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Lipschitz regularity of solutions for mixed integro-differential equations

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A B S T R A C T

We establish *new Hölder and Lipschitz* estimates for viscosity solutions of a large class of elliptic and parabolic nonlinear integro-differential equations, by the classical Ishii–Lions's method. We thus extend the Hölder regularity results recently obtained by Barles, Chasseigne and Imbert (2011). In addition, we deal with a new class of nonlocal equations that we term *mixed integro-differential equations*. These equations are particularly interesting, as they are degenerate both in the local and nonlocal term, but their overall behavior is driven by the local–nonlocal interaction, e.g. the fractional diffusion may give the ellipticity in one direction and the classical diffusion in the complementary one.

1. Introduction

Recently regularity results for integro-differential equations have been investigated by many authors: we provide below some references but the list is by no means complete. In particular, Hölder estimates for viscosity solutions of a large class of elliptic and parabolic nonlinear integro-differential equations are obtained in [1], by the classical Ishii–Lions's method.

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The aim of this article is twofold: on one hand, we extend these results to provide Lipschitz estimates in a similar framework and, on the other hand, we deal with a new class of nonlocal equations that we call *mixed integro-differential equations* for which we also give complementary Hölder estimates. The simplest example of such mixed integro-differential equations is given by

$$-\Delta_{x_1} u + (-\Delta_{x_2})^{\beta/2} u = f(x_1, x_2) \tag{1}$$

where $x_1 \in \mathbb{R}^{d_1}$, $x_2 \in \mathbb{R}^{d_2}$, and $(-\Delta_{x_2})^{\beta/2} u$ denotes the fractional Laplacian with respect to the x_2-variables

$$(-\Delta_{x_2})^{\beta/2} u = -\int_{\mathbb{R}^{d_2}} \big(u(x_1, x_2 + z_2) - u(x_1, x_2) - D_{x_2} u(x_1, x_2) \cdot z_2 1_{B^{d_2}}(z_2)\big) \frac{dz_2}{|z_2|^{d_2+\beta}}$$

where B^{d_2} is the unit ball in $\mathbb{R}^{d_2}$. In this case local diffusions occur only in the x_1-directions and fractional diffusions in the x_2-directions.

To be more specific about our approach, we first recall that Ishii and Lions introduced in [11] a simple method to prove $C^{0,\alpha}$ $(0 < \alpha \leqslant 1)$ regularity of viscosity solutions of fully nonlinear, possibly degenerate, elliptic partial differential equations, which has the double advantage of providing *explicit* $C^{0,\alpha}$ *estimates* combined with a *light localization procedure*.

This simple method, closely related to classical viscosity solutions theory, was recently explored by the first, second and fourth authors in [1] for *second order, fully nonlinear elliptic partial integro-differential equations*, dealing with a large class of integro-differential operators, whose singular measures depend on x. They prove that the solution is α-Hölder continuous for any $\alpha < \min(\beta, 1)$, where β characterizes the singularity of the measure associated with the integral operator. However, in the case $\beta \geqslant 1$ the respective ad litteram estimates do not yield Lipschitz regularity.

In order to treat a large class of nonlinear equations, the authors of [1] assume the nonlinearity satisfies a suitable ellipticity growth assumption. Roughly speaking, this assumption gives a suitable meaning to a generalized ellipticity of the equation in the sense that at each point of the domain, the ellipticity comes either from the second order term (the equation is strictly elliptic in the classical fully nonlinear sense), or from the nonlocal term (the equation is strictly elliptic in a nonlocal nonlinear sense).

In a recent study of the strong maximum principle for integro-differential equation [8], the third author introduced another type of *mixed ellipticity*: at each point, the nonlinearity may be *degenerate in the second order term, and in the nonlocal term*, but *the combination of the local and the nonlocal diffusions renders the nonlinearity uniformly elliptic*. Eq. (1) is the typical example of such *mixed integro-differential equations* since the diffusion term gives the ellipticity in certain directions, whereas it is given by the nonlocal term in the complementary directions. For this type of nondegenerate equations, the assumptions in [1] are not satisfied.

1.1. Main results

Using Ishii–Lions's viscosity method, we give both *Hölder and Lipschitz regularity results* of viscosity solutions for a *general class of mixed elliptic integro-differential equations* of the type

$$\begin{aligned} &F_0\big(u(x), Du, D^2u, \mathcal{I}[x,u]\big) + F_1\big(x_1, D_{x_1}u, D^2_{x_1x_1}u, \mathcal{I}_{x_1}[x,u]\big) \\ &\quad + F_2\big(x_2, D_{x_2}u, D^2_{x_2x_2}u, \mathcal{I}_{x_2}[x,u]\big) = f(x) \end{aligned} \tag{2}$$

as well as evolution equations

$$\begin{aligned} &u_t + F_0\big(u(x), Du, D^2u, \mathcal{I}[x,u]\big) + F_1\big(x_1, D_{x_1}u, D^2_{x_1x_1}u, \mathcal{I}_{x_1}[x,u]\big) \\ &\quad + F_2\big(x_2, D_{x_2}u, D^2_{x_2x_2}u, \mathcal{I}_{x_2}[x,u]\big) = f(x). \end{aligned} \tag{3}$$

A point in $x \in \mathbb{R}^d$ is written as $x = (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, with $d = d_1 + d_2$. The symbols u_t, Du, D^2u stand for the derivative with respect to time, respectively the gradient and the Hessian matrix with respect to x. Subsequently, we write the gradient on components as $Du = (D_{x_1}u, D_{x_2}u)$ and the Hessian matrix $D^2u \in \mathbb{S}^d$ (with $\mathbb{S}^d$ the set of real symmetric $d \times d$ matrices) as a block matrix of the form

$$D^2u = \begin{bmatrix} D^2_{x_1x_1}u & D^2_{x_1x_2}u \\ D^2_{x_2x_1}u & D^2_{x_2x_2}u \end{bmatrix}.$$

$\mathcal{I}[x,u]$ is an integro-differential operator, taken on the whole space $\mathbb{R}^d$, associated to Lévy processes

$$\mathcal{I}[x,u] = \int_{\mathbb{R}^d} \big(u(x+z) - u(x) - Du(x)\cdot z1_B(z)\big)\,\mu_x(dz)$$

where $1_B(z)$ denotes the indicator function of the unit ball B and $(\mu_x)_{x\in\mathbb{R}^d}$ is a family of Lévy measures, i.e. nonnegative, possibly singular, Borel measures on $\mathbb{R}^d$ such that

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \min\big(|z|^2, 1\big)\,\mu_x(dz) < \infty.$$

Accordingly, one has the directional integro-differential operators

$$\mathcal{I}_{x_1}[x,u] = \int_{\mathbb{R}^{d_1}} \big(u(x_1+z, x_2) - u(x_1,x_2) - D_{x_1}u(x)\cdot z1_{B^{d_1}}(z)\big)\,\mu^1_{x_1}(dz),$$

$$\mathcal{I}_{x_2}[x,u] = \int_{\mathbb{R}^{d_2}} \big(u(x_1, x_2+z) - u(x_1,x_2) - D_{x_2}u(x)\cdot z1_{B^{d_2}}(z)\big)\,\mu^2_{x_2}(dz),$$

where $(\mu^i_{x_i})_{x_i\in\mathbb{R}^{d_i}}$, $i = 1, 2$, are Lévy measures and $1_{B^{d_i}}$ is the indicator function of the unit ball B^{d_i} in $\mathbb{R}^{d_i}$. We consider as well the special class of Lévy–Itô operators, defined as follows

$$\mathcal{J}[x,u] = \int_{\mathbb{R}^d} \big(u\big(x + j(x,z)\big) - u(x) - Du(x)\cdot j(x,z)1_B(z)\big)\,\mu(dz)$$

where μ is a Lévy measure and $j(x,z)$ is the size of the jumps at x satisfying

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \min\big(\big|j(x,z)\big|^2, 1\big)\,\mu(dz) < \infty.$$

Similarly, we deal with directional Lévy–Itô integro-differential operators

$$\mathcal{J}_{x_1}[x,u] = \int_{\mathbb{R}^{d_1}} \big(u\big(x_1 + j(x_1,z), x_2\big) - u(x_1,x_2) - D_{x_1}u(x)\cdot j(x_1,z)1_{B^{d_1}}(z)\big)\,\mu^1(dz),$$

$$\mathcal{J}_{x_2}[x,u] = \int_{\mathbb{R}^{d_2}} \big(u\big(x_1, x_2 + j(x_2,z)\big) - u(x_1,x_2) - D_{x_2}u(x)\cdot j(x_2,z)1_{B^{d_2}}(z)\big)\,\mu^2(dz).$$

We assume the nonlinearities are continuous and *degenerate elliptic*, i.e.

$$F_i(\dots, X, l) \leqslant F_i(\dots, Y, l') \quad \text{if } X \geqslant Y,\ l \geqslant l',$$

for all $X, Y \in \mathbb{S}^{d_i}$ and $l, l' \in \mathbb{R}$, $i = 0, 1, 2$.

In addition, we suppose that the three nonlinearities satisfy suitable strict ellipticity and growth conditions, that we omit here for the sake of simplicity, but will be made precise in the following section. These structural growth conditions can be illustrated on the following example:

$$-a_1(x_1)\Delta_{x_1}u - a_2(x_2)\mathcal{I}_{x_2}[x, u] - \mathcal{I}[x, u] + b_1(x_1)|D_{x_1}u_1|^{k_1} + b_2(x_2)|D_{x_2}u|^{k_2} + |Du|^n + cu = f(x)$$

where the nonlocal term $\mathcal{I}_{x_2}[x, u]$ has fractional exponent $\beta \in (0, 2)$ and $a_i(x_i) > 0$, for $i = 1, 2$. Thus

$$F_0\big(u(x), Du, D^2u, \mathcal{I}[x, u]\big) = -\mathcal{I}[x, u] + |Du|^n + cu,$$
$$F_1\big(x_1, D_{x_1}u, D^2_{x_1x_1}u, \mathcal{J}_{x_1}[x, u]\big) = -a_1(x_1)\Delta_{x_1}u + b_1(x_1)|D_{x_1}u_1|^{k_1},$$
$$F_2\big(x_2, D_{x_2}u, D^2_{x_2x_2}u, \mathcal{J}_{x_2}[x, u]\big) = -a_2(x_2)\mathcal{I}_{x_2}[x, u] + b_2(x_2)|D_{x_2}u|^{k_2}.$$

When $\beta > 1$, we show that the solution is Lipschitz continuous for mixed equations with gradient terms $b_i(x_i)|D_{x_i}u|^{k_i}$ having a natural growth $k_i \leqslant \beta$ if b_i bounded. If in addition b_i are τ-Hölder continuous, then the solution remains Lipschitz for gradient terms with natural growth $k_i \leqslant \tau + \beta$. When $\beta \leqslant 1$, the solution is α-Hölder continuous for any $\alpha < \beta$. The critical case $\beta = 1$ is left open.

1.2. Known results

The classical theory for second order, uniformly elliptic integro-differential equations includes a priori estimates, weak and strong maximum principles, etc. In particular, existence and uniqueness results have been extended from elliptic partial differential equations to elliptic integro-differential equations. For results in the framework of Green functions and classical solutions we send the reader to the up-to-date book of Garroni and Menaldi [9] and the references therein.

More recently there have been many papers dealing with $C^{0,\alpha}$ estimates and regularity of solutions (not necessarily in the viscosity setting) for fully nonlinear integro-differential equations and the literature has been considerably enriched. It is not possible to give an exhaustive list of references but we next try to give the flavor of the known results.

In the framework of potential theory (hence linear equations), Bass and Levin first establish Harnack inequalities [3]. Then Kassmann [12,13] adapted the de Giorgi theory to nonlocal operators. In the same spirit, Silvestre gave in [21] an analytical proof of Hölder continuity for harmonic functions with respect to the integral operator.

In the setting of *viscosity solutions*, there are essentially two approaches for proving Hölder or Lipschitz regularity: either by the Ishii–Lions's method or by ABP estimates and Krylov–Safonov and Harnack type inequalities. These methods do not cover the same class of equations, they have different aims and each of them has its own advantages.

The powerful Harnack approach was first introduced by Krylov and Safonov [15,16] for linear equations under non-divergence form and then adapted to fully nonlinear elliptic equations by Trudinger [22] and Caffarelli [5]. This theory applies to *uniformly elliptic, fully nonlinear equations*, with *rough coefficients*. The existing theory for second order elliptic equations has been extended to integro-differential equations by Caffarelli and Silvestre in [4]. Both for local and nonlocal equations, this theory leads to further regularity such as $C^{1,\alpha}$. But as far as nonlocal equations are concerned, it requires in particular some integrability condition of the measure at infinity.

On the contrary, direct viscosity methods apply under weaker ellipticity assumptions but require Hölder continuous coefficients and do not seem to yield further regularity. Finally these methods allow measures which are only bounded at infinity.

Very recently, Cardaliaguet and Rainer showed Hölder regularity of viscosity solutions for nonlocal Hamilton–Jacobi equations with superquadratic gradient growth [7], using probabilistic representation formulas.

We would like to conclude this introduction by mentioning that this work was motivated by the study of long time behavior of periodic viscosity solutions for integro-differential equations, that we are considering in a companion paper. We point out that long time behavior comes to the resolution of the stationary ergodic problem, which is basically the cell problem in homogenization. The periodic homogenization for nonlinear integro-differential equations has been addressed by Schwab in [18]. However, it is restricted to a certain family of equations, due to a lack of fine ABP estimate. Recently, Schwab and Guillen provided [10] and ABP estimate that would help solve the homogenization for a wider class of nonlinearities.

The paper is organized as follows. In Section 2 we give the appropriate definition of viscosity solution, make precise the ellipticity growth conditions to be satisfied by the nonlinearities and list the assumptions on the nonlocal terms. Section 3 is devoted to the main results, which for the sake of clarity are given in the periodic setting. We state partial regularity results, provide the complete proof, and then present the global regularity result. In the next Section 4 we consider several significant examples and discuss the main assumptions required by the regularity results and their implications. Extensions to the non-periodic setting, parabolic versions of the equations, Bellman–Isaacs equations and multiple nonlinearities are recounted in Section 5. At last we detail in Section 6 the technical Lipschitz and Hölder estimates for the general nonlocal operators and Lévy–Itô operators, which are essentially the backbone of the main results.

2. Notations and assumptions

2.1. Viscosity solutions for integro-differential equations

To overcome the difficulties imposed by behavior at infinity of the measures $(\mu_x)_x$, as well as the singularity at the origin, we often need to split the nonlocal terms into

$$\mathcal{I}_\delta^1[x,u] = \int\limits_{|z|\leqslant\delta} \big(u(x+z)-u(x)-Du(x)\cdot z1_B(z)\big)\,\mu_x(dz),$$

$$\mathcal{I}_\delta^2[x,p,u] = \int\limits_{|z|>\delta} \big(u(x+z)-u(x)-p\cdot z1_B(z)\big)\,\mu_x(dz),$$

respectively, in the case of Lévy–Itô operators,

$$\mathcal{J}_\delta^1[x,u] = \int\limits_{|z|\leqslant\delta} \big(u\big(x+j(x,z)\big)-u(x)-Du(x)\cdot j(x,z)1_B(z)\big)\,\mu(dz),$$

$$\mathcal{J}_\delta^2[x,p,u] = \int\limits_{|z|>\delta} \big(u\big(x+j(x,z)\big)-u(x)-p\cdot j(x,z)1_B(z)\big)\,\mu(dz)$$

with $0<\delta<1$ and $p\in\mathbb{R}^d$.

One of the very first definitions of viscosity solutions for integro-differential equations was introduced by Sayah in [17]. In particular, for mixed integro-differential equations, the definition can be stated as follows.

Definition 1 *(Viscosity solutions).* An upper semi-continuous (in short usc) function $u:\mathbb{R}^d\to\mathbb{R}$ is a *sub-solution* of (2) if for any $\phi\in C^2(\mathbb{R}^d)$ such that $u-\phi$ attains a global maximum at $x\in\mathbb{R}^d$

$$
\begin{aligned}
&F_0\big(u(x), D\phi(x), D^2\phi(x), \mathcal{I}^1_\delta[x,t,\phi] + \mathcal{I}^2_\delta\big[x,t,D\phi(x,t),u\big]\big)\\
&\quad + F_1\big(x_1, D_{x_1}\phi(x), D^2_{x_1x_1}\phi(x), \mathcal{I}^1_{x_1,\delta}[x,t,\phi] + \mathcal{I}^2_{x_1,\delta}\big[x,t,D\phi(x,t),u\big]\big)\\
&\quad + F_2\big(x_2, D_{x_2}\phi(x), D^2_{x_2x_2}\phi(x), \mathcal{I}^1_{x_2,\delta}[x,t,\phi] + \mathcal{I}^2_{x_1,\delta}\big[x,t,D\phi(x,t),u\big]\big) \leqslant f(x).
\end{aligned}
$$

A lower semi-continuous (in short lsc) function $u:\mathbb{R}^d \to \mathbb{R}$ is a *sub-solution* of (2) if for any $\phi \in C^2(\mathbb{R}^d)$ such that $u-\phi$ attains a global minimum at $x \in \mathbb{R}^d$

$$
\begin{aligned}
&F_0\big(u(x), D\phi(x), D^2\phi(x), \mathcal{I}^1_\delta[x,t,\phi] + \mathcal{I}^2_\delta\big[x,t,D\phi(x,t),u\big]\big)\\
&\quad + F_1\big(x_1, D_{x_1}\phi(x), D^2_{x_1x_1}\phi(x), \mathcal{I}^1_{x_1,\delta}[x,t,\phi] + \mathcal{I}^2_{x_1,\delta}\big[x,t,D\phi(x,t),u\big]\big)\\
&\quad + F_2\big(x_2, D_{x_2}\phi(x), D^2_{x_2x_2}\phi(x), \mathcal{I}^1_{x_2,\delta}[x,t,\phi] + \mathcal{I}^2_{x_1,\delta}\big[x,t,D\phi(x,t),u\big]\big) \geqslant f(x).
\end{aligned}
$$

However, there are several equivalent definitions of viscosity solutions. Throughout this paper, we use the definition involving sub- and super-jets, which was shown in [2] to be equivalent with Definition 1. One just has to replace in the viscosity inequalities the derivatives of the test function $(D\phi, D^2\phi)$ with semi-jets (p, X). To avoid technical details due to partial derivatives with respect to x_1 and x_2 we omit it here, and just recall the notions of semi-jets.

If $u:\mathbb{R}^d \to \mathbb{R}$ and $v:\mathbb{R}^d \to \mathbb{R}$ are respectively a lsc and an usc function, we denote by $\mathcal{D}^{2,-}u(x)$ the sub-jet of u at $x \in \mathbb{R}^d$ and by $\mathcal{D}^{2,+}v(x)$ the super-jet of v at $x \in \mathbb{R}^d$. We recall that they are given by

$$
\mathcal{D}^{2,-}u(x) = \left\{(p,X) \in \mathbb{R}^d \times \mathbb{S}^d;\ u(x+z) \geqslant u(x) + p\cdot z + \frac{1}{2}Xz\cdot z + o\big(|z|^2\big)\right\},
$$

$$
\mathcal{D}^{2,+}v(x) = \left\{(p,X) \in \mathbb{R}^d \times \mathbb{S}^d;\ u(x+z) \leqslant u(x) + p\cdot z + \frac{1}{2}Xz\cdot z + o\big(|z|^2\big)\right\}.
$$

2.2. Ellipticity growth conditions

We assume that the nonlinearities F_i, with $i = 0,1,2$, satisfy (one or more of) the next assumptions. In the sequel of this subsection, the notation F stands for any of the nonlinearities F_i. The precise selection for each of the nonlinearities shall be given later on, when the regularity result is stated. Further examples and comments upon the restrictions of these nonlinearities are provided in Section 4. In the sequel of this subsection, the notation F stands for any of the nonlinearities F_i.

$(H0)$ There exists $\tilde\gamma \in \mathbb{R}$ such that for any $u, v \in \mathbb{R}$, $p \in \mathbb{R}^{\tilde d}$, $X \in \mathbb{S}^{\tilde d}$ and $l \in \mathbb{R}$

$$
F(u,p,X,l) - F(v,p,X,l) \geqslant \tilde\gamma(u-v) \quad \text{when } u \geqslant v.
$$

$(H1)$ There exist two functions $\Lambda_1, \Lambda_2 : \mathbb{R}^{\tilde d} \to [0,\infty)$ such that $\Lambda_1(x) + \Lambda_1(x) \geqslant \Lambda_0 > 0$ and some constants $k \geqslant 0$, $\tau \in (0,1]$, $\theta, \tilde\theta \in (0,1]$ such that for any $x, y \in \mathbb{R}^{\tilde d}$, $p \in \mathbb{R}^{\tilde d}$, $l \leqslant l'$ and any $\varepsilon > 0$

$$
\begin{aligned}
F\big(y,p,Y,l'\big) - F(x,p,X,l) \leqslant{}& \Lambda_1(x)\left(\big(l-l'\big) + \frac{|x-y|^{2\theta}}{\varepsilon} + |x-y|^\tau|p|^{k+\tau} + C_1|p|^k\right)\\
&+ \Lambda_2(x)\left(\operatorname{tr}(X-Y) + \frac{|x-y|^{2\tilde\theta}}{\varepsilon} + |x-y|^\tau|p|^{2+\tau} + C_2|p|^2\right)
\end{aligned}
$$

if $X, Y \in \mathbb{S}^{\tilde d}$ satisfy the inequality

$$-\frac{1}{\varepsilon}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leqslant \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqslant \frac{1}{\varepsilon}\begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix}, \tag{4}$$

with $Z = I - \omega \hat{a} \otimes \hat{a}$, for some unit vector $\hat{a} \in \mathbb{R}^{\tilde{d}}$, and $\omega \in (1,2)$.

$(H2)$ $F(\cdot, l)$ is Lipschitz continuous, uniformly with respect to all the other variables.

$(H3)$ There exists a modulus of continuity ω_F such that for any $\varepsilon > 0$

$$F\left(y, \frac{x-y}{\varepsilon}, Y, l\right) - F\left(x, \frac{x-y}{\varepsilon}, X, l\right) \leqslant \omega_F\left(\frac{|x-y|^2}{\varepsilon} + |x-y|\right)$$

for all $x, y \in \mathbb{R}^{\tilde{d}}$, $X, Y \in \mathbb{S}^{\tilde{d}}$ satisfying the matrix inequality (4) with $Z = I$ and $l \in \mathbb{R}$.

2.3. Lévy measures for general nonlocal operators

We recall that in this case, the nonlocal term $\mathcal{I}[x,u]$ is an integro-differential operator defined by

$$\mathcal{I}[x,u] = \int_{\mathbb{R}^{\tilde{d}}} \big(u(x+z) - u(x) - Du(x) \cdot z 1_B(z)\big)\, \mu_x(dz) \tag{5}$$

where 1_B denotes the indicator function of the unit ball and $(\mu_x)_x$ is a family of Lévy measures. We need to make a series of assumptions for the family of Lévy measures that we make precise now.

$(M1)$ There exists a constant $\tilde{C}_\mu > 0$ such that

$$\sup_{x \in \mathbb{R}^{\tilde{d}}} \left(\int_B |z|^2\, \mu_x(dz) + \int_{\mathbb{R}^{\tilde{d}} \setminus B} \mu_x(dz) \right) \leqslant \tilde{C}_\mu.$$

$(M2)$ There exists $\beta \in (0,2)$ such that for every $a \in \mathbb{R}^{\tilde{d}}$ there exist $0 < \eta < 1$ and a constant $C_\mu > 0$ such that the following holds for any $x \in \mathbb{R}^{\tilde{d}}$

$$\forall \delta > 0 \quad \int_{\mathcal{C}_{\eta,\delta}(a)} |z|^2\, \mu_x(dz) \geqslant C_\mu \eta^{\frac{\tilde{d}-1}{2}} \delta^{2-\beta}$$

with $\mathcal{C}_{\eta,\delta}(a) := \{z \in B_\delta;\ (1-\eta)|z||a| \leqslant |a \cdot z|\}$.

$(M3)$ There exist $\beta \in (0,2)$, $\gamma \in (0,1)$ and a constant $C_\mu > 0$ such that for any $x, y \in \mathbb{R}^{\tilde{d}}$ and all $\delta > 0$

$$\int_{B_\delta} |z|^2\, |\mu_x - \mu_y|(dz) \leqslant C_\mu |x-y|^\gamma \delta^{2-\beta}$$

and

$$\int_{B \setminus B_\delta} |z|\, |\mu_x - \mu_y|(dz) \leqslant \begin{cases} C_\mu |x-y|^\gamma \delta^{1-\beta} & \text{if } \beta \neq 1, \\ C_\mu |x-y|^\gamma |\ln \delta| & \text{if } \beta = 1. \end{cases}$$

At the same time, we assume that the directional Lévy measures satisfy similar assumptions.

Example 1. To make precise the form of $(M2)$ we consider the fractional Laplacian with exponent β and compute in $\mathbb{R}^2$

$$\int_{\mathcal{C}_{\eta,\delta}(a)} |z|^2 \frac{dz}{|z|^{2+\beta}} = \frac{\mathrm{vol}(\mathcal{C}_{\eta,\delta}(a))}{\mathrm{vol}(B_\delta)} \int_{B_\delta} |z|^2 \frac{dz}{|z|^{2+\beta}} = \frac{\mathrm{vol}(\mathcal{C}_{\eta,1}(a))}{\mathrm{vol}(B_1)} \int_{B_\delta} |z|^2 \frac{dz}{|z|^{2+\beta}}$$
$$= \delta^{2-\beta} \frac{\mathrm{vol}(\mathcal{C}_{\eta,1}(a))}{\mathrm{vol}(B_1)} \int_{B_1} |z|^2 \frac{dz}{|z|^{2+\beta}} = \delta^{2-\beta} \frac{\theta}{\pi} \int_{B_1} |z|^2 \frac{dz}{|z|^{2+\beta}},$$

where θ denotes the angle measuring the aperture of the cone. Taking into account the definition of $\mathcal{C}_{\eta,1}(a)$ we have for small angles θ

$$\eta = 1 - \cos(\theta) = \frac{\theta^2}{2} + o\big(\theta^2\big)$$

and hence $\theta \simeq \sqrt{\eta}$, from where we deduce $(M2)$.

In higher dimension $d \geqslant 3$, the volume of the cone is given in spherical coordinates, with normal direction $a = (0, 0, \dots, 1)$, polar angle $\phi_1 \in [0, \pi]$, and angular coordinates $\phi_2, \dots, \phi_{d-2} \in [0, \pi]$, $\phi_{d-1} \in [0, 2\pi]$, by the formula

$$\mathrm{vol}\big(\mathcal{C}_{\eta,1}(a)\big) = \int_0^\theta \sin^{d-2}(\phi_1)\, d\phi_1 \dots \int_0^\pi \sin(\phi_{d-2})\, d\phi_{d-2} \int_0^{2\pi} d\phi_{d-1} \int_0^1 r^{d-1}\, dr.$$

For small angles θ the volume can be approximated by

$$\mathrm{vol}\big(\mathcal{C}_{\eta,1}(a)\big) \approx \frac{\theta^{d-1}}{d-1} \int_0^\pi \sin^{d-3}(\phi_2)\, d\phi_2 \dots \int_0^\pi \sin(\phi_{d-2})\, d\phi_{d-2} \int_0^{2\pi} d\phi_{d-1} \int_0^1 r^{d-1}\, dr.$$

Therefore there exists a positive constant $C > 0$ such that

$$\frac{\mathrm{vol}(\mathcal{C}_{\eta,1}(a))}{\mathrm{vol}(B_1)} \geqslant C\theta^{d-1} = C\eta^{\frac{d-1}{2}}$$

and hence, denoting by $C_\mu = C \int_{B_1} |z|^2 \frac{dz}{|z|^{2+\beta}}$, (M_2) is satisfied

$$\int_{\mathcal{C}_{\eta,\delta}(a)} |z|^2 \frac{dz}{|z|^{2+\beta}} \geqslant C\eta^{\frac{d-1}{2}} \delta^{2-\beta} \int_{B_1} |z|^2 \frac{dz}{|z|^{2+\beta}} = C_\mu \eta^{\frac{d-1}{2}} \delta^{2-\beta}.$$

2.4. Lévy measures for Lévy–Itô operators

Lévy–Itô operators are defined by

$$\mathcal{J}[x, u] = \int_{\mathbb{R}^{\tilde{d}}} \Big(u\big(x + j(x, z)\big) - u(x) - Du(x) \cdot j(x, z) 1_B(z)\Big)\, \mu(dz). \tag{6}$$

In the sequel, we assume that the jump function(s) satisfies the following conditions.

$(J1)$ There exists a constant $\tilde{C}_\mu > 0$ such that for all $x \in \mathbb{R}^{\tilde{d}}$

$$\int_B \left| j(x,z) \right|^2 \mu(dz) + \int_{\mathbb{R}^{\tilde{d}} \setminus B} \mu(dz) \leqslant \tilde{C}_\mu.$$

$(J2)$ There exists $\beta \in (0,2)$ such that for every $a \in \mathbb{R}^{\tilde{d}}$ there exist $0 < \eta < 1$ and a constant $C_\mu > 0$ such that the following holds for any $x \in \mathbb{R}^{\tilde{d}}$

$$\forall \delta > 0 \quad \int_{\mathcal{C}_{\eta,\delta}(a)} \left| j(x,z) \right|^2 \mu(dz) \geqslant C_\mu \eta^{\frac{d-1}{2}} \delta^{2-\beta}$$

with $\mathcal{C}_{\eta,\delta}(a) := \{z;\ |j(x,z)| \leqslant \delta,\ (1-\eta)|j(x,z)||a| \leqslant |a \cdot j(x,z)|\}$.

$(J3)$ There exists $\beta \in (0,2)$ such that for $\delta > 0$ small enough

$$\int_{B \setminus B_\delta} |z| \, \mu(dz) \leqslant \begin{cases} \tilde{C}_\mu \delta^{1-\beta} & \text{if } \beta \neq 1, \\ \tilde{C}_\mu |\ln \delta| & \text{if } \beta = 1. \end{cases}$$

$(J4)$ There exist $\gamma \in (0,1]$ and two constants $c_0, C_0 > 0$ such that for any $x \in \mathbb{R}^{\tilde{d}}$ and $z \in \mathbb{R}^{\tilde{d}}$

$$c_0|z| \leqslant \left| j(x,z) \right| \leqslant C_0|z|$$

and for all $z \in B$ and $x, y \in \mathbb{R}^{\tilde{d}}$

$$\left| j(x,z) - j(y,z) \right| \leqslant C_0|z||x-y|^\gamma.$$

$(J5)$ There exist $\gamma \in (0,1]$ and a constant $\tilde{C}_0 > 0$ such that for all $z \in \mathbb{R}^{\tilde{d}} \setminus B$ and $x, y \in \mathbb{R}^{\tilde{d}}$

$$\left| j(x,z) - j(y,z) \right| \leqslant \tilde{C}_0|x-y|^\gamma.$$

When several assumptions hold simultaneously, the constants denoted similarly are considered to be the same (e.g. β, C_μ, $\tilde{C}_\mu$).

3. Lipschitz continuity of viscosity solutions

In this section we present the main regularity results for mixed integro-differential equations. We deal with *general nonlinearities* derived from the toy model, namely Eq. (1), where the fractional diffusion gives the ellipticity in certain directions and the classical diffusion in the complementary ones. We first establish partial regularity results, namely Hölder and Lipschitz regularity of the solution with respect to the x_1-variables. This is because of the lack of complete local or nonlocal diffusion. We then derive the global regularity of the solution.

For the sake of simplicity, we give the statements and proofs in the periodic setting. This yields $C^{0,\alpha}$ regularity instead of local regularity. At the same time it allows us to avoid the localization terms, meant to overcome the behavior at infinity of the solutions, which is related to the integrability of the singular measure away from the origin.

3.1. Partial regularity results

We first give partial regularity estimates, in which case we use classical regularity arguments in one set of variables, and uniqueness type arguments in the other variables. Regularity arguments apply for both general nonlocal operators and Lévy–Itô operators. However, uniqueness applies only for the latter. Consequently, we state two results: one for equations that mix general nonlocal operators with Lévy–Itô ones, and another one for equations dealing only with Lévy–Itô operators.

Theorem 2 *(Partial regularity for periodic, mixed PIDEs – general nonlocal operators). Let f be a continuous, periodic function. Assume the nonlinearities F_i, $i = 0, 1, 2$, are degenerate elliptic and that they satisfy the following:*

- *F_0 is $\mathbb{Z}^d$-periodic and satisfies assumptions $(H0)$, $(H2)$ with $\tilde{d} = d$ and some constant $\tilde{\gamma}$;*
- *F_1 is $\mathbb{Z}^{d_1}$-periodic and satisfies $(H1)$ with $\tilde{d} = d_1$, for some functions Λ_1, Λ_2 and some parameters Λ_0, $k \geqslant 0$, $\tau, \theta, \tilde{\theta} \in (0, 1]$;*
- *F_2 is $\mathbb{Z}^{d_2}$-periodic and satisfies $(H2)$, $(H3)$ with $\tilde{d} = d_2$.*

Let μ^0, $(\mu^1_{x_1})_{x_1}$ and μ^2 be Lévy measures on $\mathbb{R}^d$, $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$ respectively associated to the integro-differential operators $\mathcal{I}[x, u]$, $\mathcal{I}_{x_1}[x, u]$ and $\mathcal{J}_{x_2}[x, u]$. Suppose

- *$(\mu^1_{x_1})_{x_1}$ satisfies $(M1)$–$(M3)$ for some C_{μ^1}, $\tilde{C}_{\mu^1}$, β and γ, with $\begin{cases} k \leqslant \beta, & \beta > 1, \\ k < \beta, & \beta \leqslant 1; \end{cases}$*
- *the jump function $j(x_2, z)$ satisfies $(J1)$, $(J4)$ and $(J5)$ for some C_{μ^2}, $\tilde{C}_{\mu^2}$, and $\gamma = 1$.*

Then any periodic continuous viscosity solution u of

$$\begin{aligned} &F_0\big(u(x), Du, D^2u, \mathcal{I}[x, u]\big) + F_1\big(x_1, D_{x_1}u, D^2_{x_1x_1}u, \mathcal{I}_{x_1}[x, u]\big) \\ &\quad + F_2\big(x_2, D_{x_2}u, D^2_{x_2x_2}u, \mathcal{J}_{x_2}[x, u]\big) = f(x) \end{aligned} \tag{7}$$

(a) *is Lipschitz continuous in the x_1 variable if $\beta > 1$;*
(b) *is $C^{0,\alpha}$ continuous in the x_1 variable with $\alpha < \frac{\beta-k}{1-k}$, if $\beta \leqslant 1$.*

The Lipschitz/Hölder constant L depends on $\|u\|_\infty$, the dimension of the space d, the constants associated to the Lévy measures as well as the constants required by the growth condition $(H1)$.

Remark 1. In particular, when $d_1 = d$ and $F_0 \equiv 0$, $F_2 \equiv 0$ we extend to Lipschitz the Hölder regularity result, recently obtained by Barles, Chasseigne and Imbert in [1].

Remark 2. When $k = \beta = 1$, the solution is α-Hölder continuous, with α small enough. Unfortunately in this case we cannot characterize the Hölder exponent α.

Remark 3. When $\beta < 1$, if $C_1 = 0$ in $(H1)$ and $\beta(k + \tau) > k$, then the solution is exactly $C^{0,\beta}$.

Since the concave estimates for Lévy–Itô operators are of the same order as those for general nonlocal operators, similar regularity results hold. Namely, we have the following.

Theorem 3 *(Partial regularity for periodic, mixed PIDEs – Lévy–Itô operators). Let f and F_i, $i = 0, 1, 2$, satisfy the same assumptions as in Theorem 2. Let μ^0, μ^1 and μ^2 be Lévy measures on $\mathbb{R}^d$, $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively associated to the integro-differential operators $\mathcal{I}[x, u]$, $\mathcal{J}_{x_1}[x, u]$ and $\mathcal{J}_{x_2}[x, u]$. Suppose*

– *the jump function* $j^1(x_1,z)$ *satisfies assumptions* $(J1)$–$(J4)$, *for some parameters* β, C_{μ^1}, $\tilde{C}_{\mu^1}$, *and* $\gamma \in (1-\beta/2, 1]$, *and in addition* $\begin{cases} k \leqslant \beta, & \beta > 1, \\ k < \beta, & \beta \leqslant 1; \end{cases}$

– *the jump function* $j^2(x_2,z)$ *satisfies* $(J1)$, $(J4)$ *and* $(J5)$ *for some* C_{μ^2}, $\tilde{C}_{\mu^2}$, *and* $\gamma = 1$.

Then any periodic continuous viscosity solution u of

$$F_0\big(u(x), Du, D^2u, \mathcal{I}[x,u]\big) + F_1\big(x_1, D_{x_1}u, D^2_{x_1x_1}u, \mathcal{J}_{x_1}[x,u]\big) + F_2\big(x_2, D_{x_2}u, D^2_{x_2x_2}u, \mathcal{J}_{x_2}[x,u]\big) = f(x) \tag{8}$$

(a) *is Lipschitz continuous in the* x_1 *variable, if* $\beta > 1$;
(b) *is* $C^{0,\alpha}$ *continuous in the* x_1 *variable with* $\alpha < \frac{\beta-k}{1-k}$, *if* $\beta \leqslant 1$.

The Lipschitz/Hölder constant L depends on $\|u\|_\infty$, *the dimension d of the space, the constants associated to the Lévy measures as well as the constants required by the growth condition* $(H1)$.

Remark 4. In order to establish Lipschitz or Hölder regularity results for the solution u, we shift the function and show that the corresponding difference can be uniformly controlled by

$$\phi(t) = Lt^\alpha, \quad \text{for all } \alpha \in (0,1].$$

Roughly speaking, one has to look at the maximum of the function

$$(x,y) \mapsto u(x) - u(y) - \phi\big(|x-y|\big)$$

(see Fig. 1) and, *in the case of elliptic PDEs*, follow the uniqueness proof with a careful analysis of the matrix inequality given by Jensen–Ishii's lemma. Precise computations show that we just need ellipticity of the equation in the gradient direction. In the case of nonlocal diffusions, one has to translate in a proper way the ellipticity in the gradient direction. This is reflected in the nondegeneracy conditions $(M2)$ (respectively $(J2)$) required by the family of Lévy measures.

Proof of Theorem 2. The proof of the regularity of u consists of two steps: we first show that the solution u is $C^{0,\alpha}$ continuous for all $\alpha \in (0,1)$, then we check that in the subcritical case $\beta > 1$ this implies the Lipschitz continuity. We use the viscosity method introduced by Ishii and Lions in [11].

Step 1. We introduce the auxiliary function

$$\psi(x_1, y_1, x_2) = u(x_1, x_2) - u(y_1, x_2) - \phi(x_1 - y_1)$$

where ϕ is a radial function of the form

$$\phi(z) = \varphi\big(|z|\big)$$

with a suitable choice of a smooth increasing concave function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(0) = 0$ and $\varphi(t_0) \geqslant 2\|u\|_\infty$ for some $t_0 > 0$. Our aim is to show that for all $x_2 \in \mathbb{R}^{d_2}$

$$\psi(x_1, y_1, x_2) \leqslant 0 \quad \text{if } |x_1 - y_1| < t_0. \tag{9}$$

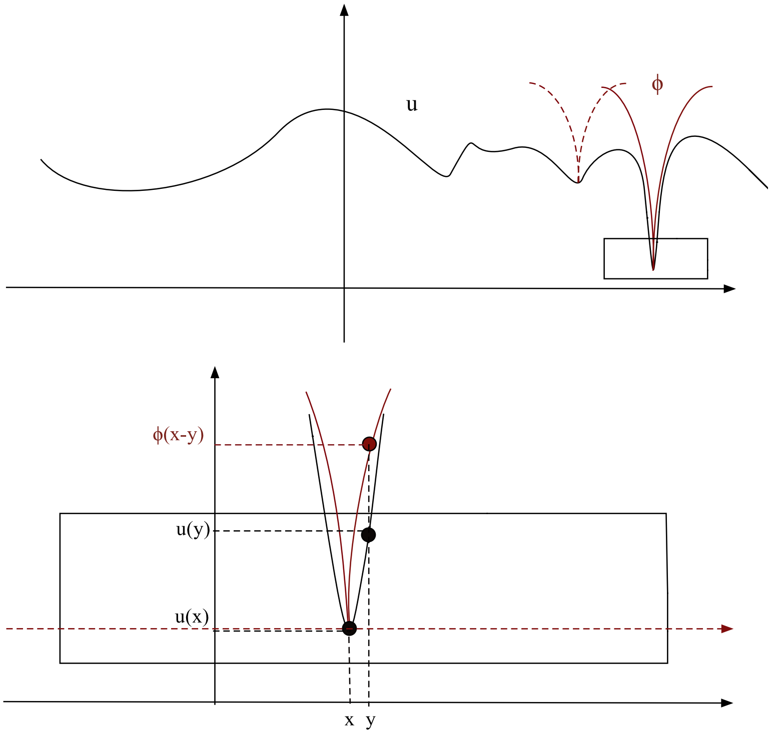

Fig. 1. Uniformly controlling the shift of u by $\phi(|x-y|) = L|x-y|^\alpha$, for all $\alpha \in (0, 1]$.

This yields the desired regularity result, for a proper choice of φ. Namely, $\varphi = Lt^\alpha$ will give the partial Hölder regularity of the solution

$$\left|u(x_1, x_2) - u(y_1, x_2)\right| \leqslant L|x_1 - y_1|^\alpha \quad \text{if } |x_1 - y_1| < t_0$$

and $\varphi = L(t - \rho t^{1+\alpha})$ the partial Lipschitz regularity

$$\left|u(x_1, x_2) - u(y_1, x_2)\right| \leqslant L|x_1 - y_1| \quad \text{if } |x_1 - y_1| < t_0.$$

Step 2. To this end, we argue by contradiction and assume that $\psi(x_1, y_1, x_2)$ has a positive strict maximum at some point $(\bar{x}_1, \bar{y}_1, \bar{x}_2)$ with $|\bar{x}_1 - \bar{y}_1| < t_0$:

$$M = \psi(\bar{x}_1, \bar{y}_1, \bar{x}_2) = \max_{\substack{x_1, y_2 \in \mathbb{R}^{d_1}, x_2 \in \mathbb{R}^{d_2} \\ |x_1 - y_1| < t_0}} \psi(x_1, y_1, x_2) > 0.$$

Denote $\bar{x} = (\bar{x}_1, \bar{x}_2)$ and $\bar{y} = (\bar{y}_1, \bar{x}_2)$. Then

$$\varphi\big(|\bar{x} - \bar{y}|\big) \leqslant u(\bar{x}) - u(\bar{y}) \leqslant \omega_u\big(|\bar{x} - \bar{y}|\big), \tag{10}$$

$$\varphi\big(|\bar{x} - \bar{y}|\big) \leqslant u(\bar{x}) - u(\bar{y}) \leqslant 2\|u\|_\infty. \tag{11}$$

To be able to extract some valuable information hereafter, we need to construct test functions defined on the whole space $\mathbb{R}^d$. For this reason, we penalize ψ around the maximum by doubling the variables, staying at the same time as close as possible to the maximum point. Therefore, we consider the auxiliary function

$$\psi_\varepsilon(x, y) = u(x_1, x_2) - u(y_1, y_2) - \phi(x_1 - y_1) - \frac{|x_2 - y_2|^2}{\varepsilon^2}$$

whose maximum is attained, say at $(x^\varepsilon, y^\varepsilon)$. Denote its maximum value by

$$M^\varepsilon = \psi_\varepsilon\big(x^\varepsilon, y^\varepsilon\big) = \max_{x, y \in \mathbb{R}^d} \psi_\varepsilon(x, y).$$

Then the following holds.

Lemma 4. *There exists $(\bar{x}, \bar{y})$ such that $M = \psi(\bar{x}_1, \bar{y}_1, \bar{x}_2)$ and up to a subsequence, the sequences of maximum points $((x^\varepsilon, y^\varepsilon))_\varepsilon$ and of maximum values $(M^\varepsilon)_\varepsilon$ satisfy as $\varepsilon \to 0$*

$$M^\varepsilon \to M, \qquad \frac{|x_2^\varepsilon - y_2^\varepsilon|^2}{\varepsilon^2} \to 0, \qquad \big(x^\varepsilon, y^\varepsilon\big) \to (\bar{x}, \bar{y}).$$

The proof of this lemma is classical and therefore omitted in this paper.

Step 3. Let $\bar{a} = (\bar{a}_1, \bar{a}_2) = \bar{x} - \bar{y}$, $p = (p_1, p_2) = (D\phi(\bar{a}_1), 0)$ and denote

$$a^\varepsilon = \big(a_1^\varepsilon, a_2^\varepsilon\big) = x^\varepsilon - y^\varepsilon, \qquad \hat{a}^\varepsilon = \frac{a^\varepsilon}{|a^\varepsilon|}, \qquad p^\varepsilon = \big(p_1^\varepsilon, p_2^\varepsilon\big) = \left(D\phi\big(a_1^\varepsilon\big), 2\frac{x_2^\varepsilon - y_2^\varepsilon}{\varepsilon^2}\right).$$

Since $x_1^\varepsilon \neq y_1^\varepsilon$, for ε small enough the function ϕ is smooth and we can apply the Jensen–Ishii's lemma for integro-differential equations [2]. This yields the existence, for each $\varepsilon > 0$, of two sequences of matrices $(X^{\varepsilon,\zeta})_\zeta, (Y^{\varepsilon,\zeta})_\zeta \subset \mathbb{S}^d$ of the form

$$X^{\varepsilon,\zeta} = \begin{bmatrix} X_1^{\varepsilon,\zeta} & 0 \\ 0 & X_2^{\varepsilon,\zeta} \end{bmatrix} \quad \text{and} \quad Y^{\varepsilon,\zeta} = \begin{bmatrix} Y_1^{\varepsilon,\zeta} & 0 \\ 0 & Y_2^{\varepsilon,\zeta} \end{bmatrix}, \tag{12}$$

which correspond to the sub-jets and super-jets of u at the points x^ε and y^ε. In addition the block diagonal matrix satisfies

$$-\frac{1}{\zeta}\begin{bmatrix} I_d & 0 \\ 0 & I_d \end{bmatrix} \leqslant \begin{bmatrix} X^{\varepsilon,\zeta} & 0 \\ 0 & -Y^{\varepsilon,\zeta} \end{bmatrix} \leqslant \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix} + o_\zeta(1), \tag{13}$$

with Z a block matrix of the form

$$\begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \tag{14}$$

with blocks

$$Z_1 = D^2\phi\big(a_1^\varepsilon\big) = \frac{\varphi'(|a_1^\varepsilon|)}{|a_1^\varepsilon|} I_{d_1} + \left(\varphi''\big(\big|a_1^\varepsilon\big|\big) - \frac{\varphi'(|a_1^\varepsilon|)}{|a_1^\varepsilon|}\right)\hat{a}_1^\varepsilon \otimes \hat{a}_1^\varepsilon,$$

$$Z_2 = \frac{2}{\varepsilon^2} I_{d_2}.$$

By Lemma 24 the triple of block matrices $(X_i^{\varepsilon,\zeta}, Y_i^{\varepsilon,\zeta}, Z_i)$ for $i=1,2$ satisfy (13). Then, by sup and inf matrix convolution (see Lemmas 25 and 26 in Appendix A) we build matrices, that we still denote by $X^{\varepsilon,\zeta}$ and $Y^{\varepsilon,\zeta}$, for which the corresponding blocks $X_i^{\varepsilon,\zeta}$ and $Y_i^{\varepsilon,\zeta}$ for $i=1,2$ satisfy uniform bounds

$$-\frac{2}{\bar{\varepsilon}}\begin{bmatrix} I_{d_1} & 0 \\ 0 & I_{d_1}\end{bmatrix} \leqslant \begin{bmatrix} X_1^{\varepsilon,\zeta} & 0 \\ 0 & -Y_1^{\varepsilon,\zeta}\end{bmatrix} \leqslant \begin{bmatrix} \tilde{Z}_1 & -\tilde{Z}_1 \\ -\tilde{Z}_1 & \tilde{Z}_1\end{bmatrix} + o_\zeta(1), \tag{15}$$

$$-\frac{4}{\varepsilon^2}\begin{bmatrix} I_{d_2} & 0 \\ 0 & I_{d_2}\end{bmatrix} \leqslant \begin{bmatrix} X_2^{\varepsilon,\zeta} & 0 \\ 0 & -Y_2^{\varepsilon,\zeta}\end{bmatrix} \leqslant \frac{4}{\varepsilon^2}\begin{bmatrix} I_{d_2} & 0 \\ 0 & I_{d_2}\end{bmatrix} + o_\zeta(1) \tag{16}$$

with $\tilde{Z}_1 = Z_1^{\frac{\bar{\varepsilon}}{2}}$, where

$$\bar{\varepsilon} = \frac{|a_1^\varepsilon|}{\varphi'(|a_1^\varepsilon|)}.$$

In addition, from the monotonicity of the sup- and inf-convolution (37) the new block matrices $X^{\varepsilon,\zeta}$ and $Y^{\varepsilon,\zeta}$ are still sub- and super-jets of u at x^ε, respectively y^ε

$$\big(p^\varepsilon, X^{\varepsilon,\zeta}\big) \in \mathcal{D}^{2,+}\big(u\big(x^\varepsilon\big)\big),$$
$$\big(p^\varepsilon, Y^{\varepsilon,\zeta}\big) \in \mathcal{D}^{2,-}\big(u\big(y^\varepsilon\big)\big).$$

Since the bounds in (15) and (16) are uniform with respect to ζ, we can let $\zeta \to 0$ and obtain two matrices X^ε and Y^ε satisfying the double inequality required by the ellipticity growth condition $(H1)$, which are still sub- and super-jets of u at x^ε and y^ε respectively. Hence, they satisfy the viscosity inequalities

$$F_0\big(u\big(x^\varepsilon\big), p^\varepsilon, X^\varepsilon, \mathcal{I}\big[x^\varepsilon, p^\varepsilon, u\big]\big) + \sum_{i=1,2} F_i\big(\bar{x}_i^\varepsilon, p_i^\varepsilon, X_i^\varepsilon, \mathcal{I}_{x_i}\big[x^\varepsilon, p_i^\varepsilon, u\big]\big) \leqslant f\big(x^\varepsilon\big),$$

$$F_0\big(u\big(y^\varepsilon\big), p^\varepsilon, Y^\varepsilon, \mathcal{I}\big[y^\varepsilon, p^\varepsilon, u\big]\big) + \sum_{i=1,2} F_i\big(\bar{y}_i^\varepsilon, p_i^\varepsilon, Y_i^\varepsilon, \mathcal{I}_{y_i}\big[y^\varepsilon, p_i^\varepsilon, u\big]\big) \geqslant f\big(y^\varepsilon\big).$$

Subtracting the above inequalities and denoting

$$E_0\big(x^\varepsilon, y^\varepsilon, u\big) = F_0\big(u\big(y^\varepsilon\big), p^\varepsilon, Y^\varepsilon, \mathcal{I}\big[y^\varepsilon, p^\varepsilon, u\big]\big) - F_0\big(u\big(x^\varepsilon\big), p^\varepsilon, X^\varepsilon, \mathcal{I}\big[x^\varepsilon, p^\varepsilon, u\big]\big) + f\big(x^\varepsilon\big) - f\big(y^\varepsilon\big),$$
$$E_i\big(\bar{x}_i^\varepsilon, \bar{y}_i^\varepsilon, u\big) = F_i\big(\bar{y}_i^\varepsilon, p_i^\varepsilon, Y_i^\varepsilon, \mathcal{I}_{y_i}\big[y^\varepsilon, p_i^\varepsilon, u\big]\big) - F_i\big(\bar{x}_i^\varepsilon, p_i^\varepsilon, X_i^\varepsilon, \mathcal{I}_{x_i}\big[x^\varepsilon, p_i^\varepsilon, u\big]\big), \quad i=1,2,$$

we get that

$$0 \leqslant E_0\big(x^\varepsilon, y^\varepsilon, u\big) + E_1\big(x_1^\varepsilon, y_1^\varepsilon, u\big) + E_2\big(x_2^\varepsilon, y_2^\varepsilon, u\big). \tag{17}$$

Step 4. In the following we estimate each of these terms as $\varepsilon \to 0$, bringing into play the ellipticity growth assumptions satisfied by each nonlinearity.

Since $u(y^\varepsilon) \leqslant u(x^\varepsilon)$, $X^\varepsilon \leqslant Y^\varepsilon$, the monotonicity assumption $(H0)$, the ellipticity (E) with respect to the second order term and the nonlocal term and the Lipschitz continuity $(H2)$ of F_0 with respect to the nonlocal term yield

$$E_0\big(x^\varepsilon, y^\varepsilon, u\big) \leqslant \tilde{\gamma}\big(u\big(y^\varepsilon\big) - u\big(x^\varepsilon\big)\big) + L_{F_0}\big(\mathcal{I}\big[x^\varepsilon, p^\varepsilon, u\big] - \mathcal{I}\big[y^\varepsilon, p^\varepsilon, u\big]\big)_+ + f\big(x^\varepsilon\big) - f\big(y^\varepsilon\big).$$

As the Lévy measures corresponding to the nonlinearity F_0 do not depend on x, we immediately deduce from the maximum condition that

$$u\big(x^\varepsilon+z\big)-v\big(y^\varepsilon+z\big)\leqslant u\big(x^\varepsilon\big)-v\big(y^\varepsilon\big)$$

renders nonpositive the difference of the nonlocal terms

$$\mathcal{I}\big[x^\varepsilon,p^\varepsilon,u\big]-\mathcal{I}\big[y^\varepsilon,p^\varepsilon,u\big]\leqslant 0.$$

Therefore, passing to the limits as $\varepsilon\to 0$ and employing Lemma 4 we have

$$\limsup_{\varepsilon\to 0} E_0\big(x^\varepsilon,y^\varepsilon,u\big)\leqslant -\tilde{\gamma}M. \tag{18}$$

The estimate of E_2 does not depend on the choice of φ and is given by the growth condition $(H3)$ and the Lipschitz continuity $(H2)$ of $F_2(\cdot,l)$, uniformly with respect to all the other variables

$$E_2\big(x_2^\varepsilon,y_2^\varepsilon,u\big)\leqslant \omega_{F_2}\left(\frac{|a_2^\varepsilon|^2}{\varepsilon^2}+\big|a_2^\varepsilon\big|\right)+L_{F_2}\big(\mathcal{I}_{x_2}\big[x^\varepsilon,p_2^\varepsilon,u\big]-\mathcal{I}_{y_2}\big[y^\varepsilon,p_2^\varepsilon,u\big]\big)_+$$

where L_{F_2} is the Lipschitz constant of $F_2(\cdot,l)$. From Proposition 20 in Section 6 the quadratic estimates for Lévy–Itô operators hold

$$\mathcal{I}_{x_2}\big[x^\varepsilon,p_2^\varepsilon,u\big]-\mathcal{I}_{y_2}\big[y^\varepsilon,p_2^\varepsilon,u\big]\leqslant C\frac{1}{\varepsilon^2}\int_{B_\delta}|z_2|^2\,\mu^2(dz_2)+CC_{\mu^2}\frac{|a_2^\varepsilon|^2}{\varepsilon^2},$$

for some positive constant C. As $\delta\to 0$, the estimate gives

$$\mathcal{I}_{x_2}\big[x^\varepsilon,p_2^\varepsilon,u\big]-\mathcal{I}_{y_2}\big[y^\varepsilon,p_2^\varepsilon,u\big]\leqslant C\tilde{C}_{\mu^2}\frac{|a_2^\varepsilon|^2}{\varepsilon^2}.$$

Letting now $\varepsilon\to 0$ and using Lemma 4 which ensures that $\frac{|a_2^\varepsilon|^2}{\varepsilon^2}\to 0$ we are finally led to

$$\limsup_{\varepsilon\to 0} E_2\big(x_2^\varepsilon,y_2^\varepsilon,u\big)\leqslant 0. \tag{19}$$

For the estimate of E_1, we use the ellipticity growth condition $(H1)$

$$\begin{aligned} E_1\big(x_1^\varepsilon,y_1^\varepsilon,u\big)\leqslant{}& \Lambda_1\big(x_1^\varepsilon\big)\left(\big(\mathcal{I}_{x_1}\big[x^\varepsilon,p_1^\varepsilon,u\big]-\mathcal{I}_{y_1}\big[y^\varepsilon,p_1^\varepsilon,u\big]\big)+\frac{|a_1^\varepsilon|^{2\theta}}{\bar{\varepsilon}}+\big|a_1^\varepsilon\big|^\tau\big|p_1^\varepsilon\big|^{k+\tau}+C_1\big|p_1^\varepsilon\big|^k\right)\\ &+\Lambda_2\big(x_1^\varepsilon\big)\left(\operatorname{tr}\big(X_1^\varepsilon-Y_1^\varepsilon\big)+\frac{|a_1^\varepsilon|^{2\tilde{\theta}}}{\bar{\varepsilon}}+\big|a_1^\varepsilon\big|^\tau\big|p_1^\varepsilon\big|^{2+\tau}+C_2\big|p_1^\varepsilon\big|^2\right) \end{aligned} \tag{20}$$

where we recall that $p_1^\varepsilon=D\phi(a_1^\varepsilon)=L\varphi'(|a_1^\varepsilon|)\hat{a}_1^\varepsilon$. The goal is to show that, for each choice of φ (measuring either the Hölder or the Lipschitz continuity), the right-hand side quantity is negative, arriving thus to a contradiction by combining (17), (18), (19) and (20).

Step 5.1 *(Hölder continuity).* In order to establish the Hölder regularity of solutions, we consider the auxiliary function

$$\varphi = Lt^{\alpha}, \quad \text{with } \alpha < \min(1, \beta).$$

In this case, we apply Corollary 10 from Section 6, to the functions $u(\cdot, x_2)$ and $u(\cdot, y_2)$, which yields the following Hölder estimate for the difference of the nonlocal terms

$$\mathcal{I}_{x_1}\big[x^{\varepsilon}, p_1^{\varepsilon}, u\big] - \mathcal{I}_{y_1}\big[y^{\varepsilon}, p_1^{\varepsilon}, u\big] \leqslant -L\big|a_1^{\varepsilon}\big|^{\alpha-\beta}\big\{\alpha C\big(\mu^1\big) - o_{|a_1^{\varepsilon}|}(1)\big\} + O(1).$$

Lemma 27 from Appendix A applies with $\tilde{Z}_1 = Z_1^{\frac{\bar{\varepsilon}}{2}}$, $\bar{\varepsilon} = (L\alpha|a_1^{\varepsilon}|^{\alpha-2})^{-1}$, $\omega = 2-\alpha$ and hence the trace is bounded by

$$\operatorname{trace}\big(X_1^{\varepsilon} - Y_1^{\varepsilon}\big) \leqslant -8\bar{\omega}\big(L\alpha\big|a_1^{\varepsilon}\big|^{\alpha-2}\big) \tag{21}$$

where $\bar{\omega} = \frac{\omega-1}{\omega+1}$ is a constant in $(0, \frac{1}{3})$. We plug these estimates into the inequality for E_1. Letting ε go to zero and employing the penalization Lemma 4 and $(H4)$ we obtain the following bound

$$\limsup_{\varepsilon\to 0} E_1\big(x_1^{\varepsilon}, y_1^{\varepsilon}, u\big) \leqslant \Lambda_0\mathcal{E}^1\big(|\bar{a}|\big) + \Lambda_0\mathcal{E}^2\big(|\bar{a}|\big) + O(1)$$

where for $2\theta + \beta > 2$

$$\begin{aligned}\mathcal{E}^1\big(|\bar{a}|\big) &= -L|\bar{a}|^{\alpha-\beta}\big(\alpha C\big(\mu^1\big) - o_{|\bar{a}|}(1)\big) + |\bar{a}|^{2\theta}\big(L\alpha|\bar{a}|^{\alpha-2}\big) + |\bar{a}|^{\tau}\big(L\alpha|\bar{a}|^{\alpha-1}\big)^{k+\tau} + C_1\big(L\alpha|\bar{a}|^{\alpha-1}\big)^{k}\\ &= -L|\bar{a}|^{\alpha-\beta}\big\{\alpha C\big(\mu^1\big) - o_{|\bar{a}|}(1) - \alpha^{k+\tau}|\bar{a}|^{\beta-k}\big(L|\bar{a}|^{\alpha}\big)^{k+\tau-1} - C_1\alpha^k|\bar{a}|^{\beta-k}\big(L|\bar{a}|^{\alpha}\big)^{k-1}\big\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}^2\big(|\bar{a}|\big) &= -8\bar{\omega}\big(L\alpha|\bar{a}|^{\alpha-2}\big) + |\bar{a}|^{2\tilde{\theta}}\big(L\alpha|\bar{a}|^{\alpha-2}\big) + |\bar{a}|^{\tau}\big(L\alpha|\bar{a}|^{\alpha-1}\big)^{2+\tau} + C_2\big(L\alpha|\bar{a}|^{\alpha-1}\big)^{2}\\ &= -L|\bar{a}|^{\alpha-2}\big\{\alpha\big(8\bar{\omega} - |\bar{a}|^{2\tilde{\theta}}\big) - \alpha^{2+\tau}\big(L|\bar{a}|^{\alpha}\big)^{1+\tau} - C_2\alpha^2L|\bar{a}|^{\alpha}\big\}.\end{aligned}$$

Using the fact that $L|\bar{a}|^{\alpha} \leqslant 2\|u\|_{\infty}$ we have

$$\mathcal{E}^2\big(|\bar{a}|\big) \leqslant -L|\bar{a}|^{\alpha-2}\big\{\alpha\big(8\bar{\omega} - |\bar{a}|^{2\tilde{\theta}}\big) - \alpha^{2+\tau}\big(2\|u\|_{\infty}\big)^{1+\tau} - C_2\alpha^2\big(2\|u\|_{\infty}\big)\big\}.$$

As far as $\mathcal{E}^1$ is concerned, we further argue differently for the subcritical and supercritical case, with respect to the Lévy exponent β, and accordingly with respect to k and τ. Namely

(a) if $1 < k \leqslant \beta$, in which case $k + \tau - 1 > 0$, $k - 1 > 0$, we have

$$\begin{aligned}\mathcal{E}^1\big(|\bar{a}|\big) \leqslant -L|\bar{a}|^{\alpha-\beta}\big\{&\alpha C\big(\mu^1\big) - o_{|\bar{a}|}(1) - \alpha^{k+\tau}|\bar{a}|^{\beta-k}\big(2\|u\|_{\infty}\big)^{k+\tau-1}\\ &- C_1\alpha^k|\bar{a}|^{\beta-k}\big(2\|u\|_{\infty}\big)^{k-1}\big\};\end{aligned}$$

(b) if $k < \min(1, \beta)$, then
(b.1) for $0 < k \leqslant 1 - \tau$ and $\beta - k + \alpha(k + \tau - 1) > 0$

$$\begin{aligned}\mathcal{E}^1\big(|\bar{a}|\big) &\leqslant -L|\bar{a}|^{\alpha-\beta}\big\{\alpha C\big(\mu^1\big) - o_{|\bar{a}|}(1) - \alpha^{k+\tau}|\bar{a}|^{\beta-k+\alpha(k+\tau-1)}L^{k+\tau-1} \\ &\quad - C_1\alpha^k|\bar{a}|^{\beta-k+\alpha(k-1)}L^{k-1}\big\} \\ &= -L|\bar{a}|^{\alpha-\beta}\big(\alpha C\big(\mu^1\big) - o_{|\bar{a}|}(1)\big).\end{aligned}$$

(b.2) for $1 - \tau < k \leqslant 1$ and $\beta - k + \alpha(k + \tau - 1) > 0$

$$\begin{aligned}\mathcal{E}^1\big(|\bar{a}|\big) &\leqslant -L|\bar{a}|^{\alpha-\beta}\big\{\alpha C\big(\mu^1\big) - o_{|\bar{a}|}(1) - \alpha^{k+\tau}\big(2\|u\|_\infty\big)^{k+\tau-1} \\ &\quad - C_1\alpha^k|\bar{a}|^{\beta-k+\alpha(k-1)}L^{k-1}\big\} \\ &= -L|\bar{a}|^{\alpha-\beta}\big\{\alpha C\big(\mu^1\big) - o_{|\bar{a}|}(1) - \alpha^{k+\tau}\big(2\|u\|_\infty\big)^{k+\tau-1}\big\}.\end{aligned}$$

This implies that for α small enough the two terms become (large) negative

$$\lim_{L\to\infty}\mathcal{E}^1\big(|\bar{a}|\big) = -\infty \quad \text{and} \quad \lim_{L\to\infty}\mathcal{E}^2\big(|\bar{a}|\big) = -\infty.$$

Hence

$$\lim_{L\to\infty}\limsup_{\varepsilon\to 0} E_1\big(x_1^\varepsilon, y_1^\varepsilon, u\big) = -\infty. \tag{22}$$

We now turn back to inequality (17), let first $\varepsilon \to 0$ and then $L \to \infty$. Plugging in the estimates (18)–(22) we arrive to a contradiction. Therefore, we have proved up to this point the $C^{0,\alpha}$ regularity of the solution, for α small enough. Note that the exponent α only depends on $\|u\|_\infty$, k and τ.

We further use this first step to provide the $C^{0,\alpha}$ regularity for all $\alpha \in (0,1)$. To this end, we estimate $L|\bar{a}|^\alpha$ with the modulus of continuity of u and get

$$\mathcal{E}^2\big(|\bar{a}|\big) \leqslant -L|\bar{a}|^{\alpha-2}\big\{\alpha\big(8\bar{\omega} - |\bar{a}|^{2\tilde{\theta}}\big) - \alpha^{2+\tau}\big(\omega_u\big(|\bar{a}|\big)\big)^{1+\tau} - C_2\alpha^2\omega_u\big(|\bar{a}|\big)\big\}.$$

Taking into account that $\omega_u(|\bar{a}|) \leqslant \bar{L}|\bar{a}|^{\bar{\alpha}}$ for some $\bar{\alpha}$ small, we come back to the original estimates in case $k > 1$ and to the estimates given in (b.1) when $k \in (0, 1-\tau)$, respectively (b.2) when $k \in (1-\tau, 1)$, where α is everywhere replaced with $\bar{\alpha}$. By similar arguments we obtain

$$\begin{aligned}\mathcal{E}^1\big(|\bar{a}|\big) &\leqslant -L|\bar{a}|^{\alpha-\beta}\big(\alpha C\big(\mu^1\big) - o_{|\bar{a}|}(1)\big), \\ \mathcal{E}^2\big(|\bar{a}|\big) &\leqslant -L|\bar{a}|^{\alpha-2}\big(\alpha C\big(\mu^1\big) - o_{|\bar{a}|}(1)\big).\end{aligned}$$

This yields (22) for L sufficiently large, and therefore completes the $C^{0,\alpha}$ regularity result.

Step 5.2 *(Lipschitz continuity).* In the case $\beta > 1$, we establish the Lipschitz regularity of solutions. Therefore, we consider the auxiliary function

$$\varphi(t) = \begin{cases} L(t - \varrho t^{1+\alpha}), & t \in [0, t_0], \\ \varphi(t_0), & t > t_0 \end{cases}$$

where $\alpha \in (0,1)$ will be chosen small enough, ρ and t_0 as in Corollary 9 in Section 6. We remind that α is related to the aperture of the cone corresponding to $\eta \sim |\bar{a}|^{2\alpha}$. In order to estimate the difference of the nonlocal terms, we apply Corollary 9, to the same choice of functions $u(\cdot, x_2)$ and $u(\cdot, y_2)$:

$$\mathcal{I}_{x_1}\big[x^\varepsilon, p_1^\varepsilon, u\big] - \mathcal{I}_{y_1}\big[y^\varepsilon, p_1^\varepsilon, u\big] \leqslant -L|a_1^\varepsilon|^{(1-\beta)+\alpha(d_1+2-\beta)}\big\{\Theta\big(\varrho, \alpha, \mu^1\big) - o_{|a_1^\varepsilon|}(1)\big\} + O(1).$$

At this point, we fix ρ such that the constant $\Theta(\varrho, \alpha, \mu^1)$ is positive. We then apply Lemma 27 in Appendix A with $\tilde{Z}_1 = Z_1^{\frac{\bar{\varepsilon}}{2}}$, where this time

$$\bar{\varepsilon} = \frac{|a_1^\varepsilon|}{\varphi'(|a_1^\varepsilon|)} = \big(L\big|a_1^\varepsilon\big|^{-1} - L\rho(1+\alpha)\big|a_1^\varepsilon\big|^{\alpha-1}\big)^{-1}.$$

Indeed $\omega = 1 - \varphi''(|a_1^\varepsilon|)\bar{\varepsilon} \in (1,2)$ for ε sufficiently small. Hence

$$\operatorname{trace}\big(X_1^\varepsilon - Y_1^\varepsilon\big) \leqslant -\frac{8}{\bar{\varepsilon}}\frac{\omega-1}{\omega+1} = \frac{8\varphi''(|a_1^\varepsilon|)}{2 - \varphi''(|a_1^\varepsilon|)\bar{\varepsilon}}.$$

Note that in this case $\frac{\omega-1}{\omega+1}$ depends on $|a_1^\varepsilon|$. However there exists a positive constant $\bar{\omega}$ such that for ε sufficiently small

$$\frac{8\varphi''(|a_1^\varepsilon|)}{2 - \varphi''(|a_1^\varepsilon|)\bar{\varepsilon}} \leqslant 8\bar{\omega}\varphi''\big(\big|a_1^\varepsilon\big|\big).$$

Hence, denoting by $c = \rho(1+\alpha)$, second order terms are bounded by

$$\operatorname{trace}\big(X_1^\varepsilon - Y_1^\varepsilon\big) \leqslant -8c\bar{\omega}\big(L\alpha\big|a_1^\varepsilon\big|^{\alpha-1}\big).$$

We plug these estimates into the inequality for E_1. Letting ε go to zero and employing Lemma 4 we arrive as before to

$$\limsup_{\varepsilon\to 0} E_1\big(x_1^\varepsilon, y_1^\varepsilon, u\big) \leqslant \Lambda_0\mathcal{E}^1\big(|\bar{a}|\big) + \Lambda_0\mathcal{E}^2\big(|\bar{a}|\big) + O(1),$$

where denoting by $C(\mu^1) = \Theta(\varrho, \alpha, \mu^1)$ the terms $\mathcal{E}^1$, $\mathcal{E}^2$ are given by

$$\begin{aligned}
\mathcal{E}^1\big(|\bar{a}|\big) &= -L|\bar{a}|^{(1-\beta)+\alpha(d_1+2-\beta)}\big(C\big(\mu^1\big) - o_{|\bar{a}|}(1)\big) + |\bar{a}|^{2\theta}\big(L|\bar{a}|^{-1}\big(1 - c|\bar{a}|^\alpha\big)\big) \\
&\quad + |\bar{a}|^\tau\big(L\big(1 - c|\bar{a}|^\alpha\big)\big)^{\beta+\tau} + C_1\big(L\big(1 - c|\bar{a}|^\alpha\big)\big)^\beta, \\
\mathcal{E}^2\big(|\bar{a}|\big) &= -8c\bar{\omega}\big(L\alpha|\bar{a}|^{\alpha-1}\big) + |\bar{a}|^{2\tilde{\theta}}\big(L|\bar{a}|^{-1}\big(1 - c|\bar{a}|^\alpha\big)\big) \\
&\quad + |\bar{a}|^\tau\big(L\big(1 - c|\bar{a}|^\alpha\big)\big)^{2+\tau} + C_2\big(L\big(1 - c|\bar{a}|^\alpha\big)\big)^2.
\end{aligned}$$

Whenever $\alpha(d_1+3-\beta) < 2\theta - 2 - \beta$ the second term in $\mathcal{E}^1$ behaves like $o(|\bar{a}|^{(1-\beta)+\alpha(d_1+2-\beta)})$. Taking $L|\bar{a}|^{(1-\beta)+\alpha(d_1+2-\beta)}$ as a common multiplier and using that $1 - c|\bar{a}|^\alpha \leqslant 1$ we have

$$\begin{aligned}
\mathcal{E}^1\big(|\bar a|\big) &\leqslant -L|\bar a|^{(1-\beta)+\alpha(d_1+2-\beta)}\big\{C\big(\mu^1\big) - o_{|\bar a|}(1) \\
&\quad - |\bar a|^{-\alpha(d_1+2-\beta)}\big(L|\bar a| - cL|\bar a|^{\alpha+1}\big)^{\beta+\tau-1} - C_1|\bar a|^{-\alpha(d_1+2-\beta)}\big(L|\bar a| - cL|\bar a|^{\alpha+1}\big)^{\beta-1}\big\} \\
&\leqslant -L|\bar a|^{(1-\beta)+\alpha(d_1+2-\beta)}\big\{C\big(\mu^1\big) - o_{|\bar a|}(1) \\
&\quad - 2|\bar a|^{-\alpha(d_1+2-\beta)}\big(\varphi\big(|\bar a|\big)\big)^{\beta+\tau-1} - 2C_1|\bar a|^{-\alpha(d_1+2-\beta)}\big(\varphi\big(|\bar a|\big)\big)^{\beta-1}\big\}.
\end{aligned}$$

On the other hand, similar techniques give us an estimate for $\mathcal{E}^2$:

$$\begin{aligned}
\mathcal{E}^2\big(|\bar a|\big) &\leqslant -L|\bar a|^{\alpha-1}\big\{8c\alpha\bar\omega - |\bar a|^{2\tilde\theta}|\bar a|^{-\alpha} \\
&\quad - |\bar a|^{-\alpha}\big(L|\bar a| - cL|\bar a|^{\alpha+1}\big)^{1+\tau} - C_2|\bar a|^{-\alpha}\big(L|\bar a| - cL|\bar a|^{\alpha+1}\big)\big\} \\
&\leqslant -L|\bar a|^{\alpha-1}\big\{8c\alpha\bar\omega - |\bar a|^{2\tilde\theta}|\bar a|^{-\alpha} \\
&\quad - 2|\bar a|^{-\alpha}\big(\varphi\big(|\bar a|\big)\big)^{1+\tau} - 2C_2|\bar a|^{-\alpha}\big(\varphi\big(|\bar a|\big)\big)\big\}.
\end{aligned}$$

When α is small enough we have $|\bar a|^{2\tilde\theta}|\bar a|^{-\alpha} = o_{|\bar a|}(1)$. Then

$$\mathcal{E}^2\big(|\bar a|\big) \leqslant -L|\bar a|^{\alpha-1}\big\{C - o_{|\bar a|}(1) - 2|\bar a|^{-\alpha}\big(\varphi\big(|\bar a|\big)\big)^{1+\tau} - 2C_2|\bar a|^{-\alpha}\big(\varphi\big(|\bar a|\big)\big)\big\}.$$

Since we have just seen that u is Hölder continuous for any $\tilde\alpha \in (0,1)$, we have

$$\varphi\big(|\bar a|\big)|\bar a|^{-\tilde\alpha} \to 0, \quad \text{as } L \to \infty.$$

Using this relation in the previous inequalities estimating $\mathcal{E}^1$ and $\mathcal{E}^2$ we get that, for L large enough

$$\begin{aligned}
\mathcal{E}^1\big(|\bar a|\big) &\leqslant -L|\bar a|^{(1-\beta)+\alpha(d_1+2-\beta)}\big(C\big(\mu^1\big) - o_{|\bar a|}(1)\big), \\
\mathcal{E}^2\big(|\bar a|\big) &\leqslant -L|\bar a|^{\alpha-1}\big(C - o_{|\bar a|}(1)\big).
\end{aligned}$$

Hence (22) holds and this further yields the desired contradiction. □

3.2. Global regularity

It follows immediately from the previous results that as long as both nonlinearities F_1 and F_2 satisfy assumptions $(H1)$–$(H3)$, the solution is global Lipschitz or Hölder continuous.

Corollary 5 *(Global regularity for periodic, mixed PIDEs). Let the nonlinearities F_i, $i = 0, 1, 2$, be degenerate elliptic, continuous and periodic, f continuous and periodic. Assume the following*:

- *F_0 satisfies assumptions $(H0)$, $(H2)$ with $\tilde d = d$ and some constant $\tilde\gamma > 0$;*
- *F_i with $i = 1, 2$ satisfy assumptions $(H1)$–$(H3)$ with $\tilde d = d_i$, for some functions Λ_i^1, Λ_i^2 and some constants $k_i \geqslant 0$, $\tau_i \in [0,1]$, $\theta_i, \tilde\theta_i \in (0,1]$.*

Let μ^0, μ^i, with $i = 1, 2$, be Lévy measures on $\mathbb{R}^d$, $\mathbb{R}^{d_i}$ respectively associated to the integro-differential operators $\mathcal{I}[x,u]$, $\mathcal{J}_{x_i}[x,u]$ and suppose the corresponding jump functions $j^i(x_i, z_i)$ satisfy assumptions $(J1)$–$(J5)$ for some constants β_i, C_{μ^i}, $\tilde C_{\mu^i}$, with $\gamma = 1$. Then any periodic continuous viscosity solution u of

$$F_0\big(u(x), Du, D^2u, \mathcal{I}[x,u]\big) + F_1\big(x_1, D_{x_1}u, D^2_{x_1x_1}u, \mathcal{J}_{x_1}[x,u]\big) + F_2\big(x_2, D_{x_2}u, D^2_{x_2x_2}u, \mathcal{J}_{x_2}[x,u]\big) = f(x) \tag{23}$$

(a) *is Lipschitz continuous, if $\beta_i > 1$ and $k_i \leqslant \beta_i$ for $i = 1, 2$;*
(b) *is $C^{0,\alpha}$ continuous with $\alpha < \min(\frac{\beta_1 - k_1}{1-k_1}, \frac{\beta_2-k_2}{1-k_2})$, if $\beta \leqslant 1$ and $k_i < \beta_i$ for $i = 1, 2$.*

The Lipschitz/Hölder constant depends on $\|u\|_\infty$, on the dimension d of the space and on the constants associated to the Lévy measures and on the constants required by the growth condition $(H1)$.

At first glance, the fact that $(H1)$ and $(H3)$ must hold simultaneously seems to exclude a large class of nonlinear equations dealing with directional gradient or drift terms such as $|D_{x_i}u|^r$ or $|b(x_i)|D_{x_i}u|^{k+\tau}$, $r, k > 0$. Indeed, taking in the ellipticity growth condition $(H1)$ $l = l'$, $p = \frac{x-y}{\varepsilon}$ and $\tilde{\theta} = \theta$ we get

$$F\left(y, \frac{x-y}{\varepsilon}, Y, l\right) - F\left(x, \frac{x-y}{\varepsilon}, X, l\right) \leqslant \Lambda(x)\left(\operatorname{tr}(X-Y) + \frac{|x-y|^{2\theta}}{\varepsilon} + \frac{|x-y|^{k+2\tau}}{\varepsilon^{k+\tau}} + \frac{|x-y|^r}{\varepsilon^r}\right).$$

Hence $(H3)$ would hold whenever $k = r = 0$, $\theta = 1$. In this case $(H1)$ and $(H3)$ could be joined together in assumption:

(H) There exist two functions $\Lambda_1, \Lambda_2 : \mathbb{R}^{\tilde{d}} \to [0,\infty)$ such that $\Lambda_1(x) + \Lambda_1(x) \geqslant \Lambda_0 > 0$ and a modulus of continuity $\omega_F(r) \to 0$, as $r \to 0$ such that for any $x, y \in \mathbb{R}^{\tilde{d}}$, $p \in \mathbb{R}^{\tilde{d}}$, $l \leqslant l'$ and any $\varepsilon > 0$

$$F\big(y, p, Y, l'\big) - F(x, p, X, l) \leqslant \Lambda_1(x)\big(l - l'\big) + \Lambda_2(x)\operatorname{tr}(X-Y) + \omega_F\left(|x-y|\big(1+|p|\big) + \frac{|x-y|^2}{\varepsilon}\right)$$

if $X, Y \in \mathbb{S}^{\tilde{d}}$ satisfy inequality (4) with $Z = I - \bar{\omega}\hat{z} \otimes \hat{z}$, for $z \in \mathbb{R}^{\tilde{d}}$ and $\bar{\omega} \geqslant 1$.

Nevertheless, one can argue under weaker growth assumptions, by a *cut-off gradients* argument for equations of the type (23) where F_i, for $i = 1, 2$, satisfy assumptions $(H1)$–$(H2)$ and F_0 satisfies $(H2)$ and $(H0)$ with $\tilde{\gamma} > 0$.

Roughly speaking, one should look at the approximated equation with $|Du|$ replaced by $|Du| \wedge R$, for $R > 0$ and remark that its solutions are Lipschitz continuous, with the Lipschitz norm independent of R, thus the solution of the original problem is also Lipschitz continuous. This is made precise by defining, for each $i = 0, 1, 2$ the following functions

$$F_i^R(\cdot, p, X, l) = \begin{cases} F_i(\cdot, p, X, l) & \text{if } |p| \leqslant R, \\ F_i(\cdot, R\frac{p}{|p|}, X, l) & \text{if } |p| \geqslant R. \end{cases}$$

Consider then the approximated problem

$$F_0^R\big(u^R(x), Du^R, D^2u^R, \mathcal{I}\big[x, u^R\big]\big) + F_1^R\big(x_1, D_{x_1}u^R, D^2_{x_1x_1}u^R, \mathcal{J}_{x_1}\big[x, u^R\big]\big) + F_2^R\big(x_2, D_{x_2}u^R, D^2_{x_2x_2}u^R, \mathcal{J}_{x_2}\big[x, u^R\big]\big) = f(x) \tag{24}$$

and remark that $(H3)$ holds. Thus the approximated problem (24) has a Lipschitz/Hölder viscosity solution, whose continuity constant depends on $\|u^R\|_\infty$ the constants required by the Lévy measures and those appearing in the ellipticity growth assumption $(H1)$.

Let

$$M := \left|F_1(0,0,0,0)\right| + \left\|F_1(x_1,0,0,0)\right\|_\infty + \left\|F_2(x_2,0,0,0)\right\|_\infty + \|f\|_\infty.$$

Since $M(\tilde{\gamma})^{-1}$ and $-M(\tilde{\gamma})^{-1}$ are respectively a super-solution and a sub-solution of the approximated problem (24), by a comparison result between sub- and super-solutions we have due to $(H0)$

$$\left\|u^R\right\|_\infty \leqslant \frac{M}{\tilde{\gamma}}.$$

Therefore, the Lipschitz constant of u^R is independent of R. Observing that for R large enough the solution u^R of the approximated problem is as well a solution of the original, we conclude.

4. Examples and discussion on assumptions

In this section, we illustrate the partial and global regularity results on several examples. We start with two examples of classical nonlinearities for which we deal with global regularity: a model equation as in [1] and the advection fractional diffusion. Then we present the partial and global regularity results for pure mixed equations: first on the toy model and then on a general nonlinearity dealing with mixed gradient terms.

4.1. *Classical nonlinearities*

As already presented in the introduction, the Lipschitz regularity result applies for equations that are strictly elliptic in a generalized sense: at each point, the nonlinearity is *either nondegenerate in the second order term*, or is *nondegenerate in the nonlocal term*. More precisely, by Theorem 2 we extend the Hölder regularity result in [1] to Lipschitz regularity when the nonlocal exponent $\beta > 1$.

4.1.1. *Model equation*

A model equation for such nondegenerate equations is

$$-\operatorname{tr}\left(A(x)D^2u\right) - c(x)\mathcal{I}[x,u] + b(x)|Du|^k + |Du|^r = 0 \quad \text{in } \mathbb{R}^d, \tag{25}$$

where A and c are continuous functions, $b \in C^{0,\tau}(\mathbb{R}^d)$, with $0 \leqslant \tau \leqslant 1$, $k, r \in (0, 2+\tau)$. $\mathcal{I}[x,u]$ is a nonlocal term of type (5) or (6) of exponent $\beta \in (0,2)$. In the following, we discuss the ellipticity growth assumption $(H1)$ and make precise the role of each term.

- One has to assume that Eq. (25) is strictly elliptic in the sense that

$$A(x) \geqslant \Lambda_1(x)I \quad \text{and} \quad c(x) \geqslant \Lambda_2(x) \quad \text{in } \mathbb{R}^d \tag{26}$$

  with

$$\Lambda_1(x) + \Lambda_2(x) \geqslant \Lambda_0 > 0.$$

  Thus the equation may be degenerate in the local or the nonlocal term as for all $x \in \mathbb{R}^d$, $A(x) \geqslant 0$ and $c(x) \geqslant 0$. However, at each point either $A(x)$ is a positive definite matrix and the equation is strictly elliptic in the classical sense, or $c(x) > 0$ and $\mathcal{I}[x,u]$ satisfies suitable nondegeneracy assumptions (that we discuss below) and the equation is strictly elliptic with respect to the integro-differential term.

- $A = \sigma^T\sigma$ *with* σ *a bounded, uniformly continuous function* which maps $\mathbb{R}^d$ into the space of $N \times p$-matrices for some $p \leqslant N$. It can be checked that

$$-\big(\mathrm{tr}\big(A(x)X\big) - \mathrm{tr}\big(A(y)Y\big)\big) \leqslant d\frac{\omega_\sigma^2(|x-y|)}{\varepsilon}$$

for any $X, Y \in \mathbb{S}^d$ satisfying inequality (4).
- The nonlocal term can be written as a general nonlocal operator

$$c(x)\mathcal{I}[x,u] = c(x)\int_{\mathbb{R}^d}\big(u(x+z) - u(x) - Du(x)\cdot z1_B(z)\big)\,\mu_x(dz)$$
$$= \int_{\mathbb{R}^d}\big(u(x+z) - u(x) - Du(x)\cdot z1_B(z)\big)c(x)\,\mu_x(dz)$$

where $(\mu_x)_x$ is a family of Lévy measures, satisfying assumptions $(M1)$–$(M3)$. When $c:\mathbb{R}^d \to \mathbb{R}$ *is* γ*-Hölder continuous* the results for general nonlocal operators literally apply for the new family of operators associated to the Lévy measures $\tilde{\mu}_x = c(x)\mu_x$.
For a Lévy–Itô type operator, the nonlocal term can be written as

$$c(x)\mathcal{I}[x,u] = c(x)\int_{\mathbb{R}^d}\big(u\big(x+j(x,z)\big) - u(x) - Du(x)\cdot j(x,z)1_B(z)\big)\,\mu(dz)$$
$$= \int_{\mathbb{R}^d}\big(u\big(x+j(x,z)\big) - u(x) - Du(x)\cdot j(x,z)1_B(z)\big)c(x)\,\mu(dz)$$

where the jump function $j(x,z)$ satisfies assumptions $(J1)$–$(J5)$. In this case, the results for general nonlocal operators do not apply ad litteram! Otherwise we could have considered Lévy–Itô operators as a particular case of general integro-differential operators. However, when c *is* γ*-Hölder continuous*, combining estimates arguments (see Section 6) used for Lévy–Itô operators with those for general nonlocal operators, we arrive to the same conclusion.
- $b:\mathbb{R}^d \to \mathbb{R}$ *is a* τ*-Hölder continuous function, or just a bounded continuous function*. The growth conditions k, r on the gradient are related to the regularity of coefficients of b.
When $\beta > 1$, the solution is Lipschitz continuous for gradient terms $b(x)|Du|^k$ with natural growth $k \leqslant \beta$ and b bounded. If in addition b is τ-Hölder continuous, then the solution remains Lipschitz for gradient terms with growth $k \leqslant \tau + \beta$. Similarly, the solution is Lipschitz for any term gradient term $|Du|^r$ with $r \leqslant \beta$.

4.1.2. Advection fractional diffusion equation

Several recent papers deal with the regularity of solutions for the advection fractional diffusion equation

$$u_t + (-\Delta_x)^{\beta/2}u + b(x)\cdot Du = f.$$

One distinguishes three cases, according to the order of fractional diffusion. The case $\beta < 1$ is known as the supercritical case, since the fractional diffusion is of lower order than the advection; conversely, $\beta > 1$ is the subcritical case. In between we have the critical value $\beta = 1$, when the drift and the diffusion are of the same order.

In the critical case, it was shown by Caffarelli and Vasseur [6] by using De Giorgi's approach that the solution is smooth for L^2 initial data, $f \equiv 0$, and divergence free vector fields b belonging to the

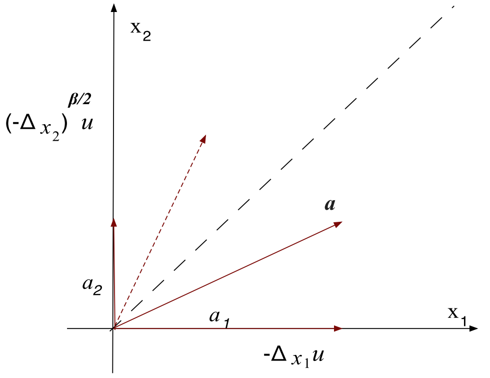

Fig. 2. Local diffusions occur only in x_1-directions and fractional diffusions in x_2-directions.

BMO class. The key step is to prove first that it is Hölder continuous. Their motivation comes from the quasi-geostrophic model in fluid mechanics. We mention that for smooth periodic initial data, Kiselev, Nazarov and Volberg [14] proved that the solution of the quasi-geostrophic equation remains smooth.

Recently, Silvestre [19] proved Hölder estimates for solutions of this equation (and nonlinear versions of it) by Harnack techniques. He also showed [20] that when $\beta \geqslant 1$ and the vector field b is $C^{1-\beta+\tau}$, the solution becomes $C^{1,\tau}$.

As we shall see in the following Section 5, our regularity results apply as well in the parabolic and/or non-periodic setting. Hence for such an equation (and nonlinear versions of it), we obtain that the solution is Lipschitz continuous in the subcritical case $\beta > 1$ with b bounded; hence the fractional diffusion is stronger than the advection and prescribes the regularity of the solution. In the supercritical case $\beta \leqslant 1$, the solution is β Hölder continuous whenever b is $C^{1-\beta+\tau}$, where $\tau > 0$.

4.2. *Mixed nonlinearities*

As discussed before, there is another interesting type of *mixed ellipticity*: at each point, the nonlinearity is *degenerate both in the second order term, and in the nonlocal term*, but *the combination of the local and the nonlocal diffusions renders the nonlinearity uniformly elliptic*. For this type of equations, partial regularity results apply first and then they are used to derive the global regularity.

4.2.1. *A toy-model for the mixed case*

The simplest example of pure mixed equations is given by

$$-\Delta_{x_1} u + (-\Delta_{x_2})^{\beta/2} u = f(x_1, x_2)$$

where $(-\Delta_{x_2})^{\beta/2} u$ denotes the fractional Laplacian with respect to the x_2-variable (see Fig. 2)

$$(-\Delta_{x_2})^{\beta/2} u = -\int_{\mathbb{R}^{d_2}} \big(u(x_1, x_2 + z_2) - u(x_1, x_2) - D_{x_2} u(x_1, x_2) \cdot z_2 1_B(z_2)\big) \frac{dz_2}{|z_2|^{d_2+\beta}}.$$

It is clear that the equation is degenerate both with respect to the local and the nonlocal term, as both the Laplacian and the fractional Laplacian are incomplete. Indeed, the directional classical Laplacian has all of the eigenvalues corresponding to the x_2 variable equal to zero, and therefore the nonlinearity F is degenerate with respect to the second order term $D^2 u$. On the other hand, the degeneracy with respect to the nonlocal term comes from the fact that

$$\mu(dz_2) = \frac{dz_2}{|z_2|^{d_2+\beta}}$$

could be viewed as the restriction of the fractional Laplacian to the subspace $\{z_1 = 0\}$

$$\nu(dz) = 1_{\{z_1=0\}}(dz_1)\,\mu(dz_2).$$

Therefore, for a cone whose direction a is orthogonal to the x_2-direction, we have

$$\int_{\mathcal{C}^d_{\eta,\delta}} |z|^2\,\nu(dz) = \int_{\mathcal{C}^{d_2}_{\eta,\delta}} |z_2|^2\,\mu(dz_2) = 0$$

where $\mathcal{C}^{d_2}_{\eta,\delta} = \{z_2 \in B^{d_2}_\delta;\ (1-\eta)|z_2||a| \leqslant |a_2 \cdot z_2|\}$. Thus, $(M2)$ and $(J2)$ fail and the Hölder regularity results of [1] do not apply.

Instead, the partial regularity results of Theorem 2 hold: the solution is Lipschitz continuous with respect to the x_2 variable when $\beta \geqslant 1$ and Hölder continuous when $\beta < 1$, and Lipschitz continuous with respect to the x_1 variable.

Remark 5. If we try to argue directly in $\mathbb{R}^d$ and apply the regularity result as if we had only one nonlinearity defined on the whole space, then the best result we can get is Hölder regularity of the solution, except for the diagonal direction, i.e. for all $\varepsilon \in (0,1]$ the following holds for all $\alpha \in (0,\varepsilon)$

$$u(x) - u(y) \leqslant C|x-y|^\alpha, \quad \forall x, y \in \mathbb{R}^d \quad \text{s.t.} \quad \max_{i=1,2} \frac{|x_i - y_i|}{|x-y|} \geqslant \sqrt{\frac{1}{2-\varepsilon}}.$$

In addition, the further we go from the diagonal, the better the regularity of the solution is.

Let us check that when the gradient direction is the diagonal between x_1 and x_2 it is not possible to retrieve Hölder continuity directly. For this purpose, consider two matrices X, Y satisfying inequality (4), with $Z = D\phi(a)$, where $\phi(z) = L|z|^\alpha$. Let $a = (a_1, a_2) = \bar{x} - \bar{y}$ be the gradient direction. The matrix inequality can be rewritten as follows

$$Xz \cdot z - Yz' \cdot z' \leqslant D^2\phi(a)\big(z - z'\big) \cdot \big(z - z'\big). \tag{27}$$

Estimate of the diffusion terms. Applying (27) to $z = -z' = e_1 = \frac{1}{|a_1|}(a_1, 0)$ and to $z = z' = (e, 0)$ for any unit vector e orthogonal to e_1 we obtain

$$\operatorname{tr}(X_1 - Y_1) \leqslant 4D^2\phi(a)e_1 \cdot e_1.$$

Therefore taking into account the expression for $D^2\phi(a) = \varphi'(|a|)\frac{1}{|a|}(I - \hat{a} \otimes \hat{a}) + \varphi''(|a|)\hat{a} \otimes \hat{a}$, we get

$$\operatorname{tr}(X_1 - Y_1) \leqslant 4\frac{\varphi'(|a|)}{|a|}\left(1 - \frac{|a_1|^2}{|a|^2}\right) + 4\varphi''\big(|a|\big)\frac{|a_1|^2}{|a|^2}.$$

Using that $\phi(z) = L|z|^\alpha$ with $\alpha \in (0,\varepsilon)$ and $L > 0$ the previous inequality reads

$$\operatorname{tr}(X_1 - Y_1) \leqslant 4L\alpha|a|^{\alpha-2}\left(1 + (\alpha - 2)\frac{|a_1|^2}{|a|^2}\right). \tag{28}$$

This expression is negative only if

$$\frac{|a_1|^2}{|a|^2} > \frac{1}{2-\varepsilon}.$$

Hence, when the gradient direction is "closer" to the x_1-axis, the classical diffusion gains and the regularity is driven by the classical Laplacian.

Estimate of the nonlocal terms. As already made precise, the ellipticity of the equation comes in this case from the nondegeneracy assumption $(M2)$ with respect to the Lévy measures. Accordingly, the estimate that renders the nonlocal difference negative comes from the evaluation on the cone in the gradient direction. In view of $(M2)$ we have by rough approximations (see Proposition 8 and its corollaries) that for $e_2 = \frac{1}{|a_2|}(0, a_2)$

$$\begin{aligned}
\mathcal{I}_{x_2}[\bar{x}, u] - \mathcal{I}_{x_2}[\bar{y}, u] &\leqslant \int_{\mathcal{C}_{\eta,\delta}} \sup_{|s|<1} \big(D^2_{a_2a_2}\phi\big(a + s(0, z_2)\big)z_2 \cdot z_2\big)\, \mu(dz_2) + cL\alpha|a|^{\alpha-2} \\
&= \int_{\mathcal{C}_{\eta,\delta}} \sup_{|s|<1} \bigg(\big(1-\tilde{\eta}^2\big)\frac{\varphi'(|a + s(0, z_2)|)}{|a + s(0, z_2)|} + \tilde{\eta}^2\varphi''\big(\big|a + s(0, z_2)\big|\big)\bigg)|z_2|^2\, \mu(dz) \\
&\quad + cL\alpha|a|^{\alpha-2} \\
&\leqslant C\bigg(\big(1-\tilde{\eta}^2\big)\frac{\varphi'(|a|)}{|a|}\bigg(1 - \frac{|a_2|^2}{|a|^2}\bigg) + \tilde{\eta}^2\varphi''\big(|a|\big)\frac{|a_2|^2}{|a|^2}\bigg) + cL\alpha|a|^{\alpha-2} \\
&= CL\alpha|a|^{\alpha-2}\bigg(1 + \tilde{\eta}^2(\alpha-2)\frac{|a_2|^2}{|a|^2}\bigg) + cL\alpha|a|^{\alpha-2}.
\end{aligned}$$

This expression is negative only if

$$\frac{|a_1|^2}{|a|^2} > \frac{1}{\tilde{\eta}^2(2-\varepsilon)}.$$

Similarly, when the gradient direction is "closer" to the x_2-axis, the fractional diffusion gains and the regularity is driven by the (directional) fractional Laplacian.

4.2.2. Mixed integro-differential equations with first order terms

Partial and global, Hölder and Lipschitz regularity results apply for a general class of mixed integro-differential equations. As pointed out in the previous theorems, the three nonlinearities must satisfy suitable strict ellipticity and growth conditions. The typical examples one can solve under those assumptions can be summed up by the following equation

$$-a_1(x_1)\Delta_{x_1}u - a_2(x_2)\mathcal{I}_{x_2}[x, u] - \mathcal{I}[x, u] + b_1(x_1)|D_{x_1}u_1|^{k_1} + b_2(x_2)|D_{x_2}u|^{k_2} + |Du|^n + cu = f(x)$$

where for $i = 1, 2$, $a_i(x_i) \geqslant 0$ and $a_i \in C^{0,\gamma}(\mathbb{R}^{d_i})$, $b_i \in C^{0,\tau}(\mathbb{R}^{d_i})$ with $0 \leqslant \tau \leqslant 1$, $k_i \in (0, 2+\tau)$, $n \geqslant 0$ and $c > 0$. We have thus considered

$$\begin{aligned}
F_0\big(u(x), Du, D^2u, \mathcal{I}[x, u]\big) &= -\mathcal{I}[x, u] + |Du|^n + cu, \\
F_1\big(x_1, D_{x_1}u, D^2_{x_1x_1}u, \mathcal{J}_{x_1}[x, u]\big) &= -a_1(x_1)\Delta_{x_1}u + b_1(x_1)|D_{x_1}u_1|^{k_1}, \\
F_2\big(x_2, D_{x_2}u, D^2_{x_2x_2}u, \mathcal{J}_{x_2}[x, u]\big) &= -a_2(x_2)\mathcal{I}_{x_2}[x, u] + b_2(x_2)|D_{x_2}u|^{k_2}.
\end{aligned}$$

Let us have a look at each of these terms and see the assumptions they have to satisfy, in order to ensure partial or global regularity of solutions. To fix ideas, suppose the nonlocal term $\mathcal{I}_{x_2}[x,u]$ is an integro-differential operator of fractional exponent $\beta \in (0,2)$.

In both situations, the nonlocal term $\mathcal{I}[x,u]$ can either be a general nonlocal operator associated to some Lévy measures μ^0 or a Lévy–Itô operator. We emphasize the fact that the associated Lévy measure has no x-dependency. This explains as well the lack of any coefficient $a_0(x)$ in front of the nonlocal term $\mathcal{I}[x,u]$. The gradient term $|Du|^n$ is allowed to have any possible growth $n \geqslant 0$.

As far as we are interested in partial regularity results, the constant c may be any real number, since we just need cu to be bounded. Yet, when combining the partial regularity results to obtain global regularity, F_1 and F_2 are submitted to rather restrictive assumptions, due to the uniqueness requirements. Thus, when b_1 and b_2 depend explicitly on x_1, respectively x_2 the corresponding gradient terms are restrained to sublinear growth. To turn around this difficulty and obtain regularity of solutions in superlinear cases, one can argue by approximation, truncating the gradient terms and using Corollary 5 for obtaining uniform gradient bounds. To perform this program, c must be positive: $c > 0$.

We first discuss the *partial regularity* of the solution with respect to each of its variables. To this end, we need classical regularity assumptions in one set of variables, and uniqueness type assumptions in the other variables.

Partial regularity in x_2-variable requires ellipticity of the equation in x_2 direction:

$$\forall x_1 \in \mathbb{R}^{d_1},\ x_2 \in \mathbb{R}^{d_2} \quad a_1(x_1) \geqslant 0 \quad \text{and} \quad a_2(x_2) > 0.$$

To ensure the uniqueness argument in x_1-variable, we must take $a_1(x) = \sigma_1(x)^2$ with σ_1 a Lipschitz continuous function. The nonlocal term $\mathcal{I}_{x_2}[x,u]$ is either a general integro-differential operator or a Lévy–Itô operator.

When $\beta > 1$, the solution is Lipschitz continuous in the x_2 variable for directional gradient terms $b_2(x_2)|D_{x_2}u|^{k_2}$ having a natural growth $k_2 \leqslant \beta$ if b_2 is bounded *and* directional gradient terms $b_1(x_1)|D_{x_1}u|^{k_1}$ with linear growth $k_1 = 1$ if b_1 is Lipschitz (or sublinear growth $k_1 < 1$ if $b_1 \in C^{0,k_1}$). If in addition b_2 is τ-Hölder continuous, then the solution remains Lipschitz for gradient terms up to growth $k_2 \leqslant \tau + \beta$. When $\beta \leqslant 1$, the solution is α-Hölder continuous for any $\alpha < \frac{\beta - k_2}{1 - k_2}$.

Partial regularity in x_1-variable requires nondegeneracy of the equation in x_1 direction

$$a_1(x_1) > 0, \quad \forall x_1 \in \mathbb{R}^{d_1}.$$

In this case, in the x_2 variable, we can only deal with nonlocal operators of Lévy–Itô type $\mathcal{I}_{x_2}[x,u] = \mathcal{J}_{x_2}[x,u]$, for which the jump function is Lipschitz continuous and satisfies the structural conditions $(J1)$, $(J4)$ and $(J5)$. The uniqueness constraint with respect to x_2 does not allow any x_2-dependence of the Lévy measure associated to the nonlocal term, and hence $a_2(x_2)$ should be a constant function.

Then the solution is Lipschitz in the x_1 variable, for directional gradient terms $b_1(x_1)|D_{x_1}u|^{k_1}$ having a natural growth $k_1 \leqslant 2 + \tau$ with $b_1 \in C^{0,\tau}(\mathbb{R}^{d_1})$, $0 \leqslant \tau \leqslant 1$. Once again, the uniqueness hypothesis forces directional gradient terms $b_2(x_2)|D_{x_2}u|^{k_2}$ to have growth $k_2 = 1$ and b_2 is Lipschitz continuous.

Global regularity holds under slightly weaker assumptions than the partial regularity. It follows by interchanging the roles of x_1 and x_2. Accordingly, the equation must be strongly elliptic both in the local and nonlocal term

$$a_1(x_1) > 0 \quad \text{and} \quad a_2(x_2) > 0, \quad \forall x_1 \in \mathbb{R}^{d_1},\ x_2 \in \mathbb{R}^{d_2}.$$

The nonlocal term $\mathcal{I}_{x_2}[x,u]$ is necessarily a Lévy–Itô operator, satisfying the nondegeneracy assumption $(J2)$, as well as the rest of structural conditions $(J1)$–$(J5)$. In addition

$$a_1(x_1) = \sigma_1(x_1)^2 > 0$$

with σ_1 Lipschitz continuous and $a_2(x) \equiv a_2 > 0$ constant function.

Joining the partial Lipschitz regularity results, we get Lipschitz continuity of the solution whenever b_1 and b_2 are Lipschitz continuous for linear, directional gradient terms $b_1(x_1)|D_{x_1}u|$ and $b_2(x_2)|D_{x_2}u|$. The linear growth is constraint by the uniqueness argument.

However, looking at the approximated equations with $|Du|$ replaced by $|Du| \wedge R$, for $R > 0$ and noting that the solutions are Lipschitz continuous, with the Lipschitz norm independent of R when $c > 0$, we obtain Lipschitz continuous viscosity solutions for general equations, dealing with gradient terms of growth $k_1 \leqslant 2, k_2 \leqslant \tau + \beta$, when $b_2 \in C^{0,\tau}(\mathbb{R}^{d_2})$. Similarly, we get α-Hölder continuous solutions, for any $\alpha < \frac{\beta - k_2}{1-k_2} \leqslant 1$.

5. Extensions

5.1. Non-periodic setting

Theorem 6. *Let f be continuous, the nonlinearities F_i, $i = 0, 1, 2$, be degenerate elliptic, continuous, such that F_0 satisfies $(H0)$ with $\tilde{\gamma} > 0$ and $(H2)$, and that both F_i, for $i = 1, 2$, satisfy assumptions $(H2)$ and $(H1')$, with $\tilde{d} = d_i$, for some functions Λ_i^1, Λ_i^2 and some constants $k_i \geqslant 0$, $\tau_i, \theta_i, \tilde{\theta}_i \in (0, 1]$, where:*

$(H1')$ *There exist two functions $\Lambda^1, \Lambda^2 : \mathbb{R}^{\tilde{d}} \to [0, \infty)$ such that $\Lambda^1(x) + \Lambda^1(x) \geqslant \Lambda^0 > 0$ and for each $0 < R < \infty$ there exist some constants $k \geqslant 0$, $\tau, \theta, \tilde{\theta} \in (0, 1]$ such that for any $x, y \in \mathbb{R}^{\tilde{d}}$, $p, q \in \mathbb{R}^{\tilde{d}}$, $|q| < R$, $l \leqslant l'$ and any $\varepsilon > 0$*

$$\begin{aligned} &F\big(y, p, Y, l'\big) - F(x, p+q, X, l) \\ &\quad \leqslant \Lambda_1(x)\left(\big(l - l'\big) + \frac{|x-y|^{2\theta}}{\varepsilon} + |x-y|^{\tau}|p|^{k+\tau} + C^1|p|^k\right) \\ &\qquad + \Lambda_2(x)\left(\operatorname{tr}(X-Y) + \frac{|x-y|^{2\tilde{\theta}}}{\varepsilon} + |x-y|^{\tau}|p|^{2+\tau} + C^2|p|^2\right) + O(K, R) \end{aligned}$$

if $X, Y \in \mathbb{S}^{\tilde{d}}$ satisfy inequality

$$-\frac{1}{\varepsilon}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leqslant \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqslant \frac{1}{\varepsilon}\begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix} + K\begin{bmatrix} I & 0 \\ -0 & 0 \end{bmatrix},$$

for some $Z = I - \omega\hat{a} \otimes \hat{a}$, with $\hat{a} \in \mathbb{R}^d$ a unit vector, and $\omega \in (1, 2)$.

Let μ^0, μ^i, with $i = 1, 2$ and $j^i(x_i, z_i)$ satisfy assumptions $(J1)$–$(J5)$ for some constants β_i, C_{μ^i}, $\tilde{C}_{\mu^i}$, with $\gamma = 1$ in $(J3)$. Then any bounded continuous viscosity solution u of (23) is

(a) *locally Lipschitz continuous, if $\beta_i > 1$ and $k_i \leqslant \beta_i$ for $i = 1, 2$, and*
(b) *locally $C^{0,\alpha}$ continuous with $\alpha < \min(\frac{\beta_1 - k_1}{1-k_1}, \frac{\beta_2 - k_2}{1-k_2})$, if $\beta \leqslant 1$ and $k_i < \beta_i$ for $i = 1, 2$.*

The Lipschitz/Hölder constant depends on $\|u\|_\infty$, on the dimension d of the space and on the constants associated to the Lévy measures and on the constants required by the growth condition $(H1)$.

Sketch of the proof. The fact that the solution is not periodic anymore, requires a localization term when measuring the shift of the solution. Thus, in order to prove the local continuity of the solution, either if it refers to Hölder or Lipschitz, we need to show that for each x^0 in the domain, there exists a constant K, depending on x^0, such that for a proper choice of α (both in the Hölder in the Lipschitz case) there exists a constant L, depending on x^0, large enough such that the auxiliary function

$$\psi(x_1, y_1, x_2) = u(x_1, x_2) - u(y_1, x_2) - L\varphi\big(|x_1 - y_1|\big) - \frac{K}{2}\big|(x_1, x_2) - \big(x_1^0, x_2^0\big)\big|^2$$

attains a nonpositive maximum. The proof is technically the same, except that here there will be an additional contribution in the estimate of the nonlocal terms, coming from the localization term. The point is to show that this contribution is of order $O(K)$. □

5.2. Parabolic integro-differential equations

The techniques previously developed apply literally to parabolic integro-differential equations.

Corollary 7. *Let f, the nonlinearities F_i and the jump functions $j^i(x_i, z_i)$ satisfy the assumptions of Corollary 5. If, for some $T > 0$, $u : [0, T) \times \mathbb{R}^d \to \mathbb{R}$ is an x-periodic, continuous viscosity solution of*

$$\begin{aligned} &u_t + F_0\big(u(x), Du, D^2u, \mathcal{I}[x, u]\big) + F_1\big(x_1, D_{x_1}u, D^2_{x_1x_1}u, \mathcal{I}_{x_1}[x, u]\big) \\ &\quad + F_2\big(x_2, D_{x_2}u, D^2_{x_2x_2}u, \mathcal{I}_{x_2}[x, u]\big) = f(x) \quad \text{in } (0, T) \times \mathbb{R}^d. \end{aligned} \tag{29}$$

(a) *If $\beta_i > 1$, $k_i \leqslant \beta_i$ for $i = 1, 2$ and if $u_0 \in Lip(\mathbb{R}^d)$, then u is Lipschitz continuous with respect to x on $[0, T]$.*
(b) *If $\beta \leqslant 1$, $k_i < \beta_i$ for $i = 1, 2$ and if $u_0 \in C^{0,\alpha}(\mathbb{R}^d)$, then u is $C^{0,\alpha}$ with respect to x on $[0, T]$, with $\alpha < \min(\frac{\beta_1 - k_1}{1 - k_1}, \frac{\beta_2 - k_2}{1 - k_2})$.*

The Lipschitz/Hölder constant depends on $\|u\|_\infty$, on the dimension d of the space and on the constants associated to the Lévy measures and on the constants required by the growth condition $(H1)$.

Sketch of the proof. The key difference with the previous proof consists in considering the space–time auxiliary function

$$\psi(t, x_1, y_1, x_2) = u(t, x_1, x_2) - u(t, y_1, x_2) - \phi(x_1 - y_1)$$

and show that $\max_{t, x_1, x_2, y_2} \psi(t, x_1, y_1, x_2) < 0$. By small space–time perturbations

$$\psi_{\varepsilon,\varsigma}(x, y, s, t) = u(t, x_1, x_2) - u(s, y_1, y_2) - \phi(x_1 - y_1) - \frac{|x_2 - y_2|^2}{\varepsilon^2} - \frac{(t - s)^2}{\varsigma^2},$$

this leads to considering in the nonlocal Jensen–Ishii's lemma the parabolic sub- and super-jets

$$\begin{aligned} &\big(r^{\varepsilon,\varsigma}, p^{\varepsilon,\varsigma}, X^{\varepsilon,\varsigma}\big) \in \mathcal{D}_p^{2,+}\big(u\big(x^{\varepsilon,\varsigma}\big)\big), \\ &\big(r^{\varepsilon,\varsigma}, p^{\varepsilon,\varsigma}, Y^{\varepsilon,\varsigma}\big) \in \mathcal{D}_p^{2,-}\big(u\big(y^{\varepsilon,\varsigma}\big)\big) \end{aligned}$$

with $r^{\varepsilon,\varsigma} = 2\frac{t-s}{\varsigma^2}$. Writing down the viscosity inequalities, note that the $r^{\varepsilon,\varsigma}$ is the common term corresponding to the first order time-derivative, and hence it vanishes by subtraction. Therefore, when passing to the limits in inequality (17), we can first let ς go to zero. The rest of the proof is literally the same. □

5.3. Bellman–Isaacs equations

These results can be extended to fully nonlinear equations, that arise naturally in stochastic control problems for jump-diffusion processes. The following Bellman–Isaacs type equation arises

$$\sup_{\gamma\in\Gamma}\inf_{\delta\in\Delta}\big(F_0^{\gamma,\delta}\big(\ldots,\mathcal{J}^{\gamma,\delta}[x,u]\big)+F_1^{\gamma,\delta}\big(\ldots,\mathcal{J}_{x_1}^{\gamma,\delta}[x,u]\big)+F_2^{\gamma,\delta}\big(\ldots,\mathcal{J}_{x_2}^{\gamma,\delta}[x,u]\big)-f^{\gamma,\delta}(x)\big)=0$$

where $\mathcal{J}^{\gamma,\delta}[x,u]$ is a family of Lévy–Itô operators associated with a common Lévy measure μ^0 and a family of jump functions $j_0^{\gamma,\delta}(x,z)$, respectively $\mathcal{J}_{x_i}^{\gamma,\delta}[x,u]$ are families of Lévy–Itô operators associated with the Lévy measures μ^i and the families of jump functions $j_i^{\gamma,\delta}(x_i,z)$, for $i=1,2$.

A typical (and practical) example is

$$F_0^{\gamma,\delta}=cu-\frac{1}{2}\operatorname{tr}\big(A^{\gamma,\delta}(x)D^2u\big)-\mathcal{J}^{\gamma,\delta}[x,u]-b^{\gamma,\delta}(x)\cdot Du,$$
$$F_i^{\gamma,\delta}=\quad-\frac{1}{2}\operatorname{tr}\big(a_i^{\gamma,\delta}(x_i)D^2_{x_ix_i}u\big)-\mathcal{J}_{x_i}^{\gamma,\delta}[x,u]-b_i^{\gamma,\delta}(x)\cdot D_{x_i}u.$$

Similar techniques to the previous ones yield the Hölder and Lipschitz continuity of solutions of Bellman–Isaacs equations, provided that the structure condition $(H1)$ is uniformly satisfied by $F_i^{\gamma,\delta}$, for $i=1,2$, as well as the assumptions $(J1)$–$(J5)$ by the family of jump functions $j_i^{\gamma,\delta}(x_i,z)$. In occurrence, the constants and functions appearing therein must be independent of γ and δ. For the above example, it is sufficient that $A^{\gamma,\delta}(x), a_i^{\gamma,\delta}(x), b_i^{\gamma,\delta}(x), f^{\gamma,\delta}(x)$ are bounded in $W^{1,\infty}$, uniformly in γ and δ.

The proof is based on the classical inequality

$$\sup_{\gamma}\inf_{\delta}\big(F^{\gamma,\delta}\big(\ldots,\mathcal{J}^{\gamma,\delta}[x,u]\big)\big)-\sup_{\gamma}\inf_{\delta}\big(F^{\gamma,\delta}\big(\ldots,\mathcal{J}^{\gamma,\delta}[y,u]\big)\big)$$
$$\leqslant\sup_{\gamma,\delta}\big(F^{\gamma,\delta}\big(\ldots,\mathcal{J}^{\gamma,\delta}[x,u]\big)-F^{\gamma,\delta}\big(\ldots,\mathcal{J}^{\gamma,\delta}[y,u]\big)\big).$$

5.4. Multiple nonlinearities

The problem can be easily generalized to multiple nonlinearities

$$F_0\big(u(x),Du,D^2u,\mathcal{I}[x,u]\big)+\sum_{i\in I}F_i\big(x_i,D_{x_i}u,D^2_{x_ix_i}u,\mathcal{J}_{x_i}[x,u]\big)=f(x). \tag{30}$$

The proof can be reduced to the previous one, by grouping all the variables for which we employ uniqueness type arguments.

6. Estimates for integro-differential operators

All these results are based on a series of estimates for the nonlocal terms, that we make precise in the following. They are similar to those in [1]. As we have seen, the proof of the Lipschitz regularity of solutions uses Hölder continuity of solutions for small orders $\alpha\in(0,\frac{1}{d+1})$, where d is the dimension of the space. For this reason, the estimates below are first given in a general form, such that they can be used for both regularity proofs. We then state as corollaries their precise form for Lipschitz and Hölder case.

6.1. General nonlocal operators

We first give some estimates for general nonlocal operators

$$\mathcal{I}[x,u] = \int_{\mathbb{R}^d} \big(u(x+z) - u(x) - Du(x)\cdot z1_B\big)\,\mu_x(dz).$$

We begin with a general result on concave estimates for these integro-differential operators, under quite general assumptions. We then derive finer estimates in the particular case of Lipschitz and Hölder control functions. However, these special forms will hold for family of Lévy measures $(\mu_x)_x$ which satisfy some additional assumptions.

Proposition 8 *(Concave estimates – general nonlocal operators). Assume condition* $(M1)$ *holds. Let* u, v *be two bounded functions and* $\varphi : [0,\infty) \to \mathbb{R}$ *be a smooth increasing concave function. Define*

$$\psi(x,y) = u(x) - v(y) - \varphi\big(|x-y|\big)$$

and assume the maximum of ψ *is positive and reached at* $(\bar{x},\bar{y})$, *with* $\bar{x} \neq \bar{y}$. *Let*

$$a = \bar{x} - \bar{y}, \qquad \hat{a} = a/|a|, \qquad p = \varphi'\big(|a|\big)\hat{a}.$$

Then the following holds

$$\begin{aligned}
&\mathcal{I}[\bar{x},p,u] - \mathcal{I}[\bar{y},p,v] \\
&\quad\leqslant 4\tilde{C}_\mu \max\big(\|u\|_\infty, \|v\|_\infty\big) \\
&\qquad + \frac{1}{2}\int_{\mathcal{C}_{\eta,\delta}(a)} \sup_{|s|\leqslant 1}\left(\big(1-\tilde{\eta}^2\big)\frac{\varphi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\varphi''\big(|a+sz|\big)\right)|z|^2\,(\mu_{\bar{x}}+\mu_{\bar{y}})(dz) \\
&\qquad + 2\varphi'\big(|a|\big)\int_{B\setminus B_\delta} |z||\mu_{\bar{x}}-\mu_{\bar{y}}|(dz) + \int_{B_\delta\setminus\mathcal{C}_{\eta,\delta}(a)} \sup_{|s|\leqslant 1}\frac{\varphi'(|a+sz|)}{|a+sz|}|z|^2\,|\mu_{\bar{x}}-\mu_{\bar{y}}|(dz),
\end{aligned}$$

where

$$\mathcal{C}_{\eta,\delta}(a) = \big\{z \in B_\delta;\ (1-\eta)|z||a| \leqslant |a\cdot z|\big\}$$

and $\delta = |a|\delta_0 > 0$, $\tilde{\eta} = \frac{1-\eta-\delta_0}{1+\delta_0} > 0$ *with* $\delta_0 \in (0,1)$, $\eta \in (0,1)$ *small enough.*

Remark 6. The aperture of the cone is given by η and changes according to $|a|$. In order to ensure Lipschitz continuity of solutions, η must be chosen to behave like a power of $|a|$, i.e. $\eta \sim |a|^\alpha$, and thus is diminishing as the modulus of the gradient approaches zero: $\lim_{|a|\to 0}\eta(|a|) = 0$. Remark that as $|a| \to 0$, $\mathcal{C}_{\eta,\delta}(a)$ degenerates to the line whose direction is given by the gradient. This will be made precise when proving Corollary 10 below.

Corollary 9 *(Lipschitz estimates). Let* $(M1)$–$(M3)$ *hold, with* $\beta > 1$. *Under the assumptions of Proposition* 8 *with*

$$\varphi(t) = \begin{cases} L(t - \varrho t^{1+\alpha}), & t \in [0,t_0], \\ \varphi(t_0), & t > t_0 \end{cases}$$

where $\alpha \in (0, \min(\frac{\gamma}{d+1}, \frac{\beta-1}{d+2-\beta}))$, ϱ *is a constant such that* $\varrho\alpha 2^{\alpha-1} > 1$, $t_0 = \max_t(t - \varrho t^{1+\alpha}) = \sqrt[\alpha]{\frac{1}{\rho(1+\alpha)}}$ *and* $L > \frac{(\|u\|_\infty+\|v\|_\infty)(\alpha+1)}{t_0\alpha}$, *the following holds*: *there exists a positive constant* $C = C(\mu)$ *such that for* $\Theta(\varrho, \alpha, \mu) = C(\rho\alpha 2^{\alpha-1} - 1)$ *we have*

$$\mathcal{I}[\bar{x}, p, u] - \mathcal{I}[\bar{y}, p, v] \leqslant -L|a|^{(1-\beta)+\alpha(d+2-\beta)}\left\{\Theta(\varrho, \alpha, \mu) - o_{|a|}(1)\right\} + O(\tilde{C}_\mu).$$

Corollary 10 *(Hölder estimates). Let* (*M*1)–(*M*3) *hold, with* $\beta \in (0, 2)$. *Under the assumptions of Proposition* 8 *with*

$$\varphi(t) = \begin{cases} Lt^\alpha, & t \in [0, t_0], \\ \varphi(t_0), & t > t_0 \end{cases}$$

where $\alpha \in (0, \min(\beta, 1))$, $t_0 > 0$, *and* $L > \frac{\|u\|_\infty+\|v\|_\infty}{t_0{}^\alpha}$, *the following holds*: *there exists a positive constant* $C(\mu) > 0$ *such that*

$$\mathcal{I}[\bar{x}, p, u] - \mathcal{I}[\bar{y}, p, v] \leqslant -L|a|^{\alpha-\beta}\left\{\alpha C(\mu) - o_{|a|}(1)\right\} + O(\tilde{C}_\mu).$$

Proof of Proposition 8. We split the domain of integration into three pieces and take the integrals on each of these domains. Namely we part the ball B_δ of radius δ into the subset $\mathcal{C}_{\eta,\delta}(a)$ with $\eta = \eta(|a|)$ and $\delta = \delta(|a|)$, and its complementary $B_\delta \setminus \mathcal{C}_{\eta,\delta}(a)$. We write the difference of the nonlocal terms, corresponding to the maximum point $(\bar{x}, \bar{y})$, as the sum

$$\mathcal{I}[\bar{x}, p, u] - \mathcal{I}[\bar{y}, p, v] = \mathcal{T}^1(\bar{x}, \bar{y}) + \mathcal{T}^2(\bar{x}, \bar{y}) + \mathcal{T}^3(\bar{x}, \bar{y})$$

where

$$\mathcal{T}^1(\bar{x}, \bar{y}) = \int_{\mathbb{R}^d \setminus B} \big(u(\bar{x}+z) - u(\bar{x})\big)\,\mu_{\bar{x}}(dz) - \int_{\mathbb{R}^d \setminus B} \big(v(\bar{y}+z) - v(\bar{y})\big)\,\mu_{\bar{y}}(dz),$$

$$\mathcal{T}^2(\bar{x}, \bar{y}) = \int_{\mathcal{C}_{\eta,\delta}(a)} \big(u(\bar{x}+z) - u(\bar{x}) - p\cdot z\big)\,\mu_{\bar{x}}(dz) - \int_{\mathcal{C}_{\eta,\delta}(a)} \big(v(\bar{y}+z) - v(\bar{y}) - p\cdot z\big)\,\mu_{\bar{y}}(dz),$$

$$\mathcal{T}^3(\bar{x}, \bar{y}) = \int_{B \setminus \mathcal{C}_{\eta,\delta}(a)} \big(u(\bar{x}+z) - u(\bar{x}) - p\cdot z\big)\,\mu_{\bar{x}}(dz) - \int_{B \setminus \mathcal{C}_{\eta,\delta}(a)} \big(v(\bar{y}+z) - v(\bar{y}) - p\cdot z\big)\,\mu_{\bar{y}}(dz).$$

Let $\phi(z) = \varphi(|z|)$. Then $p = D\phi(a)$. Since $(\bar{x}, \bar{y})$ is a maximum point of $\psi(\cdot,\cdot)$, we have that

$$u(\bar{x}+z)-u(\bar{x})-p\cdot z\leqslant v\big(\bar{y}+z'\big)-v(\bar{y})-p\cdot z' \\ +\phi\big(a+z-z'\big)-\phi(a)-D\phi(a)\cdot\big(z-z'\big). \tag{31}$$

In the following we give estimates for each of these integral terms, using inequality (31) and properties of the Lévy measures $(\mu_x)_x$.

Lemma 11. *$\mathcal{T}^1(\bar{x},\bar{y})$ is uniformly bounded with respect to all parameters. More precisely*

$$\mathcal{T}^1(\bar{x},\bar{y})\leqslant 4\max\big(\|u\|_\infty,\|v\|_\infty\big)\sup_{x\in\mathbb{R}^d}\mu_x\big(\mathbb{R}^d\setminus B\big).$$

Proof. Since the functions u and v are bounded, we immediately deduce that

$$\mathcal{T}^1(\bar{x},\bar{y})\leqslant 2\|u\|_\infty\int_{\mathbb{R}^d\setminus B}\mu_{\bar{x}}(dz)+2\|v\|_\infty\int_{\mathbb{R}^d\setminus B}\mu_{\bar{y}}(dz).$$

We conclude by recalling that the measures μ_x are uniformly bounded away from the origin, by assumption $(M1)$. □

Lemma 12. *Let $\delta=|a|\delta_0$ with $\delta_0\in(0,1)$ small, η be small enough such that $1-\eta-\delta_0>0$ and*

$$\tilde{\eta}=\frac{1-\eta-\delta_0}{1+\delta_0}.$$

Then the nonlocal term $\mathcal{T}^2$ satisfies

$$\mathcal{T}^2(\bar{x},\bar{y})\leqslant\frac{1}{2}\int_{\mathcal{C}_{\eta,\delta}(a)}\sup_{|s|\leqslant 1}\left(\big(1-\tilde{\eta}^2\big)\frac{\varphi'(|a+sz|)}{|a+sz|}+\tilde{\eta}^2\varphi''\big(|a+sz|\big)\right)|z|^2\,(\mu_{\bar{x}}+\mu_{\bar{y}})(dz).$$

Remark 7. The previous notations have been introduced to simplify the form of the estimates. It is important to note however that the coefficients appearing in the convex combination of the derivatives of φ depend explicitly on $\tilde{\eta}$ and not on the aperture of the cone, given in terms of η. We eventually set $\eta\sim|a|^{2\alpha}$ and $\delta_0\sim|a|^\alpha$, thus we expect to have $\tilde{\eta}\simeq 1$. Consequently, the second derivative of φ would dominate the nonlocal difference and would render $\mathcal{T}^2(\bar{x},\bar{y})$ as negative as needed.

Proof of Lemma 12. Taking $z'=0$ and $z=0$ in inequality (31) we have

$$u(\bar{x}+z)-u(\bar{x})-p\cdot z\leqslant\phi(a+z)-\phi(a)-D\phi(a)\cdot z,$$
$$-\big(v\big(\bar{y}+z'\big)-v(\bar{y})-p\cdot z'\big)\leqslant\phi\big(a-z'\big)-\phi(a)+D\phi(a)\cdot z'.$$

Therefore

$$\mathcal{T}^2(\bar{x},\bar{y})\leqslant\int_{\mathcal{C}_{\eta,\delta}(a)}\big(\phi(a+z)-\phi(a)-D\phi(a)\cdot z\big)\,\mu_{\bar{x}}(dz) \\ +\int_{\mathcal{C}_{\eta,\delta}(a)}\big(\phi\big(a-z'\big)-\phi(a)+D\phi(a)\cdot z'\big)\,\mu_{\bar{y}}\big(dz'\big).$$

Using Taylor's formula with integral reminder, the right-hand side can be rewritten as

$$\frac{1}{2}\int_0^1 (1-s)\,ds \int_{\mathcal{C}_{\eta,\delta}(a)} \big(D^2\phi(a+sz)z\cdot z\big)\,\mu_{\bar{x}}(dz) + \frac{1}{2}\int_{-1}^0 (1+s)\,ds \int_{\mathcal{C}_{\eta,\delta}(a)} \big(D^2\phi(a+sz)z\cdot z\big)\,\mu_{\bar{y}}(dz).$$

Remark that the first and second derivatives of $\phi(z) = \varphi(|z|)$ are given by the formulas

$$D\phi(z) = \varphi'\big(|z|\big)\hat{z},$$
$$D^2\phi(z) = \frac{\varphi'(|z|)}{|z|}(I - \hat{z}\otimes\hat{z}) + \varphi''\big(|z|\big)\hat{z}\otimes\hat{z},$$

and in particular

$$D^2\phi(a+sz)z\cdot z = \frac{\varphi'(|a+sz|)}{|a+sz|}\big(|z|^2 - \big|\widehat{(a+sz)}\cdot z\big|^2\big) + \varphi''\big(|a+sz|\big)\big|\widehat{(a+sz)}\cdot z\big|^2.$$

On the set $\mathcal{C}_{\eta,\delta}(a)$ we have the following upper and lower bounds

$$|a+sz| \geqslant |a| - |s||z| \geqslant |a| - \delta = |a|(1-\delta_0),$$
$$|a+sz| \leqslant |a| + |s||z| \leqslant |a| + \delta = |a|(1+\delta_0),$$
$$\big|(a+sz)\cdot z\big| \geqslant |a\cdot z| - s|z|^2 \geqslant |a\cdot z| - \delta|z| \geqslant (1-\eta-\delta_0)|z||a|. \tag{32}$$

Hence we deduce that for all $s \in (-1,1)$

$$\big|\widehat{(a+sz)}\cdot z\big| \geqslant \tilde{\eta}|z| \quad \text{with } \tilde{\eta} = \frac{1-\eta-\delta_0}{1+\delta_0}. \tag{33}$$

Recalling that φ is increasing and concave, we get

$$D^2\phi(a+sz)z\cdot z \leqslant \big(1-\tilde{\eta}^2\big)\frac{\varphi'(|a+sz|)}{|a+sz|}|z|^2 + \tilde{\eta}^2\varphi''\big(|a+sz|\big)|z|^2.$$

This implies that the integral terms corresponding to ϕ are bounded by

$$\frac{1}{2}\int_{\mathcal{C}_{\eta,\delta}(a)} \sup_{|s|\leqslant 1}\left(\big(1-\tilde{\eta}^2\big)\frac{\varphi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\varphi''\big(|a+sz|\big)\right)|z|^2\,(\mu_{\bar{x}}+\mu_{\bar{y}})(dz),$$

which concludes the proof of the lemma. □

Lemma 13. *The following estimate holds*

$$\mathcal{T}^3(\bar{x},\bar{y}) \leqslant \int_{B_\delta\setminus\mathcal{C}_{\eta,\delta}(a)} \sup_{|s|\leqslant 1}\frac{\varphi'(|a+sz|)}{|a+sz|}|z|^2\,|\mu_{\bar{x}}-\mu_{\bar{y}}|(dz) + 2\varphi'\big(|a|\big)\int_{B\setminus B_\delta} |z|\,|\mu_{\bar{x}}-\mu_{\bar{y}}|(dz).$$

Proof. When estimating the nonlocal term outside the cone, one has to keep it as small as possible, though positive. Therefore we consider, as in [1] the signed measure $\mu = \mu_{\bar{x}} - \mu_{\bar{y}}$. Consider its Jordan decomposition $\mu = \mu^+ - \mu^-$ and denote by $|\mu|$ the corresponding total variation measure. Then, if K is the support of the positive variation μ^+, one can define the minimum of the two measures as

$$\mu_* = \mathbf{1}_K \mu_{\bar{y}} + (1 - \mathbf{1}_K)\mu_{\bar{x}}.$$

But then, the measures $\mu_{\bar{x}}$ and $\mu_{\bar{y}}$ can be rewritten as $\mu_{\bar{x}} = \mu_* + \mu^+$ and $\mu_{\bar{y}} = \mu_* + \mu^-$. With these notations in mind, we rewrite the nonlocal term $\mathcal{T}^3$ as

$$\begin{aligned}\mathcal{T}^3(\bar{x}, \bar{y}) = &\int_{B \setminus \mathcal{C}_{\eta,\delta}(a)} \big(u(\bar{x} + z) - u(\bar{x}) - p \cdot z - \big(v(\bar{y} + z) - v(\bar{y}) - p \cdot z\big)\big)\, \mu_*(dz) \\ &+ \int_{B \setminus \mathcal{C}_{\eta,\delta}(a)} \big(u(\bar{x} + z) - u(\bar{x}) - p \cdot z\big)\, \mu^+(dz) \\ &- \int_{B \setminus \mathcal{C}_{\eta,\delta}(a)} \big(v(\bar{y} + z) - v(\bar{y}) - p \cdot z\big)\, \mu^-(dz).\end{aligned}$$

Choosing successively $z' = z$, $z' = 0$ and $z = 0$ in (31) and noting that

$$u(\bar{x} + z) - u(\bar{x}) - p \cdot z \leqslant v(\bar{y} + z) - v(\bar{y}) - p \cdot z$$

we deduce that

$$\begin{aligned}\mathcal{T}^3(\bar{x}, \bar{y}) \leqslant &\int_{B \setminus \mathcal{C}_{\eta,\delta}(a)} \big(\phi(a + z) - \phi(a) - D\phi(a) \cdot z\big)\, \mu^+(dz) \\ &+ \int_{B \setminus \mathcal{C}_{\eta,\delta}(a)} \big(\phi(a - z) - \phi(a) + D\phi(a) \cdot z\big)\, \mu^-(dz).\end{aligned}$$

For estimating the integral terms corresponding to ϕ, we split the domain of integration into $B \setminus B_\delta$ and $B_\delta \setminus \mathcal{C}_{\eta,\delta}(a)$. On the first set, from the monotonicity and the concavity of φ we have

$$\begin{aligned}\phi(a + z) - \phi(a) - D\phi(a) \cdot z &\leqslant \varphi\big(|a| + |z|\big) - \varphi\big(|a|\big) - \varphi'\big(|a|\big)\hat{a} \cdot z \\ &\leqslant 2\varphi'\big(|a|\big)|z|.\end{aligned}$$

On $B_\delta \setminus \mathcal{C}_{\eta,\delta}(a)$ we use a second order Taylor expansion and we take into account that φ is smooth, $\varphi' \geqslant 0$ and $\varphi'' \leqslant 0$ to obtain the upper bound

$$\begin{aligned}\sup_{|s| \leqslant 1} \big(\phi(a + sz) - \phi(a) - D\phi(a) \cdot z\big) &\leqslant \sup_{|s| \leqslant 1} D^2\phi(a + sz)z \cdot z \\ &\leqslant \sup_{|s| \leqslant 1} \frac{\varphi'(|a + sz|)}{|a + sz|}|z|^2.\end{aligned}$$

Therefore we get the estimate

$$\mathcal{T}^3(\bar{x},\bar{y}) \leqslant 2\varphi'\big(|a|\big) \int_{B\setminus B_\delta} |z|\,|\mu_{\bar{x}} - \mu_{\bar{y}}|(dz) + \int_{B_\delta\setminus \mathcal{C}_{\eta,\delta}(a)} \sup_{|s|\leqslant 1} \frac{\varphi'(|a+sz|)}{|a+sz|}|z|^2\,|\mu_{\bar{x}} - \mu_{\bar{y}}|(dz). \qquad \square$$

From the three above lemmas, we obtain the final estimate for the nonlocal term. □

Proof of Corollary 9. Remark that $|a| \leqslant t_0$. Indeed, since the maximum of ψ is positive and in view of the lower bound on L, we have

$$\varphi\big(|a|\big) < \|u\|_\infty + \|v\|_\infty \leqslant Lt_0\frac{\alpha}{1+\alpha} = \varphi(t_0)$$

which by the strict monotonicity of φ implies the desired inequality. We first evaluate the estimate that renders the integral difference negative, namely:

$$\begin{aligned}
&\sup_{|s|\leqslant 1}\left(\big(1-\tilde{\eta}^2\big)\frac{\varphi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\varphi''\big(|a+sz|\big)\right)\\
&\quad = L\sup_{|s|\leqslant 1}\left(\big(1-\tilde{\eta}^2\big)\frac{1-\varrho(1+\alpha)|a+sz|^\alpha}{|a+sz|} - \tilde{\eta}^2\varrho\alpha(1+\alpha)|a+sz|^{\alpha-1}\right)\\
&\quad \leqslant L\sup_{|s|\leqslant 1}\left(\frac{1-\tilde{\eta}^2}{|a+sz|} - \varrho(1+\alpha)\big(1-\tilde{\eta}^2+\alpha\tilde{\eta}^2\big)|a+sz|^{\alpha-1}\right).
\end{aligned}$$

Using the fact that $\tilde{\eta}^2 \leqslant 1 \leqslant \frac{1}{1-\alpha^2}$ we have that $(1+\alpha)(1-\tilde{\eta}^2+\alpha\tilde{\eta}^2) \geqslant \alpha$ which further implies

$$\sup_{|s|\leqslant 1}\left(\big(1-\tilde{\eta}^2\big)\frac{\varphi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\varphi''\big(|a+sz|\big)\right) \leqslant L\sup_{|s|\leqslant 1}\left(\frac{1-\tilde{\eta}^2}{|a+sz|} - \varrho\alpha|a+sz|^{\alpha-1}\right).$$

But this quantity has to be integrated over the cone $\mathcal{C}_{\eta,\delta}(a)$, in which case $|a+sz|$ satisfies

$$|a|(1-\delta_0) \leqslant |a+sz| \leqslant |a|(1+\delta_0).$$

Thus, observing that $1-\tilde{\eta}^2 \leqslant 2(1-\tilde{\eta})$, the previous inequality takes the form

$$\sup_{|s|\leqslant 1}\left(\big(1-\tilde{\eta}^2\big)\frac{\varphi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\varphi''\big(|a+sz|\big)\right) \leqslant L\left(\frac{2(1-\tilde{\eta})}{|a|(1-\delta_0)} - \varrho\alpha(1+\delta_0)^{\alpha-1}|a|^{\alpha-1}\right).$$

Let $\tilde{\eta}$ be of the form

$$1-\tilde{\eta} = |a|^\alpha\tilde{\eta}_0$$

with small $\tilde{\eta}_0 < \frac{1}{4}$. Choose accordingly δ_0 and η of the form

$$\delta_0 = c_1|a|^{\alpha_1}, \qquad \eta = c_2|a|^{\alpha_2}.$$

Recalling that $\tilde{\eta} = \frac{1-\delta_0-\eta}{1+\delta_0}$ we get that c_1, c_2, α_1 and α_2 must satisfy

$$c_2|a|^{\alpha_2} + 2c_1|a|^{\alpha_1} = c_1\tilde{\eta}_0|a|^{\alpha+\alpha_1} + \tilde{\eta}_0|a|^{\alpha}.$$

Identifying the coefficients we obtain

$$\delta_0 = \frac{1}{2}|a|^{\alpha}\tilde{\eta}_0 \quad \text{and} \quad \eta = \frac{1}{2}|a|^{2\alpha}\tilde{\eta}_0^2.$$

Subsequently, the choice of parameters η, δ_0 and $\tilde{\eta}_0$ gives us

$$\sup_{|s|\leqslant 1}\left(\left(1-\tilde{\eta}^2\right)\frac{\varphi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\varphi''\big(|a+sz|\big)\right) \leqslant -L\big(\varrho\alpha 2^{\alpha-1}-1\big)|a|^{\alpha-1}.$$

This leads to a negative upper bound of the integral term taken over the cone $\mathcal{C}_{\eta,\delta}(a)$:

$$\int_{\mathcal{C}_{\eta,\delta}(a)} \sup_{|s|\leqslant 1}\left(\left(1-\tilde{\eta}^2\right)\frac{\varphi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\varphi''\big(|a+sz|\big)\right)|z|^2\,\mu_{\bar{x}}(dz)$$
$$\leqslant -L\big(\varrho\alpha 2^{\alpha-1}-1\big)|a|^{\alpha-1}\int_{\mathcal{C}_{\eta,\delta}(a)} |z|^2\,\mu_{\bar{x}}(dz).$$

Let $\Theta(\varrho,\alpha) = \varrho\alpha 2^{\alpha-1} - 1 > 0$ and use $(M2)$ and the fact that $\delta = |a|\delta_0$ to finally get

$$\int_{\mathcal{C}_{\eta,\delta}(a)} \sup_{|s|\leqslant 1}\left(\left(1-\tilde{\eta}^2\right)\frac{\varphi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\varphi''\big(|a+sz|\big)\right)|z|^2\,\mu_{\bar{x}}(dz)$$
$$\leqslant -L\Theta(\varrho,\alpha)|a|^{\alpha-1}C_\mu\eta^{\frac{d-1}{2}}\delta^{2-\beta}$$
$$= -L\Theta(\varrho,\alpha)C_\mu^1|a|^{\alpha-1}|a|^{\alpha(d-1)}|a|^{(1+\alpha)(2-\beta)}.$$

Less technical estimates give us similar upper bounds for the other two integrals. More precisely, we have in view of assumption $(M3)$

$$2\varphi'\big(|a|\big)\int_{B\setminus B_\delta} |z|\,|\mu_{\bar{x}}-\mu_{\bar{y}}|(dz) \leqslant 2LC_\mu|a|^{\gamma}\delta^{1-\beta}$$
$$= LC_\mu^2|a|^{\gamma}|a|^{(1+\alpha)(1-\beta)}$$

and

$$\int_{B_\delta\setminus\mathcal{C}_{\eta,\delta}(a)} \sup_{|s|\leqslant 1}\frac{\varphi'(|a+sz|)}{|a+sz|}|z|^2\,|\mu_{\bar{x}}-\mu_{\bar{y}}|(dz) \leqslant L\frac{C_\mu|a|^{\gamma}\delta^{2-\beta}}{|a|(1-\delta_0)}$$
$$\leqslant LC_\mu^3|a|^{\gamma-1}|a|^{(1+\alpha)(2-\beta)}.$$

For $\beta > 1$ and $\alpha > 0$ such that $\gamma > \alpha(d+1)$ the difference of the two nonlocal terms becomes negative:

$$\begin{aligned}
&\mathcal{I}[\bar{x}, p, u] - \mathcal{I}[\bar{y}, p, v] \\
&\quad \leqslant -L|a|^{1-\beta}\bigl\{C^1_\mu \Theta(\varrho, \alpha, \mu)|a|^{\alpha(d+2-\beta)} - C^2_\mu |a|^{\gamma+\alpha(1-\beta)} - C^3_\mu |a|^{\gamma+\alpha(2-\beta)}\bigr\} + O(\tilde{C}_\mu) \\
&\quad = -L|a|^{(1-\beta)+\alpha(d+2-\beta)}\bigl\{C^1_\mu \Theta(\varrho, \alpha, \mu) - o_{|a|}(1)\bigr\} + O(\tilde{C}_\mu). \qquad \square
\end{aligned}$$

Proof of Corollary 10. Estimating the integrand of the nonlocal difference $\mathcal{T}^2$ we get

$$\begin{aligned}
&\sup_{|s|\leqslant 1}\left(\bigl(1-\tilde{\eta}^2\bigr)\frac{\varphi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\varphi''\bigl(|a+sz|\bigr)\right) \\
&\quad = L\alpha\bigl(1-(2-\alpha)\tilde{\eta}^2\bigr)\inf_{|s|\leqslant 1}\bigl(|a+sz|^{\alpha-2}\bigr) \\
&\quad \leqslant -L\alpha\bigl((2-\alpha)\tilde{\eta}^2 - 1\bigr)(1+\delta_0)^{\alpha-2}|a|^{\alpha-2}.
\end{aligned}$$

Choose η and δ_0 sufficiently small such that $\delta_0 < \frac{1}{2}$

$$(2-\alpha)\tilde{\eta}^2 = (2-\alpha)\left(\frac{1-\eta-\delta_0}{1+\delta_0}\right)^2 > \frac{1}{2}.$$

Remark that, contrary to the Lipschitz case, η and δ_0 do not depend on $|a|$. We then obtain due to $(M2)$ a negative bound of the integral term over the cone $\mathcal{C}_{\eta,\delta}(a)$, for $\delta = |a|\delta_0$:

$$\begin{aligned}
&\int_{\mathcal{C}_{\eta,\delta}(a)} \sup_{|s|\leqslant 1}\left(\bigl(1-\tilde{\eta}^2\bigr)\frac{\varphi'(|a+sz|)}{|a+sz|} + \tilde{\eta}^2\varphi''\bigl(|a+sz|\bigr)\right)|z|^2\,\mu_{\bar{x}}(dz) \\
&\quad \leqslant -L\frac{\alpha}{2}(1+\delta_0)^{\alpha-2}|a|^{\alpha-2}\int_{\mathcal{C}_{\eta,\delta}(a)} |z|^2\,\mu_{\bar{x}}(dz) \\
&\quad \leqslant -L\alpha C(\mu)|a|^{\alpha-\beta}.
\end{aligned}$$

In addition, in view of $(M3)$ we have the estimates of the other two integral terms, when $\beta \neq 1$

$$\begin{aligned}
2\varphi'\bigl(|a|\bigr)\int_{B\setminus B_\delta} |z|\,|\mu_{\bar{x}} - \mu_{\bar{y}}|(dz) &\leqslant 2L\alpha|a|^{\alpha-1}C_\mu|a|^\gamma\delta^{1-\beta} \\
&= L\alpha C^2_\mu|a|^\gamma|a|^{\alpha-\beta}
\end{aligned}$$

and for $\beta = 1$

$$2\varphi'\bigl(|a|\bigr)\int_{B\setminus B_\delta} |z|\,|\mu_{\bar{x}} - \mu_{\bar{y}}|(dz) \leqslant L\alpha C^2_\mu|a|^\gamma\bigl|\ln\bigl(|a|\delta_0\bigr)\bigr||a|^{\alpha-\beta}.$$

Similarly

$$\int_{B_\delta\setminus\mathcal{C}_{\eta,\delta}(a)} \sup_{|s|\leqslant 1} \frac{\varphi'(|a+sz|)}{|a+sz|}|z|^2\,|\mu_{\bar{x}}-\mu_{\bar{y}}|(dz)$$
$$\leqslant L\alpha\big(|a|(1-\delta_0)\big)^{\alpha-2} \int_{B_\delta\setminus\mathcal{C}_{\eta,\delta}(a)} |z|^2\,|\mu_{\bar{x}}-\mu_{\bar{y}}|(dz)$$
$$\leqslant L\alpha C_\mu^3|a|^\gamma\,|a|^{\alpha-\beta}.$$

Therefore the difference of the nonlocal term becomes negative, as bounded from above by

$$\mathcal{I}[\bar{x},p,u]-\mathcal{I}[\bar{y},p,v]\leqslant -L|a|^{\alpha-\beta}\big(\alpha C(\mu)-o_{|a|}(1)\big)+O(\tilde{C}_\mu). \qquad \square$$

6.2. Lévy–Itô operators

We now establish similar results for Lévy–Itô operators

$$\mathcal{J}[x,u]=\int_{\mathbb{R}^d}\big(u\big(x+j(x,z)\big)-u(x)-Du(x)\cdot j(x,z)1_B(z)\big)\,\mu(dz).$$

As before, we give a general result on concave estimates for the difference of two Lévy–Itô operators. Then we present the Lipschitz and Hölder estimates as corollaries. In addition, we provide the quadratic estimates that are used in the uniqueness argument, in the proof of the partial regularity result, Theorem 2.

Proposition 14 *(Concave estimates – Lévy–Itô operators). Assume conditions (J1) and (J4) hold. Let u, v be two bounded functions, $\varphi:[0,\infty)\to\mathbb{R}$ be a smooth increasing concave function and define*

$$\psi(x,y)=u(x)-v(y)-\varphi\big(|x-y|\big).$$

Assume that ψ attains a positive maximum at $(\bar{x},\bar{y})$, with $\bar{x}\neq\bar{y}$. Let $a=\bar{x}-\bar{y}$, $\hat{a}=a/|a|$ and $p=\varphi'(|a|)\hat{a}$. Then the following holds

$$\begin{aligned}
&\mathcal{J}[\bar{x},p,u]-\mathcal{J}[\bar{y},p,v]\\
&\quad\leqslant 4\tilde{C}_\mu\max\big(\|u\|_\infty,\|v\|_\infty\big)\\
&\qquad+\frac{1}{2}\int_{\mathcal{C}}\sup_{\substack{|s|\leqslant 1\\ x=\bar{x},\bar{y}}}\left(\left(\big(1-\tilde{\eta}^2\big)\frac{\varphi'(|a+sj(x,z)|)}{|a+sj(x,z)|}+\tilde{\eta}^2\varphi''\big(\big|a+sj(x,z)\big|\big)\right)\big|j(x,z)\big|^2\right)\mu(dz)\\
&\qquad+2\varphi'\big(|a|\big)\int_{\substack{B\setminus\mathcal{C}\\ |\Delta(z)|\geqslant\delta}}\big|\Delta(z)\big|\,\mu(dz)+\int_{\substack{B\setminus\mathcal{C}\\ |\Delta(z)|\leqslant\delta}}\sup_{|s|\leqslant 1}\frac{\varphi'(|a+s\Delta(z)|)}{|a+s\Delta(z)|}\big|\Delta(z)\big|^2\,\mu(dz)
\end{aligned}$$

where $\Delta(z)=j(\bar{x},z)-j(\bar{y},z)$,

$$\mathcal{C}=\left\{z;\ \left|j\left(\frac{\bar{x}+\bar{y}}{2},z\right)\right|\leqslant\frac{\delta}{2}\ \text{and}\ \left|j\left(\frac{\bar{x}+\bar{y}}{2},z\right)\cdot\hat{a}\right|\geqslant\left(1-\frac{\eta}{2}\right)\left|j\left(\frac{\bar{x}+\bar{y}}{2},z\right)\right|\right\},$$
$$\left(\frac{|a|}{2}\right)^{\gamma}\leqslant\frac{c_0}{C_0}\frac{\eta}{4-\eta},\qquad\delta=|a|\delta_0>0,\qquad\tilde{\eta}=\frac{1-\eta-\delta_0}{1+\delta_0}>0$$

with $\delta_0\in(0,1)$ and $\eta\in(0,1)$ both sufficiently small.

Corollary 15 *(Lipschitz estimates). Let $\beta>1\geqslant 2(1-\gamma)$ and assume that conditions $(J1)$–$(J4)$ hold. Under the assumptions of Proposition* 14 *with*

$$\varphi(t)=\begin{cases}L(t-\varrho t^{1+\alpha}), & t\in[0,t_0],\\ \varphi(t_0), & t>t_0\end{cases}$$

where $\alpha\in(0,\min(\frac{\gamma\beta}{d+1},\frac{\beta-1}{d+2-\beta}))$, ϱ is a constant such that $\varrho\alpha 2^{\alpha-1}>1$, $t_0=\max_t(t-\varrho t^{1+\alpha})=\sqrt[\alpha]{\frac{1}{\rho(1+\alpha)}}$ and $L>\frac{(\|u\|_\infty+\|v\|_\infty)(\alpha+1)}{t_0\alpha}$, the following holds: there exists a positive constant $C=C(\mu)$ such that for $\Theta(\varrho,\alpha,\mu)=C(\rho\alpha 2^{\alpha-1}-1)$ we have

$$\mathcal{J}[\bar{x},p,u]-\mathcal{J}[\bar{y},p,v]\leqslant -L|a|^{(1-\beta)+\alpha(d+2-\beta)}\left\{\Theta(\varrho,\alpha,\mu)-o_{|a|}(1)\right\}+O(\tilde{C}_\mu).$$

Remark 8. The condition $\beta>2(1-\gamma)$ connects the singularity of the measure with the regularity of the jumps. It says that the more singular the measure is, the less regular the jumps can be.

Corollary 16 *(Hölder estimates). Let $\beta>2(1-\gamma)$ and assume that conditions $(J1)$–$(J4)$ hold. Under the assumptions of Proposition* 8 *with*

$$\varphi(t)=\begin{cases}Lt^{\alpha}, & t\in[0,t_0],\\ \varphi(t_0), & t>t_0\end{cases}$$

where $\alpha\in(0,\min(\beta,1))$, $t_0>0$, and $L>\frac{\|u\|_\infty+\|v\|_\infty}{t_0{}^{\alpha}}$, the following holds: there exists a positive constant $C(\mu)>0$ such that

$$\mathcal{J}[\bar{x},p,u]-\mathcal{J}[\bar{y},p,v]\leqslant -L|a|^{\alpha-\beta}\left\{\alpha C(\mu)-o_{|a|}(1)\right\}+O(\tilde{C}_\mu).$$

Proof of Proposition 14. In this case, the difference of the nonlocal terms reads

$$\mathcal{J}[\bar{x},p,u]-\mathcal{J}[\bar{y},p,v]=\int_{\mathbb{R}^d}\left(u\left(\bar{x}+j(\bar{x},z)-u(\bar{x})-p\cdot j(\bar{x},z)1_B(z)\right)\right)\mu(dz)$$
$$-\int_{\mathbb{R}^d}\left(v\left(\bar{y}+j(\bar{y},z)-v(\bar{y})-p\cdot j(\bar{y},z)1_B(z)\right)\right)\mu(dz).$$

Similarly to general nonlocal operators we split the domain of integration into the cone $\mathcal{C}$, its complementary in the unit ball $B\setminus\mathcal{C}$ and the region away from the origin $\mathbb{R}^d\setminus B$. Remark that the cone has the property (see Fig. 3)

$$\mathcal{C}:=\mathcal{C}_{\delta/2,\eta/2}\left(\frac{\bar{x}+\bar{y}}{2}\right)\subset\mathcal{C}_{\delta,\eta}(\bar{x})\cap\mathcal{C}_{\delta,\eta}(\bar{y}).\tag{34}$$

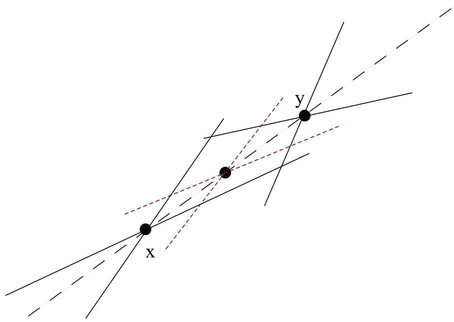

Fig. 3. The middle cone $\mathcal{C}_{\delta/2,\eta/2}(\frac{\bar{x}+\bar{y}}{2}) \subset \mathcal{C}_{\delta,\eta}(\bar{x}) \cap \mathcal{C}_{\delta,\eta}(\bar{y})$.

Indeed, for $|a|$ sufficiently small such that $(\frac{|a|}{2})^\gamma \leqslant \frac{c_0}{C_0}$, if $z \in \mathcal{C}$ then

$$\begin{aligned} \big|j(\bar{x}, z)\big| &\leqslant \left|j\left(\frac{\bar{x}+\bar{y}}{2}, z\right) - j(\bar{x}, z)\right| + \left|j\left(\frac{\bar{x}+\bar{y}}{2}, z\right)\right| \\ &\leqslant C_0|z|\left(\frac{|a|}{2}\right)^\gamma + \frac{\delta}{2} \leqslant \frac{\delta}{2}\frac{C_0}{c_0}\left(\frac{|a|}{2}\right)^\gamma + \frac{\delta}{2} \leqslant \delta \end{aligned}$$

since $c_0|z| \leqslant |j(\frac{\bar{x}+\bar{y}}{2}, z)| \leqslant \frac{\delta}{2}$. At the same time, we use the fact that $(\frac{|a|}{2})^\gamma \leqslant \frac{c_0}{C_0}\frac{\eta}{4-\eta}$, to get from $(J4)$

$$\begin{aligned} \big|j(\bar{x}, z)\cdot\hat{a}\big| &\geqslant \left|j\left(\frac{\bar{x}+\bar{y}}{2}, z\right)\cdot\hat{a}\right| - \left|j\left(\frac{\bar{x}+\bar{y}}{2}, z\right) - j(\bar{x}, z)\right| \\ &\geqslant \left(1-\frac{\eta}{2}\right)\left|j\left(\frac{\bar{x}+\bar{y}}{2}, z\right)\right| - \left|j\left(\frac{\bar{x}+\bar{y}}{2}, z\right) - j(\bar{x}, z)\right| \\ &\geqslant \left(1-\frac{\eta}{2}\right)\big|j(\bar{x}, z)\big| - \left(2-\frac{\eta}{2}\right)\left|j\left(\frac{\bar{x}+\bar{y}}{2}, z\right) - j(\bar{x}, z)\right| \\ &\geqslant \left(1-\frac{\eta}{2}\right)\big|j(\bar{x}, z)\big| - \left(2-\frac{\eta}{2}\right)C_0|z|\left(\frac{|a|}{2}\right)^\gamma \\ &\geqslant \left(1-\frac{\eta}{2}\right)\big|j(\bar{x}, z)\big| - \left(2-\frac{\eta}{2}\right)\frac{C_0}{c_0}\big|j(\bar{x}, z)\big|\left(\frac{|a|}{2}\right)^\gamma \geqslant (1-\eta)\big|j(\bar{x}, z)\big|. \end{aligned}$$

Let $\phi(z) = \varphi(|z|)$. Then $p = D\phi(a)$. Accordingly, we write the previous difference as the sum

$$\mathcal{J}[\bar{x}, p, u] - \mathcal{J}[\bar{y}, p, v] = \mathcal{T}^1(\bar{x}, \bar{y}) + \mathcal{T}^2(\bar{x}, \bar{y}) + \mathcal{T}^3(\bar{x}, \bar{y}),$$

where

$$\begin{aligned} \mathcal{T}^1(\bar{x}, \bar{y}) = &\int_{\mathbb{R}^d\setminus B} \big(u\big(\bar{x} + j(\bar{x}, z)\big) - u(\bar{x})\big)\,\mu(dz) \\ &- \int_{\mathbb{R}^d\setminus B} \big(v\big(\bar{y} + j(\bar{y}, z)\big) - v(\bar{y})\big)\,\mu(dz), \end{aligned}$$

$$\mathcal{T}^2(\bar{x},\bar{y}) = \int_{\mathcal{C}} \big(u\big(\bar{x}+j(\bar{x},z)\big) - u(\bar{x}) - p\cdot j(\bar{x},z)\big)\,\mu(dz) - \int_{\mathcal{C}} \big(v\big(\bar{y}+j(\bar{y},z)\big) - v(\bar{y}) - p\cdot j(\bar{y},z)\big)\,\mu(dz),$$

$$\mathcal{T}^3(\bar{x},\bar{y}) = \int_{B\setminus\mathcal{C}} \big(u\big(\bar{x}+j(\bar{x},z)\big) - u(\bar{x}) - p\cdot j(\bar{x},z)\big)\,\mu(dz) - \int_{B\setminus\mathcal{C}} \big(v\big(\bar{y}+j(\bar{y},z)\big) - v(\bar{y}) - p\cdot j(\bar{y},z)\big)\,\mu(dz).$$

As before, we next estimate each of these integral terms. The first lemma is straightforward.

Lemma 17. *$\mathcal{T}^1(\bar{x},\bar{y})$ is uniformly bounded with respect to all the parameters, namely*

$$\mathcal{T}^1(\bar{x},\bar{y}) \leqslant 4\max\big(\|u\|_\infty, \|v\|_\infty\big) \sup_{x\in\mathbb{R}^d} \mu_x\big(\mathbb{R}^d \setminus B\big).$$

Lemma 18. *Let $\delta = |a|\delta_0$ and $\eta \in (0, \frac{1}{2})$ such that $1-\eta-\delta_0 \geqslant 0$. We have*

$$\mathcal{T}^2(\bar{x},\bar{y}) \leqslant \int_{\mathcal{C}} \sup_{\substack{|s|\leqslant 1,\\ x=\bar{x},\bar{y}}} \left(\left(\big(1-\tilde{\eta}^2\big) \frac{\varphi'(|a+sj(x,z)|)}{|a+sj(x,z)|} + \tilde{\eta}^2 \varphi''\big(\big|a+sj(x,z)\big|\big)\right) \big|j(x,z)\big|^2 \right) \mu(dz)$$

where $\tilde{\eta} = (1-\eta-\delta_0)(1+\delta_0)^{-1}$.

Proof. Writing the maximum inequality at points $\bar{x}$, $\bar{y}$ for the pair $(z,z') = (j(\bar{x},z),0)$ and $(z,z') = (0, j(\bar{y},z))$ respectively, we have

$$u\big(\bar{x}+j(\bar{x},z)\big) - u(\bar{x}) - p\cdot j(\bar{x},z) \leqslant \phi\big(a+j(\bar{x},z)\big) - \phi(a) - D\phi(a)\cdot j(\bar{x},z),$$
$$-\big(v\big(\bar{y}+j(\bar{y},z)\big) - v(\bar{y}) - p\cdot j(\bar{y},z)\big) \leqslant \phi\big(a-j(\bar{y},z)\big) - \phi(a) + D\phi(a)\cdot j(\bar{y},z).$$

Therefore

$$\mathcal{T}^2(\bar{x},\bar{y}) \leqslant \int_{\mathcal{C}} \big(\phi\big(a+j(\bar{x},z)\big) - \phi(a) - D\phi(a)\cdot j(\bar{x},z)\big)\,\mu(dz) + \int_{\mathcal{C}} \big(\phi\big(a-j(\bar{y},z)\big) - \phi(a) + D\phi(a)\cdot j(\bar{y},z)\big)\,\mu(dz).$$

Taking into account that the set $\mathcal{C}$ is included in both $\mathcal{C}_{\eta,\delta}(\bar{x})$ and $\mathcal{C}_{\eta,\delta}(\bar{y})$ (see (34)) we have, similarly to (32) and (33), the following upper and lower bounds for the jumps

$$|a|(1-\delta_0) \geqslant \big|a+sj(\bar{x},z)\big| \geqslant |a|(1-\delta_0),$$
$$\big|\big(\widehat{a+sj(\bar{x},z)}\big)\cdot z\big| \geqslant \tilde{\eta}\big|j(\bar{x},z)\big|.$$

We then conclude as we did for general nonlocal operators, within the proof of Lemma 12. □

Lemma 19. *Denote by* $\Delta(z) = j(\bar{x}, z) - j(\bar{y}, z)$*. Then*

$$\mathcal{T}^3(\bar{x}, \bar{y}) \leqslant 2\varphi'\big(|a|\big) \int\limits_{\{z \in B \setminus \mathcal{C};\ |\Delta(z)| \geqslant \delta\}} \big|\Delta(z)\big|\, \mu(dz) + \int\limits_{\{z \in B \setminus \mathcal{C};\ |\Delta(z)| \leqslant \delta\}} \sup_{|s| \leqslant 1} \frac{\varphi'(|a + s\Delta(z)|)}{|a + s\Delta(z)|} \big|\Delta(z)\big|^2\, \mu(dz).$$

Proof. We use again the maximum inequality to obtain the bound

$$\begin{aligned}
&\big(u\big(\bar{x} + j(\bar{x}, z)\big) - u(\bar{x}) - p \cdot j(\bar{x}, z)\big) - \big(v\big(\bar{y} + j(\bar{y}, z)\big) - v(\bar{y}) - p \cdot j(\bar{y}, z)\big) \\
&\quad \leqslant \phi\big(a + j(\bar{x}, z) - j(\bar{y}, z)\big) - \phi(a) - D\phi(a) \cdot \big(j(\bar{x}, z) - j(\bar{y}, z)\big)
\end{aligned}$$

which in particular implies

$$\mathcal{T}^3(\bar{x}, \bar{y}) \leqslant \int\limits_{B \setminus \mathcal{C}} \big(\phi\big(a + j(\bar{x}, z) - j(\bar{y}, z)\big) - \phi(a) - D\phi(a) \cdot \big(j(\bar{x}, z) - j(\bar{y}, z)\big)\big)\, \mu(dz).$$

In order to estimate the integral terms corresponding to ϕ, we split the integral in two parts, as follows

$$\int\limits_{\{z \in B \setminus \mathcal{C};\ |\Delta(z)| \geqslant \delta\}} \big(\phi\big(a + \Delta(z)\big) - \phi(a) - D\phi(a) \cdot \Delta(z)\big)\, \mu(dz) + \int\limits_{\{z \in B \setminus \mathcal{C};\ |\Delta(z)| \leqslant \delta\}} \big(\phi\big(a + \Delta(z)\big) - \phi(a) - D\phi(a) \cdot \Delta(z)\big)\, \mu(dz).$$

On the first set we use the monotonicity and the concavity of φ to deduce that

$$\phi\big(a + \Delta(z)\big) - \phi(a) - D\phi(a) \cdot \Delta(z) \leqslant 2\varphi'\big(|a|\big)\big|\Delta(z)\big|.$$

On $\{z \in B \setminus \mathcal{C};\ |\Delta(z)| \leqslant \delta\}$ we use a second order Taylor expansion and we take into account that φ is a smooth increasing function with $\varphi'' \leqslant 0$ to obtain the upper bound

$$\begin{aligned}
\sup_{|s| \leqslant 1} \big(\phi\big(a + s\Delta(z)\big) - \phi(a) - D\phi(a) \cdot \Delta(z)\big) &\leqslant \frac{1}{2} \sup_{|s| \leqslant 1} D^2\phi\big(a + s\Delta(z)\big)\Delta(z) \cdot \Delta(z) \\
&\leqslant \frac{1}{2} \sup_{|s| \leqslant 1} \frac{\varphi'(|a + s\Delta(z)|)}{|a + s\Delta(z)|} \big|\Delta(z)\big|^2.
\end{aligned}$$

Therefore we get the desired estimate. □

The lemmas above yield the global estimate of the difference of the nonlocal terms. □

Proof of Corollary 15. We first evaluate, as for general nonlocal operators, the expression

$$\sup_{|s|\leqslant 1}\left(\left(1-\tilde{\eta}^2\right)\frac{\varphi'(|a+sj(x,z)|)}{|a+sj(x,z)|}+\tilde{\eta}^2\varphi''\left(|a+sj(x,z)|\right)\right)$$
$$\leqslant L\left(\frac{2(1-\tilde{\eta})}{|a|(1-\delta_0)}-\varrho\alpha(1+\delta_0)^{\alpha-1}|a|^{\alpha-1}\right).$$

For $\tilde{\eta}=1-|a|^\alpha\tilde{\eta}_0$ with $\tilde{\eta}_0<\frac{1}{4}$, consider the constant $\Theta(\varrho,\alpha)=\varrho\alpha 2^{\alpha-1}-1>0$. Then, by $(J2)$ we have

$$\int_{\mathcal{C}}\sup_{|s|\leqslant 1}\left(\left(1-\tilde{\eta}^2\right)\frac{\varphi'(|a+sj(\bar{x},z)|)}{|a+sj(\bar{x},z)|}+\tilde{\eta}^2\varphi''\left(|a+sj(\bar{x},z)|\right)\right)\left|j(\bar{x},z)\right|^2\mu(dz)$$
$$\leqslant -L\Theta(\varrho,\alpha)|a|^{\alpha-1}\int_{\mathcal{C}}\left|j(\bar{x},z)\right|^2\mu(dz)$$
$$\leqslant -L\Theta(\varrho,\alpha,\mu)|a|^{(1-\beta)+\alpha(d+2-\beta)}.$$

Similarly, taking into account assumptions $(J3)$–$(J4)$ and that $\delta=|a|\delta_0\sim|a|^{\alpha+1}$ we obtain

$$\varphi'\left(|a|\right)\int_{\{z\in B\setminus\mathcal{C};\ |\Delta(z)|\geqslant\delta\}}\left|\Delta(z)\right|\mu(dz)\leqslant LC_0|a|^\gamma\int_{\{z\in B\setminus\mathcal{C};\ \mathbb{R}^d\setminus B_\delta|a|^{-\gamma}\}}|z|\,\mu(dz)$$
$$\leqslant LC_\mu^2|a|^\gamma|a|^{(1+\alpha-\gamma)(1-\beta)}$$

and

$$\int_{\{z\in B\setminus\mathcal{C};\ |\Delta(z)|\leqslant\delta\}}\sup_{|s|\leqslant 1}\frac{\varphi'(|a+s\Delta(z)|)}{|a+s\Delta(z)|}\left|\Delta(z)\right|^2\mu(dz)\leqslant\frac{L}{|a|(1-\delta_0)}\int_{\{z\in B\setminus\mathcal{C};\ |\Delta(z)|\leqslant\delta\}}\left|\Delta(z)\right|^2\mu(dz)$$
$$\leqslant LC_\mu^3|a|^{2\gamma-1}.$$

Since $\beta>2(1-\gamma)$, $\gamma\beta>\alpha(d+1)$ and $2\gamma-2+\beta>\alpha(d+2-\beta)$ the difference of the nonlocal terms is negative, being bounded from above by

$$\mathcal{J}[\bar{x},p,u]-\mathcal{J}[\bar{y},p,v]$$
$$\leqslant -L|a|^{1-\beta}\left\{\Theta(\varrho,\alpha,\mu)|a|^{\alpha(d+2-\beta)}-C_\mu^2|a|^{\gamma+(\alpha-\gamma)(1-\beta)}-C_\mu^3|a|^{2\gamma-2+\beta}\right\}+O(\tilde{C}_\mu)$$
$$=-L|a|^{(1-\beta)+\alpha(d+2-\beta)}\left\{\Theta(\varrho,\alpha,\mu)-o_{|a|}(1)\right\}+O(\tilde{C}_\mu).\qquad\square$$

Proof of Corollary 16. Similarly to general nonlocal operators, we use $(J2)$ to get

$$\int_{\mathcal{C}}\sup_{|s|\leqslant 1}\left(\left(1-\tilde{\eta}^2\right)\frac{\varphi'(|a+sj(x,z)|)}{|a+sj(x,z)|}+\tilde{\eta}^2\varphi''\left(|a+sj(x,z)|\right)\right)\left|j(\bar{x},z)\right|^2\mu(dz)$$
$$\leqslant -L\alpha(1-\alpha)2^{\alpha-3}|a|^{\alpha-2}\int_{\mathcal{C}}|z|^2\mu(dz)$$
$$\leqslant -L\alpha C(\mu)|a|^{\alpha-\beta}.$$

In addition, from $(J3)$–$(J4)$ we have the estimates

$$\begin{aligned}\varphi'\big(|a|\big)\int\limits_{\{z\in B\setminus\mathcal{C};\ |\Delta(z)|\geqslant\delta\}}\big|\Delta(z)\big|\,\mu(dz) &\leqslant L\alpha|a|^{\alpha-1}C_0|a|^{\gamma}\int\limits_{B\setminus\mathcal{C};\,\mathbb{R}^d\setminus B_{\delta|a|^{-\gamma}}}|z|\,\mu(dz)\\ &\leqslant L\alpha C_\mu^2|a|^{\alpha-\beta+\gamma\beta}\end{aligned}$$

if $\beta\neq 1$, respectively

$$\varphi'\big(|a|\big)\int\limits_{\{z\in B\setminus\mathcal{C};\ |\Delta(z)|\geqslant\delta\}}\big|\Delta(z)\big|\,\mu(dz)\leqslant L\alpha C_\mu^2|a|^{\alpha-\beta}|a|^{\gamma}\ln\big(|a|\delta_0\big)$$

for $\beta=1$. Finally, using again $(J3)$–$(J4)$ we get

$$\begin{aligned}&\int\limits_{\{z\in B\setminus\mathcal{C};\ |\Delta(z)|\leqslant\delta\}}\sup_{|s|\leqslant 1}\frac{\varphi'(|a+s\Delta(z)|)}{|a+s\Delta(z)|}\big|\Delta(z)\big|^2\,\mu(dz)\\ &\quad\leqslant L\alpha\big(|a|(1-\delta_0)\big)^{\alpha-2}\int\limits_{\{z\in B\setminus\mathcal{C};\ |\Delta(z)|\leqslant\delta\}}\big|\Delta(z)\big|^2\,\mu(dz)\\ &\quad\leqslant L\alpha C_\mu^3|a|^{2\gamma-2+\beta}|a|^{\alpha-\beta}.\end{aligned}$$

For α sufficiently small we thus have

$$\mathcal{J}[\bar{x},p,u]-\mathcal{J}[\bar{y},p,v]\leqslant -L|a|^{\alpha-\beta}\big(\alpha C(\mu)-o_{|a|}(1)\big)+O(\tilde{C}_\mu).\qquad\square$$

Proposition 20 *(Quadratic estimates – Lévy–Itô operators). Let $(J1)$, $(J4)$ and $(J5)$ hold. Let u, v be two bounded functions and assume the auxiliary function*

$$\psi_\varepsilon(x,y)=u(x)-v(y)-\frac{|x-y|^2}{\varepsilon^2}$$

attains a positive maximum at $(\bar{x},\bar{y})$, with $\bar{x}\neq\bar{y}$. Denote $a=\bar{x}-\bar{y}$ and $p=2\frac{\bar{x}-\bar{y}}{\varepsilon^2}$. Then the following holds

$$\mathcal{J}[\bar{x},p,u]-\mathcal{J}[\bar{y},p,u]\leqslant 2C_0^2\frac{1}{\varepsilon^2}\int\limits_{B_\delta}|z|^2\,\mu(dz)+C_0^2\frac{|a|^{2\gamma}}{\varepsilon^2}\tilde{C}_\mu+2C_0\frac{|a|^{\gamma+1}}{\varepsilon^2}\tilde{C}_\mu.$$

Proof. By definition of $(\bar{x},\bar{y})$, we have

$$u\big(\bar{x}+j(\bar{x},z)\big)-v\big(\bar{y}+j\big(\bar{y},z'\big)\big)-\frac{|\bar{x}+j(\bar{x},z)-\bar{y}-j(\bar{y},z')|^2}{\varepsilon^2}\leqslant u(\bar{x})-v(\bar{y})-\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}.\qquad(35)$$

We split the difference of the integral terms into

$$\mathcal{J}[\bar{x},p,u]-\mathcal{J}[\bar{y},p,u]=\mathcal{T}_q^1(\bar{x},\bar{y})+\mathcal{T}_q^2(\bar{x},\bar{y})+\mathcal{T}_q^3(\bar{x},\bar{y})$$

where this time the integrals are taken over the ball B_δ, the ring $B \setminus B_\delta$ and the exterior of the unit ball $\mathbb{R}^d \setminus B$:

$$\mathcal{T}_q^1(\bar{x}, \bar{y}) = \int_{B_\delta} \big(u\big(\bar{x} + j(\bar{x}, z)\big) - u(\bar{x}) - p \cdot j(\bar{x}, z)\big)\, \mu(dz) - \int_{B_\delta} \big(v\big(\bar{y} + j(\bar{y}, z)\big) - v(\bar{y}) - p \cdot j(\bar{y}, z)\big)\, \mu(dz),$$

$$\mathcal{T}_q^2(\bar{x}, \bar{y}) = \int_{B \setminus B_\delta} \big(u\big(\bar{x} + j(\bar{x}, z)\big) - u(\bar{x}) - p \cdot j(\bar{x}, z)\big)\, \mu(dz) - \int_{B \setminus B_\delta} \big(v\big(\bar{y} + j(\bar{y}, z)\big) - v(\bar{y}) - p \cdot j(\bar{y}, z)\big)\, \mu(dz),$$

$$\mathcal{T}_q^3(\bar{x}, \bar{y}) = \int_{\mathbb{R}^d \setminus B} \big(u\big(\bar{x} + j(\bar{x}, z)\big) - u(\bar{x})\big)\, \mu(dz) - \int_{\mathbb{R}^d \setminus B} \big(v\big(\bar{y} + j(\bar{y}, z)\big) - v(\bar{y})\big)\, \mu(dz).$$

Lemma 21. *The following estimate holds*

$$\mathcal{T}_q^1(\bar{x}, \bar{y}) \leqslant 2C_0^2 \frac{1}{\varepsilon^2} \int_{B_\delta} |z|^2\, \mu(dz).$$

Proof. Taking $z' = 0$ and $z = 0$ in inequality (35), we have respectively $j(\bar{y}, z') = 0$, $j(\bar{x}, z) = 0$. Hence, by direct computations and $(J4)$ we have

$$\begin{aligned} u\big(\bar{x} + j(\bar{x}, z)\big) - u(\bar{x}) - p \cdot j(\bar{x}, z) &\leqslant \frac{|\bar{x} + j(\bar{x}, z) - \bar{y}|^2}{\varepsilon^2} - \frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} - p \cdot j(\bar{x}, z) \\ &= \frac{|j(\bar{x}, z)|^2}{\varepsilon^2} \leqslant C_0^2 \frac{|z|^2}{\varepsilon^2} \end{aligned}$$

and

$$\begin{aligned} -\big(v\big(\bar{y} + j\big(\bar{y}, z'\big)\big) - v(\bar{y}) - p \cdot j\big(\bar{y}, z'\big)\big) &\leqslant \frac{|\bar{x} - \bar{y} - j(\bar{y}, z')|^2}{\varepsilon^2} - \frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} + p \cdot j\big(\bar{y}, z'\big) \\ &= \frac{|j(\bar{y}, z)|^2}{\varepsilon^2} \leqslant C_0^2 \frac{|z|^2}{\varepsilon^2}. \end{aligned}$$

Integrating on B_δ we get the desired estimate. □

Lemma 22. *The following estimate holds*

$$\mathcal{T}_q^2(\bar{x}, \bar{y}) \leqslant C_0^2 \frac{|a|^{2\gamma}}{\varepsilon^2} \int_{B \setminus B_\delta} |z|^2\, \mu(dz).$$

Proof. Taking $z=z'$ in inequality (35), subtracting the corresponding gradients and using $(J4)$ we obtain the inequality

$$\begin{aligned}&\big(u\big(\bar x+j(\bar x,z)\big)-u(\bar x)-p\cdot j(\bar x,z)\big)-\big(v\big(\bar y+j(\bar y,z)\big)-v(\bar y)-p\cdot j(\bar y,z)\big)\\&\quad\leqslant\frac{|\bar x+j(\bar x,z)-\bar y-j(\bar y,z)|^2}{\varepsilon^2}-\frac{|\bar x-\bar y|^2}{\varepsilon^2}-p\cdot\big(j(\bar x,z)-j(\bar y,z)\big)\\&\quad=\frac{|j(\bar x,z)-j(\bar y,z)|^2}{\varepsilon^2}\leqslant C_0^2\frac{|z|^2|\bar x-\bar y|^{2\gamma}}{\varepsilon^2}.\end{aligned}$$

Integrating on the ring $B\setminus B_\delta$, we get the desired estimate. □

Lemma 23. *The following estimate holds*

$$\mathcal{T}_q^3(\bar x,\bar y)\leqslant C_0^2\frac{|a|^{2\gamma}}{\varepsilon^2}\int\limits_{\mathbb{R}^d\setminus B}\mu(dz)+2C_0\frac{|a|^{\gamma+1}}{\varepsilon^2}\int\limits_{\mathbb{R}^d\setminus B}\mu(dz).$$

Proof. Once again, for $z=z'$ in inequality (35) we obtain the inequality

$$\begin{aligned}&\big(u\big(\bar x+j(\bar x,z)\big)-u(\bar x)\big)-\big(v\big(\bar y+j(\bar y,z)\big)-v(\bar y)\big)\\&\quad\leqslant\frac{|\bar x+j(\bar x,z)-\bar y-j(\bar y,z)|^2}{\varepsilon^2}-\frac{|\bar x-\bar y|^2}{\varepsilon^2}.\end{aligned}$$

Integrating on $\mathbb{R}^d\setminus B$ and computing the right-hand side we get

$$\mathcal{T}_q^3(\bar x,\bar y)\leqslant\int\limits_{\mathbb{R}^d\setminus B}\left(|p|\big|j(\bar x,z)-j(\bar y,z)\big|+\frac{|j(\bar x,z)-j(\bar y,z)|^2}{\varepsilon^2}\right)\mu(dz).$$

Taking into account $(J5)$ we get the desired estimate. □

From the three above lemmas and $(J1)$ we conclude. □

Appendix A

Lemma 24. *Let X, Y and Z be block matrices of the form*

$$A=\begin{bmatrix}A_1&0\\0&A_2\end{bmatrix}$$

such that they satisfy the inequality

$$\begin{bmatrix}X&0\\0&-Y\end{bmatrix}\leqslant\begin{bmatrix}Z&-Z\\-Z&Z\end{bmatrix}. \tag{36}$$

Then the block matrices X_i, Y_i satisfy inequality (36) *where Z is replaced with Z_i, for $i=1,2$.*

Proof. The previous matrix inequality can be rewritten in the form

$$Xz\cdot z - Yz'\cdot z' \leqslant Z\big(z-z'\big)\cdot\big(z-z'\big).$$

Due to the form of the block matrices, namely the secondary diagonal null, we can write the inequality on components, for $z=(z_1,z_2)$, $z'=(z_1',z_2')$

$$\sum_{i=1,2}\big(X_i z_i\cdot z_i - Y_i z_i'\cdot z_i'\big) \leqslant \sum_{i=1,2}\big(Z_i\big(z_i-z_i'\big)\cdot\big(z_i-z_i'\big)\big).$$

Thus, taking $z=(z_1,0)$ and $z'=(z_1',0)$, respectively $z=(0,z_2)$ and $z'=(0,z_2')$ we get the corresponding inequality for the block matrices X_i, Y_i, Z_i. □

In the next lemma, for a symmetric matrix A, $\|A\|$ denotes $\max_{|\xi|\leqslant 1}|A\xi\cdot\xi|$.

Lemma 25. *Let X, Y and Z be symmetric matrices satisfying inequality* (36). *Consider the sup-convolution X^ε of X and the inf-convolution Y^ε of Y, defined by*

$$X^\varepsilon z\cdot z = \sup_{\xi\in\mathbb{R}^d}\left\{X\xi\cdot\xi - \frac{|z-\xi|^2}{\varepsilon}\right\} \quad \text{and} \quad Y_\varepsilon z\cdot z = \inf_{\xi\in\mathbb{R}^d}\left\{Y\xi\cdot\xi + \frac{|z-\xi|^2}{\varepsilon}\right\}.$$

Then there exists $\varepsilon_0 = (\max(\|X\|,\|Y\|,2\|Z\|))^{-1} > 0$ such that for any $\varepsilon\in(0,\varepsilon_0)$, X^ε, Y_ε and $Z^{2\varepsilon}$ satisfy as well inequality (36). *In addition we have*

$$-\frac{1}{\varepsilon}I, X \leqslant X^\varepsilon \quad \text{and} \quad Y_\varepsilon \leqslant Y, \frac{1}{\varepsilon}I. \tag{37}$$

Proof. Consider ε as in the statement of the lemma. Then the ε-sup-convolutions of the two quadratic forms associated with the matrix inequality (36) are finite. It must be checked that it gives the above mentioned inequality. As far as the left-hand side is concerned, writing matrix inequalities in terms of quadratic forms, we have for all $\zeta,\alpha\in\mathbb{R}^d$,

$$\sup_{\xi,\eta}\left\{X(\xi-\zeta)\cdot(\xi-\zeta) - Y(\eta-\alpha)\cdot(\eta-\alpha) - \frac{1}{\varepsilon}|\xi|^2 - \frac{1}{\varepsilon}|\eta|^2\right\} = X^\varepsilon\zeta\cdot\zeta - Y_\varepsilon\alpha\cdot\alpha.$$

As far as the right-hand side is concerned, we get

$$\begin{aligned}
&\sup_{\xi,\eta}\left\{Z(\xi-\eta)\cdot(\xi-\eta) - \frac{1}{\varepsilon}|\zeta-\xi|^2 - \frac{1}{\varepsilon}|\alpha-\eta|^2\right\} \\
&= \sup_{\tilde\xi}\left\{Z\tilde\xi\cdot\tilde\xi - \inf_{\tilde\eta}\left\{\frac{1}{\varepsilon}|\zeta-\tilde\xi-\tilde\eta-\alpha|^2 + \frac{1}{\varepsilon}|\tilde\eta|^2\right\}\right\} = \sup_{\tilde\xi}\left\{Z\tilde\xi\cdot\tilde\xi - \frac{1}{2\varepsilon}|\zeta-\alpha-\tilde\xi|^2\right\} \\
&= Z^{2\varepsilon}(\zeta-\alpha)\cdot(\zeta-\alpha)
\end{aligned}$$

where we changed ξ in $\tilde\xi = \xi-\eta$ and η in $\tilde\eta = \eta-\alpha$. The additional matrix inequalities come directly from the definition of the inf/sup-convolution. The proof of the lemma is now complete. □

Lemma 26. *Let $Z = \frac{1}{\alpha}(I - \omega\hat{a}\otimes\hat{a})$, where $\hat{a}\in\mathbb{S}^{d-1}$, $\alpha > 0$ and $\omega \geqslant 0$. Then the following holds*

$$Z^{\frac{\alpha}{2}} = \frac{2}{\alpha}\left(I - \frac{2\omega}{1+\omega}\hat{a}\otimes\hat{a}\right). \tag{38}$$

Proof. By definition

$$Z^{\frac{\alpha}{2}} z\cdot z = \sup_{\xi}\left\{Z\xi\cdot\xi - 2\frac{|z-\xi|^2}{\alpha}\right\}$$

and the supremum is attained at points $\bar{\xi}$ satisfying $Z\bar{\xi} = \frac{2}{\alpha}(\bar{\xi} - z)$, or equivalently

$$(I - \omega\hat{a}\otimes\hat{a})\bar{\xi} = 2(\bar{\xi} - z).$$

Taking the inner product with $\hat{a}$ in this identity, we have

$$\bar{\xi}\cdot\hat{a} = \frac{2}{1+\omega} z\cdot\hat{a}.$$

Taking now the inner product with z in the same identity, we have

$$\bar{\xi}\cdot z = 2|z|^2 - \omega(z\cdot\hat{a})(\bar{\xi}\cdot\hat{a}) = 2|z|^2 - \frac{2\omega}{1+\omega}(z\cdot\hat{a})^2.$$

Therefore

$$\begin{aligned} Z^{\frac{\alpha}{2}} z\cdot z &= \frac{2}{\alpha}\big((\bar{\xi} - z)\cdot\bar{\xi} - |z-\bar{\xi}|^2\big) \\ &= \frac{2}{\alpha}\big((\bar{\xi} - z)\cdot z\big) = \frac{2}{\alpha}\left(|z|^2 - \frac{2\omega}{1+\omega}(z\cdot\hat{a})^2\right). \qquad \square \end{aligned}$$

Lemma 27. *Let X, Y, $Z^{\frac{\alpha}{2}}$ satisfy the block inequality* (36), *with $Z^{\frac{\alpha}{2}}$ given by Eq.* (38), *for some $\omega \geqslant 1$. Then the following holds*:

$$\operatorname{trace}(X - Y) \leqslant -\frac{8(\omega-1)}{\alpha(1+\omega)}.$$

Proof. Rewrite the matrix inequality in the form

$$Xz\cdot z - Yz'\cdot z' \leqslant Z^{\frac{\alpha}{2}}\big(z - z'\big)\cdot\big(z - z'\big).$$

Taking $z = -z' = \hat{a}$ we have

$$X\hat{a}\cdot\hat{a} - Y\hat{a}\cdot\hat{a} \leqslant 4Z^{\frac{\alpha}{2}}\hat{a}\cdot\hat{a}$$

whereas for any vector z orthogonal to $\hat{a}$

$$Xz\cdot z - Yz\cdot z \leqslant 0.$$

Therefore

$$\mathrm{trace}(X-Y)\leqslant\frac{8}{\alpha}\left(|\hat{a}|^2-\frac{2\omega}{1+\omega}|\hat{a}|^2\right)=-\frac{8(\omega-1)}{\alpha(\omega+1)}.\qquad\square$$

References

[1] G. Barles, E. Chasseigne, C. Imbert, Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations, J. Eur. Math. Soc. (JEMS) 13 (2011) 1–26.
[2] G. Barles, C. Imbert, Second-order elliptic integro-differential equations: viscosity solutions' theory revisited, Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008) 567–585.
[3] R.F. Bass, D.A. Levin, Harnack inequalities for jump processes, Potential Anal. 17 (2002) 375–388.
[4] L. Caffarelli, L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math. 62 (2009) 597–638.
[5] L.A. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, Ann. of Math. (2) 130 (1989) 189–213.
[6] L.A. Caffarelli, A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math. 2 (2010) 1903–1930.
[7] P. Cardaliaguet, C. Rainer, Hölder regularity for viscosity solutions of fully nonlinear, local or nonlocal, Hamilton–Jacobi equations with super-quadratic growth in the gradient, arXiv:1003.1059v1, 2010.
[8] A. Ciomaga, On the strong maximum principle for second order nonlinear parabolic integro-differential equations, arXiv: 1006.2607v1, 2010.
[9] M.G. Garroni, J.L. Menaldi, Second Order Elliptic Integro-differential Problems, Chapman & Hall/CRC Res. Notes Math., vol. 430, Chapman & Hall/CRC, Boca Raton, FL, 2002.
[10] N. Guillen, R. Schwab, Aleksandrov–Bakelman–Pucci type estimates for integro-differential equations, arXiv:1101.0279.
[11] H. Ishii, P.-L. Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Differential Equations 83 (1990) 26–78.
[12] M. Kassmann, The theory of De Giorgi for non-local operators, C. R. Math. Acad. Sci. Paris 345 (2007) 621–624.
[13] M. Kassmann, A priori estimates for integro-differential operators with measurable kernels, Calc. Var. Partial Differential Equations 34 (2009) 1–21.
[14] A. Kiselev, F. Nazarov, A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, Invent. Math. 167 (2007) 445–453.
[15] N.V. Krylov, M.V. Safonov, An estimate for the probability of a diffusion process hitting a set of positive measure, Dokl. Akad. Nauk SSSR 245 (1979) 18–20.
[16] N.V. Krylov, M.V. Safonov, A property of the solutions of parabolic equations with measurable coefficients, Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980) 161–175, 239.
[17] A. Sayah, Équations d'Hamilton–Jacobi du premier ordre avec termes intégro-différentiels. II. Existence de solutions de viscosité, Comm. Partial Differential Equations 16 (1991) 1075–1093.
[18] R.W. Schwab, Periodic homogenization for nonlinear integro-differential equations, SIAM J. Math. Anal. 42 (2010) 2652–2680.
[19] L. Silvestre, Hölder estimates for advection fractional-diffusion equations, arXiv:1009.5723.
[20] L. Silvestre, On the differentiability of the solution to an equation with drift and fractional diffusion, arXiv:1012.2401.
[21] L. Silvestre, Hölder estimates for solutions of integro-differential equations like the fractional Laplace, Indiana Univ. Math. J. 55 (2006) 1155–1174.
[22] N.S. Trudinger, Comparison principles and pointwise estimates for viscosity solutions of nonlinear elliptic equations, Rev. Mat. Iberoam. 4 (1988) 453–468.