeta' - (\xi_1' + \xi_2') \\ \zeta'' - \frac{\xi_1''+\xi_2''}{2} \end{pmatrix} \\
&\quad + \frac{1}{\varepsilon}|\xi_1'' - \xi_2''|^2 + \frac{1}{2\nu}|\xi_1'|^2 + \frac{1}{2\nu}|\xi_1''|^2 + \frac{1}{2\nu}|\xi_2'|^2 + \frac{1}{2\nu}|\xi_2''|^2 \Bigg\} \\
&= \inf_{\xi', \xi'', \bar\xi', \bar\xi''} \Bigg\{ B \begin{pmatrix} \zeta' - \xi' \\ \zeta'' - \xi'' \end{pmatrix} \cdot \begin{pmatrix} \zeta' - \xi' \\ \zeta'' - \xi'' \end{pmatrix} + \frac{1}{\varepsilon}|\bar\xi''|^2 + \frac{1}{4\nu}|\xi'|^2 + \frac{1}{4\nu}|\xi''|^2 + \frac{1}{4\nu}|\bar\xi'|^2 + \frac{1}{4\nu}|\bar\xi''|^2 \Bigg\} \\
&= \inf_{\xi', \xi''} \Bigg\{ B \begin{pmatrix} \zeta' - \xi' \\ \zeta'' - \xi'' \end{pmatrix} \cdot \begin{pmatrix} \zeta' - \xi' \\ \zeta'' - \xi'' \end{pmatrix} + \frac{1}{4\nu}|\xi'|^2 + \frac{1}{4\nu}|\xi''|^2 \Bigg\} = B_{4\nu}\zeta\cdot\zeta
\end{aligned}$$

and choose $\nu = \varepsilon/4$. The proof is now complete. □

Noticing that $\nabla_{x''}u_{\lambda_i}(x', x'') = \lambda_i \nabla_{x''}u(x'/\lambda_i, x'')$ and $D^2 u_{\lambda_i}(x) = \lambda_i \odot' D^2 u(x'/\lambda_i, x'')$, a straightforward consequence of this lemma is the following proposition:

Proposition 6. *Let $\Omega = \mathbb{R}^{n'} \times \Omega''$ with Ω'' open subset of $\mathbb{R}^{n''}$ and let $u : \overline{\Omega} \to \mathbb{R}$ be lsc and satisfy* (18). *Then for $(x', x'') \in \overline{\Omega}$, consider $(p, A) \in D^{2,-}_{\Omega} u^{\star\star}(x', x'')$. Consider the points x_i' and $x_j'^{\infty}$ and the real numbers λ_i such that* (19) *hold true. Then for every $\varepsilon > 0$, there exist $x_{i,\varepsilon}, x^{\infty}_{j,\varepsilon}$, $(p'_\varepsilon, p''_{i,\varepsilon}), (p'_\varepsilon, p''^{\infty}_{j,\varepsilon}) \in \mathbb{R}^n$ and $A_i, B_j \in \mathbb{S}^n$, $B \in \mathbb{S}^{2n}$ such that*

$$\begin{cases}
((p'_\varepsilon, p''_{i,\varepsilon}), A_i) \in \overline{D}^{2,-}u(x'_{i,\varepsilon}, x''_{i,\varepsilon}), \quad ((p'_\varepsilon, p''^{\infty}_{j,\varepsilon}), B_j) \in \overline{D}^{2,-}\underline{u_\infty}(x'^{\infty}_{j,\varepsilon}, x''^{\infty}_{j,\varepsilon}), \\
B_\varepsilon \leqslant ((\square')_{i=1}^p \lambda_i \odot' A_i) \,\square'\, ((\square')_{j=1}^q B_j), \\
\sum_{i=1}^p \lambda_i p''_{i,\varepsilon} + \sum_{j=1}^q p''^{\;\infty}_{j,\varepsilon} = p''_\varepsilon \to p'' \quad \text{as } \varepsilon \to 0, \\
(x'_{i,\varepsilon}, x''_{i,\varepsilon}, x'^{\infty}_{j,\varepsilon}, x''^{\infty}_{j,\varepsilon}) \to (x_i', x'', x_j^{\infty}, x'') \quad \text{as } \varepsilon \to 0, \\
(u(x_{i,\varepsilon}), u_\infty(x^{\infty}_{j,\varepsilon})) \to (u(x_i), u_\infty(x_j^{\infty})) \quad \text{as } \varepsilon \to 0, \\
(p'_\varepsilon, B) \to (p', A) \quad \text{as } \varepsilon \to 0.
\end{cases} \tag{20}$$

In view of this proposition, it is clear that in order to ensure partial convexity of a viscosity solution, the analogous of (3) is the following condition: for any $p' \in \mathbb{R}^n$ and x'', for any $(x', r, p'', A), (y', s, q'', B) \in \overline{\Omega} \times \mathbb{R} \times \times \mathbb{R}^{n''} \times \mathbb{S}^n_+$ and $\lambda \in]0, 1[$,

$$\lambda F\big((x', x''), r, (p', p''), A\big) + (1-\lambda) F\big((y', x''), s, (p', q''), B\big) \leqslant F\big(\big(\lambda x' + (1-\lambda) y', x''\big), \lambda r + (1-\lambda) s, \big(p', \lambda p'' + (1-\lambda) q''\big), \lambda \odot' A \,\square'\, (1-\lambda) \odot' B\big). \tag{21}$$

One can check that Condition (30) in [1, p. 282] implies ours. A difficulty arises when dedoubling the variables: the point x'' moves a bit. Of course, for C^2 solutions, one can pass to the limit and obtain the result described in [1].

Proposition 7. (See [1].) *Let $\Omega = \mathbb{R}^{n'} \times \Omega''$ with Ω'' open subset of $\mathbb{R}^{n''}$ and let F be continuous, degenerate elliptic and satisfy* (21). *Let $u : \overline{\Omega} \to \mathbb{R}$ be a C^2 supersolution of* (1) *that satisfies* (18). *Then $u^{\star\star}$ is a* (*lsc*) *supersolution of* (1).

In order to avoid further technicalities, we assume that F depends on x'' via a source term, i.e., we consider:

$$H\big(x', u(x', x''), (\nabla_{x'} u, \nabla_{x''} u), D^2_{(x', x'')} u(x', x'')\big) = f(x', x'') \tag{22}$$

Therefore, $F = H - f$. Notice that this Hamiltonian does depend on all the (first and second) derivatives of u and on the function u itself. We can now state a result corresponding to Proposition 3.

Proposition 8. *Let $\Omega = \mathbb{R}^{n'} \times \Omega''$ with Ω'' open subset of $\mathbb{R}^{n''}$ and let H be continuous, degenerate elliptic, that f is convex and that H satisfies* (21). *Let $u : \overline{\Omega} \to \mathbb{R}$ be a supersolution of* (22) *that satisfies* (18). *Then so is the partial convex envelope $u^{\star\star}$.*

Proof. We follow the lines of the proof of Proposition 5 and we omit details. The analogous of Lemmata 1 and 2 can be stated for general Hamiltonians F.

Lemma 4. *Let u be a solution of* (1). *Then $u_{\underline{\infty}}$ is a solution of*:

$$F^{\overline{\infty}}\big(x, u_\infty, \nabla u_\infty, D^2 u_\infty\big) = 0 \quad \text{in } \Omega_{\underline{\infty}},$$

with $\Omega_{\underline{\infty}} = \{(y', y'') \in \mathbb{R}^{n'} \times \Omega'' \colon \exists (t_n, y'_n) \in (0; +\infty) \times \mathbb{R}^{n'},\ t_n \to 0,\ t_n y'_n \to y'\}$ *and*

$$F^{\overline{\infty}}\big(y', y'', v, (p', p''), B\big) = \limsup_{\substack{(z', v', q'', B') \to (y', v, p'', B) \\ t \to 0^+}} t F\left(\frac{z'}{t}, y'', \frac{v'}{t}, \left(p', \frac{q''}{t}\right), \frac{1}{t} \odot' B'\right).$$

Lemma 5. *The function $F^{\overline{\infty}}$ satisfies*:

$$F^{\overline{\infty}}(ty', y'', tv, p, t \odot' B) = t F^{\overline{\infty}}(y', y'', v, p, B) \quad \text{for } t > 0,$$
$$F^{\overline{\infty}}(y', y'', v, p, B) + F(x', y'', u, p, A) \leqslant F(x' + y', y'', u + v, p, A \,\square'\, B),$$
$$F^{\overline{\infty}}(y', y'', v, p, B) + F^{\overline{\infty}}(z', y'', w, q, C) \leqslant F^{\overline{\infty}}(y' + z', y'', v + w, p, B \,\square'\, C).$$

Now we use the lemmata for our special Hamiltonian $F = H - f$. Consider $(p, A) \in D^{2,-} u^{\star\star}(x)$ for $x \in \overline{\Omega}$. By Proposition 6, for every $\varepsilon > 0$, there exist $x_{i,\varepsilon}, x^\infty_{j,\varepsilon}, (p'_\varepsilon, p''_{i,\varepsilon}), (p'_\varepsilon, p''^\infty_{j,\varepsilon}) \in \mathbb{R}^n$ and $A_i, B_j \in \mathbb{S}^n$, $B \in \mathbb{S}^{2n}$ such that (20) holds true. Since u is a supersolution of $F = 0$ at $(x'_{i,\varepsilon}, x''_{i,\varepsilon})$ and u_∞ is a supersolution of F^∞ at $(x'^\infty_{j,\varepsilon}, x''^\infty_{j,\varepsilon})$ (Lemma 1), we have:

$$F\left(\left(\sum_{i=1}^p \lambda_i x'_{i,\varepsilon} + \sum_{j=1}^q x'^\infty_{j,\varepsilon}, x''\right), \sum_{i=1}^p \lambda_i u(x_{i,\varepsilon}) + \sum_{j=1}^q u_\infty(x^\infty_{j,\varepsilon}), \left(p'_\varepsilon, \sum_{i=1}^p \lambda_i p''_{i,\varepsilon} + \sum_{j=1}^q p''^\infty_{j,\varepsilon}\right), A_\varepsilon\right)$$
$$\geqslant F\big(\{\ldots\}, \big((\square')_{i=1}^p \lambda_i \odot' A_i\big) \,\square'\, \big((\square')_{j=1}^q B_j\big)\big)$$
$$\geqslant F\left(\left(\sum_{i=1}^p \lambda_i x'_{i,\varepsilon}, x''\right), \sum_{i=1}^p \lambda_i u(x_{i,\varepsilon}), \left(p'_\varepsilon, \sum_{i=1}^p \lambda_i p''_{i,\varepsilon} + \sum_{j=1}^q p''^\infty_{j,\varepsilon}\right), (\square')_{i=1}^p \lambda_i \odot' A_i\right)$$

$$+ F^{\infty}\left(\left(\sum_{j=1}^{q} x'^{\infty}_{j,\varepsilon}, x''\right), \sum_{j=1}^{q} u_{\infty}(x^{\infty}_{j,\varepsilon}), \left(p'_{\varepsilon}, \sum_{j=1}^{q} p''^{\infty}_{j,\varepsilon}\right), (\Box')_{j=1}^{q} B_j\right)$$
$$\geqslant \sum_{i=1}^{p} \lambda_i F\big((x'_{i,\varepsilon}, x''), u(x_{i,\varepsilon}), (p'_{\varepsilon}, p''_{i,\varepsilon}), A_i\big) + \sum_{j=1}^{q} F^{\infty}\big((x'^{,\infty}_{j,\varepsilon}, x''), u_{\infty}(x^{\infty}_{j,\varepsilon}), (p'_{\varepsilon}, p''^{\infty}_{j,\varepsilon}), B_j\big)$$
$$\geqslant \sum_{i=1}^{p} \lambda_i \big[f(x'_{i,\varepsilon}, x''_{i,\varepsilon}) - f(x'_{i,\varepsilon}, x'')\big] + \sum_{j=1}^{q} \big[f(x'^{\infty}_{j,\varepsilon}, x''^{\infty}_{j,\varepsilon}) - f(x'^{\infty}_{j,\varepsilon}, x'')\big].$$

Letting $\varepsilon \to 0$ and using (20) yields $F(x, u(x), p', p'', A) \geqslant 0$ and the proof is complete. □

We next give an example of how to use this result. Let us translate Assumptions (A1)–(A5) for Eq. (22).

(B1) $H : \mathbb{R}^{n'} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ continuous;
(B2) There exists $\gamma > 0$ such that for any $x' \in \mathbb{R}^{n'}$, $p \in \mathbb{R}^n$ and $A \in \mathbb{S}^n$,

$$H(x', u, p, A) - H(x', v, p, A) \geqslant \gamma(u - v);$$

(B3) There exists a modulus of continuity $m(\cdot)$ such that for any $x = (x', x'')$, $y = (y', y'') \in \mathbb{R}^n$ and any $X \in \mathbb{S}^n$,

$$H\left(y', u, \frac{x-y}{\varepsilon}, X^{\varepsilon/3}\right) - H\left(x', u, \frac{x-y}{\varepsilon}, X\right) \leqslant m\left(\frac{|x-y|^2}{\varepsilon} + |x-y|\right); \tag{23}$$

(B4) $H(x', u, p, X)$ is uniformly continuous in (p, X), uniformly in (x', u);
(B5) $H(x', 0, 0, 0)$ and f are uniformly continuous.

Theorem 2. *Let F satisfy Assumptions* (B1)–(B5) *and* (21). *Then the unique uniformly continuous viscosity solution of* (22) *is convex w.r.t. x'.*

3. $C^{1,1}$ regularity of convex solutions of strictly elliptic equations

This section is concerned with $C^{1,1}$ estimates for convex solutions of equations that are strictly elliptic. A new proof of the Alexandroff estimate obtained by Caffarelli [4] is also provided.

3.1. A general $C^{1,1}$ estimate

The fact that convex viscosity solution of □-concave strictly elliptic equations in $\mathbb{R}^n$ are $C^{1,1}$ is a consequence of the following simple idea: since the function u is convex, it is enough to prove that it is semi-concave, that is to say it is enough to prove that there exists C_R such that the function $x \mapsto u(x) - \frac{C_R}{2}|x|^2$ is concave on B_R for any $R > 0$. Using for instance [1, Lemma 1, p. 268], it suffices to prove that for any $(p, A) \in D^{2,-}u(x)$ and any $x \in B_R$, we have: $A \leqslant C_R I$.

Let us first recall what are strictly elliptic equations (see for instance [10]). The Hamiltonian F is *strictly elliptic* if

- There exists $\lambda > 0$ such that for any $X, Y \in \mathbb{S}^n$, $X \leqslant Y$,

$$F(x, u, p, Y) \leqslant F(x, u, p, X) - \lambda \operatorname{Tr}(Y - X). \tag{24}$$

  In the linear case, i.e., when $F(x, u, p, X) = -\operatorname{Tr}(AX)$, this condition is equivalent to $A\xi \cdot \xi \geqslant \lambda|\xi|^2$. This condition is weaker than the *uniform ellipticity* of F that is defined as follows in the nonlinear case:
- There exists $\lambda, \Lambda > 0$ such that for any $X, Y \in \mathbb{S}^n$,

$$F(x, u, p, Y) \leqslant F(x, u, p, X) - \lambda \operatorname{Tr}(Y - X)^+ + \Lambda \operatorname{Tr}(Y - X)^- \tag{25}$$

  which reads in the linear case: $\lambda|\xi|^2 \leqslant A\xi \cdot \xi \leqslant \Lambda|\xi|^2$ for any $\xi \in \mathbb{R}^n$.

We can now state a generalization of a result of Caffarelli about the $C^{1,1}$ regularity of viscosity solution of $F(D^2u)=0$ with F concave and uniformly elliptic. See the discussion following [5, Theorem 6.6].

Theorem 3. *Suppose that F satisfies* (A0)–(A5), (24) *and* (3) *and consider the unique Lipschitz continuous convex viscosity solution u of* (15) *given by Theorem* 1. *Then u is $C^{1,1}$ and for a.e. $x \in \mathbb{R}^n$,*

$$\left\|D^2u(x)\right\| \leqslant \frac{1}{\lambda}\left|F\left(x, u(x), \nabla u(x), 0\right)\right|.$$

Theorem 3 is a straightforward consequence of the following $C^{1,1}$ estimate. We only assume the strict ellipticity of F. In particular, we neither assume that F is concave or convex, nor that it is Lipschitz continuous w.r.t. D^2u.

Theorem 4 *($C^{1,1}$ estimate). Consider the general equation* (1) *and suppose that F is strictly elliptic, i.e., it satisfies* (24). *Then any convex supersolution of* (1) *in $\mathbb{R}^n$ is $C^{1,1}$ in $\mathbb{R}^n$ and for a.e. $x \in \mathbb{R}^n$:*

$$\left\|D^2u(x)\right\| \leqslant \frac{1}{\lambda}\left|F\left(x, u(x), \nabla u(x), 0\right)\right|. \tag{26}$$

Proof. We first claim that u is a supersolution of the following equation:

$$-\lambda \operatorname{Tr} D^2u + F\left(x, u(x), \nabla u(x), 0\right) = 0 \quad \text{in } \overline{\Omega}.$$

This is a consequence of the strict ellipticity of F, i.e., (24) with $Y=0$ and, once again, of a result of [1], namely their Lemma 3. This lemma asserts that in order to prove that the lsc convex function u is a supersolution of (26), it suffices to consider superjets (p, A) of u such that $A \geqslant 0$. But for $A \geqslant 0$, $A \leqslant (\operatorname{Tr} A)I$. Hence, we obtain:

$$A \leqslant \frac{1}{\lambda}\left|F\left(x, u(x), p, 0\right)\right| I.$$

Hence we are done. □

3.2. A new proof of Alexandroff estimate

The ideas of the preceding section can be used to prove the Alexandroff estimate obtained by Trudinger in [16] and by Caffarelli in [4] for fully nonlinear elliptic equations. The Hamiltonian F must satisfy: for any $x \in B_d$, any $r \in \mathbb{R}$ and any $p \in \mathbb{R}^n$,

$$F(x, r, p, 0) \leqslant F(x, r, 0, 0) + \gamma_d(x)|p|, \tag{27}$$

where $\gamma_d(\cdot)$ is continuous.

Theorem 5 *(Alexandroff estimate). Let F verify* (A1), (A2) *with $\gamma = 0$,* (24) *and* (27) *and u be a* (*lsc*) *supersolution of* (1) *in B_d. Then*

$$\sup_{B_d} u \leqslant M_\partial + Cd\left(\int_{B_d \cap \{u+M_\partial = \Gamma(u)\}} (f^+)^n\right)^{1/n}, \tag{28}$$

where $M_\partial = \sup_{\partial B_d} u^-$, $\Gamma(u)$ is the convex envelope of $\min(u+M_\partial, 0)$ *extended by* 0 *on B_{2d}, $C = C(n, d, \lambda, \|\gamma_d\|_n)$ and* $f(x) = F(x, M_\partial, 0, 0)$.

Remark 4.

(1) Note that we do not assume that F is strictly elliptic and not uniformly elliptic (see (24) and (25)). But if one wants to prove a comparison result or use twice this estimate in order to prove a Harnack type inequality, F will need to be uniformly elliptic.
(2) We do not assume that u is continuous but merely lsc.
(3) The result can be extended to more general domains Ω.

Before proving this result, we recall that the main difficulty in adapting the classical proof of [9, Theorem 9.1, p. 220] to fully nonlinear elliptic equations, as explained in [5], is to prove that $\Gamma(u)$ is $C^{1,1}$ on B_d and to get an estimate of $D^2\Gamma(u)$ on the contact set. In [4], the author does so by using a suitable "barrier"; see [5, Lemma 3.3]. Moreover, the size of the balls $\tilde{B}$ on which the function $\Gamma(u)$ is dominated by a convex paraboloids has to be controlled; see [5, estimates (3.12), (3.13), p. 27].

Sketch of the proof of Theorem 5. First, we reduce to the case where $u \geqslant 0$ at the boundary. In order to do so, we consider $v = u + M_\partial$; it is a solution of $G(D^2v, \nabla v, v, x) = 0$ with $G(x, r, p, X) = F(x, r + M_\partial, p, X)$. Then $\Gamma(u)$ is the convex envelope of $-v^- = \min(v, 0)$.

In order to prove that v^{**} is $C^{1,1}$ on B_d, it is enough to prove that v^{**} is semi-concave; in virtue for instance of Lemma 1 in [1], it is enough to prove that there exists a constant C such that for any $x \in B_d$ and $(p, A) \in D^{2,-}v^{**}(x)$, $A \leqslant CI$. We first suppose that $A \geqslant 0$. We then distinguish two cases. First, study a contact point $x \in \{v = v^{**}\}$. In such a case, $(p, A) \in D^{2,-}v(x) = D^{2,-}u(x)$, so that $F(x, u(x), p, A) \geqslant 0$ and (24) yields:

$$-\lambda \operatorname{Tr} A + \gamma_d(x)|p| + f^+(x) \geqslant 0,$$

and since $A \geqslant 0$, we conclude that

$$A \leqslant \frac{1}{\lambda}\big(\gamma_d(x)|p| + f^+(x)\big)I \tag{29}$$

and the right-hand side is bounded on B_d since v^{**} is Lipschitz continuous and γ_d and f^+ are continuous. Let us denote $A \leqslant CI$. Remark that the previous inequality also holds true for A such that $(p, A) \in \bar{D}^{2,+}u(x)$, $A \geqslant 0$, since the equation is also satisfied for limiting semi-jets. Next, we consider a point $x \in B_d \setminus \{v = v^{**}\}$. There then exist $x_i \in \bar{B}_d$ and $\lambda_i \in (0, 1)$ such that (33) holds true (where $u = v$). Remark that there are no points at infinity. We know that there is at most one point on ∂B_{2d} and the others are in B_d; if not, $v^{**} \equiv 0$ and there is nothing to prove. Then by Proposition 3, for any $\varepsilon > 0$, there exist p matrices $\lambda_i \odot A_i \geqslant A_\varepsilon \geqslant 0$ such that $\square_{i=1}^{p}\lambda_i \odot A_i \geqslant A_\varepsilon$ and $(p, A_i) \in \bar{D}^{2,+}v(x_i) = \bar{D}^{2,+}u(x_i)$. If there are no points on ∂B_{2d}, then for any i, $A_i \leqslant CI$ and $A_\varepsilon \leqslant CI$ follows. If $x_p \in \partial B_{2d}$, say, then we deduce from (33) that $\lambda_p \leqslant 2/3$; hence, there exists $i \in \{1, \dots, p-1\}$ such that $\lambda_i \geqslant 1/3n$. For instance $i = 1$. Then we conclude that

$$A_\varepsilon \leqslant \frac{1}{\lambda_1}A_1 \leqslant 3nCI.$$

Passing to the limit on ε, we obtain $A \leqslant CI$ (changing the constant C). Now consider a semi-jet (p, A) with A not necessarily nonnegative. Classically, by using the fact that v^{**} is convex, we can construct $(p_\varepsilon, B_\varepsilon) \in D^{2,+} \cap D^{2,-}v^{**}(x_\varepsilon)$ with $x_\varepsilon \to x$ as $\varepsilon \to 0$ with $A \leqslant B_\varepsilon + o_\varepsilon(1)$. Then $B_\varepsilon \geqslant 0$ and by the previous case, $B_\varepsilon \leqslant CI$; passing to the limit, we conclude once again that $A \leqslant CI$.

We therefore have proved that v^{**} is $C^{1,1}$ on the ball and (29) implies that a.e. on the contact set, we have:

$$D^2\Gamma(u) \leqslant \frac{1}{\lambda}\big(\gamma_d(x)\big|\nabla u(x)\big| + f^+(x)\big)I.$$

We now use the same technique as in [9, pp. 223–224]; first, we consider $g(p) = (|p|^{n/n-1} + \mu^{n/n-1})^{1-n}$ and we write:

$$|\gamma_d||\nabla u| + f^+ \leqslant \frac{(|\gamma_d|^n + \mu^{-n}(f^+)^n)^{1/n}}{g^{1/n}}. \tag{30}$$

We conclude from [9, Lemma 9.4] that (28) holds with

$$C = \lambda\left\{\exp\left(\frac{2^{n-2}}{\omega_n}\left[1 + \int_{\{v=\Gamma(u)\}} (\gamma_d/\lambda)^n\right]\right) - 1\right\}^{1/n}. \qquad \square$$

Appendix A. Useful facts from convex analysis

Let us recall some facts about convex analysis. The *recession function* associated with a lsc function $f:\Omega\to\mathbb{R}$ is denoted f_∞ and is defined as follows:

$$f_\infty(d)=\liminf_{t\to 0^+,\ d'\to d} tf\left(\frac{d'}{t}\right).$$

This function is positively homogeneous. When f is convex, it is also sub-additive and the following equality holds true:

$$f_\infty(d)=\sup_{x\in\mathbb{R}^n}\left\{f(x+d)-f(x)\right\}. \tag{31}$$

See for instance [14, p. 66]. We only use the following consequence: if f is convex, then for any $x,d\in\mathbb{R}^n$,

$$f_\infty(d)+f(x)\geqslant f(x+d). \tag{32}$$

In Section 2, we needed the following proposition.

Proposition 9. (See [2].) *Let Ω be a convex open set and $u:\overline{\Omega}\to\mathbb{R}$ be lsc and epi-pointed. For $x\in\overline{\Omega}$, consider $(p,A)\in D^{2,-}_{\Omega}u^{**}(x)$. There then exist $x_1,\dots,x_p\in\overline{\Omega}$, $p\leqslant n$, $\lambda_1,\dots,\lambda_p\in[0,1]$, $\sum_{i=1}^p\lambda_i=1$ and $x_1^\infty,\dots,x_q^\infty\in\mathbb{R}^n$, $q\leqslant n+1-p$ such that*

$$\begin{cases} x=\sum_{i=1}^p\lambda_i x_i+\sum_{j=1}^q x_j^\infty,\\ u^{**}(x)=\sum_{i=1}^p\lambda_i u(x_i)+\sum_{j=1}^q u_\infty(x_j^\infty). \end{cases} \tag{33}$$

One says that the points x_i, x_j^∞ are called by x. For 1-coercive functions, there are no points at infinity: $q=0$.

Appendix B. Subjets of an inf-convolution

The following proposition can be seen as a generalization of Lemma 2.14 of [11] (see also Proposition 4.3 in [6]).

Proposition 10.

(1) *Consider two lower semicontinuous functions $u,v:\mathbb{R}^n\to(-\infty,+\infty]$ and consider their inf-convolution:*

$$u\,\square\,v(z)=\inf_{y\in\Omega}\left\{u(y)+v(z-y)\right\}.$$

Suppose it is finite for $x\in\mathbb{R}^n$ and that this infimum is attained at $\bar{y}$. Then for any $(p,A)\in D^{2,-}u\,\square\,v(x)$, we have:

$$(p,A)\in D^{2,-}u(\bar{y})\cap D^{2,-}v(x-\bar{y}).$$

(2) *If $v(x)=\frac{1}{2}Bx\cdot x$, then $p=B(x-\bar{y})$ and $A\leqslant B$ and $(p,A\boxminus B)\in\overline{D}^{2,-}u(\bar{y})$.*
(3) *If v is a C^2 function and $(p,A)\in\bar{D}^{2,-}u\,\square\,v(x)$ then $p=\nabla v(x-\bar{y})$ and $A\leqslant D^2v(x-\bar{y})$ and $(p,A\boxminus D^2v(x-\bar{y}))\in\overline{D}^{2,-}u(\bar{y})$.*

Remark 5.

- We will see in the proof of Proposition 10 that if $A-B$ is not degenerate, then we can even ensure that $(p,A\boxminus B)\in D^{2,-}u(\bar{y})$.
- We will see that (3) is a simple consequence of (2) by an approximation argument.

Proof. For any $z\in\mathbb{R}^n$ and h small enough, we have:

$$u(z) + v(x+h-z) \geqslant u \Box v(x+h) \geqslant u \Box v(x) + p \cdot h + \frac{1}{2} Ah \cdot h + o(|h|^2)$$
$$\geqslant u(\bar{y}) + v(x-\bar{y}) + p \cdot h + \frac{1}{2} Ah \cdot h + o(|h|^2). \quad (1)$$

Choosing $z = \bar{y} + h$, we find that $(p, A) \in D^{2,-}u(\bar{y})$ and since the problem is symmetric in u and v, the first part of the result is now proved.

Next, we consider the case $v(x) = \frac{1}{2}Bx \cdot x$; the first part implies that $p = B(x - \bar{y})$ and $A \leqslant B$ so that (1) reads with $z = \bar{y} + \delta$, $\delta \in \mathbb{R}^n$,

$$u(\bar{y}+\delta) \geqslant u(\bar{y}) + \frac{1}{2}B(x-\bar{y}) \cdot (x-\bar{y}) - \frac{1}{2}B(x-\bar{y}+h-\delta) \cdot (x-\bar{y}+h-\delta) + B(x-\bar{y}) \cdot h + \frac{1}{2}Ah \cdot h + o(|h|^2)$$
$$\geqslant u(\bar{y}) + p \cdot \delta + \frac{1}{2}\big(Ah \cdot h - B(h-\delta) \cdot (h-\delta)\big) + o(|h|^2).$$

Let us first suppose that $A - B$ is not degenerate. Choosing now $h = T\delta$, we obtain that $(p, C) \in D^{2,-}u(\bar{y})$ with $C = T^*(A-B)T + T^*B + BT - B$. Since $A - B$ is not degenerate, we can consider $T_\varepsilon = (B-A)^{-1}B$; the associated C is $B(B-A)^{-1}A = A \boxminus B$.

If now $A - B$ is degenerate, then for any $\varepsilon > 0$, $(p, A - \varepsilon I) \in D^{2,-}u \Box v(x)$ and by the previous case, we have $(p, (A - \varepsilon I) \boxminus B) \in D^{2,-}u(\bar{y})$. Passing to the limit as $\varepsilon \to 0$ permits to get the result. The proof is now complete. □

Let us state without proof the proposition associated with sup-convolutions.

Proposition 11.

(1) *Consider two lower semicontinuous functions $u, v : \Omega \to \mathbb{R}$ respectively bounded from above and below and consider their sup-convolution*:

$$u \boxminus v(x) = \sup_{y \in \mathbb{R}^n} \{u(y) - v(x-y)\}.$$

Suppose that this supremum is attained at $\bar{y}$ and consider $(p, A) \in D^{2,+}u \Box v(x)$. Then

$$(p, A) \in D^{2,+}u(\bar{y}) \cap \big[-D^{2,-}v(x-\bar{y})\big];$$

(2) *If $v(x) = \frac{1}{2}Bx \cdot x$, then $p = -B(x-\bar{y})$ and $A + B \geqslant 0$ and $(p, A \Box B) \in \overline{D}^{2,+}u(\bar{y})$;*

(3) *If v is a C^2 function and $(p, A) \in \bar{D}^{2,-}u \Box v(x)$ then $p = -\nabla v(x-\bar{y})$ and $A + D^2v(x-\bar{y}) \geqslant 0$ and $(p, A \Box D^2v(x-\bar{y})) \in \overline{D}^{2,+}u(\bar{y})$.*

Remark 6. Lemma 2.14 of [11] corresponds to the case $B = \varepsilon \odot I$.

References

[1] O. Alvarez, J.-M. Lasry, P.-L. Lions, Convex viscosity solutions and state constraints, J. Math. Pures Appl. (9) 76 (3) (1997) 265–288.

[2] J. Benoist, J.-B. Hiriart-Urruty, What is the subdifferential of the closed convex hull of a function?, SIAM J. Math. Anal. 27 (6) (1996) 1661–1679.

[3] L. Caffarelli, M.G. Crandall, M. Kocan, A. Swięch, On viscosity solutions of fully nonlinear equations with measurable ingredients, Comm. Pure Appl. Math. 49 (4) (1996) 365–397.

[4] L.A. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, Ann. of Math. (2) 130 (1) (1989) 189–213.

[5] L.A. Caffarelli, X. Cabré, Fully Nonlinear Elliptic Equations, Amer. Math. Soc. Colloq. Publ., vol. 43, Amer. Math. Soc., Providence, RI, 1995.

[6] M.G. Crandall, M. Kocan, P. Soravia, A. Święch, On the equivalence of various weak notions of solutions of elliptic PDEs with measurable ingredients, in: Progress in Elliptic and Parabolic Partial Differential Equations (Capri, 1994), in: Pitman Res. Notes Math. Ser., vol. 350, Longman, Harlow, 1996, pp. 136–162.

[7] M.G. Crandall, H. Ishii, P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.) 27 (1) (1992) 1–67.

[8] M.G. Crandall, P.-L. Lions, Quadratic growth of solutions of fully nonlinear second order equations in $\mathbf{R}^n$, Differential Integral Equations 3 (4) (1990) 601–616.

[9] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
[10] H. Ishii, P.-L. Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Differential Equations 83 (1) (1990) 26–78.
[11] R. Jensen, Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations, Indiana Univ. Math. J. 38 (3) (1989) 629–667.
[12] A.U. Kennington, Power concavity and boundary value problems, Indiana Univ. Math. J. 34 (3) (1985) 687–704.
[13] N.J. Korevaar, Convex solutions to nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J. 32 (4) (1983) 603–614.
[14] R.T. Rockafellar, Convex Analysis, Princeton Math. Ser., vol. 28, Princeton Univ. Press, Princeton, NJ, 1970.
[15] H.M. Soner, Optimal control with state-space constraint. I, SIAM J. Control Optim. 24 (3) (1986) 552–561.
[16] N.S. Trudinger, Comparison principles and pointwise estimates for viscosity solutions of nonlinear elliptic equations, Rev. Mat. Iberoamericana 4 (3–4) (1988) 453–468.