Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 246 (2009) 1057–1097
Journal of Differential Equations
www.elsevier.com/locate/jde

Homogenization of fully overdamped Frenkel–Kontorova models

N. Forcadel [a,b,c], C. Imbert [d,*], R. Monneau [a]

[a] *CERMICS, Paris Est-ENPC, 6 & 8 avenue Blaise Pascal, Cité Descartes, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France*
[b] *Projet Commands, CMAP-INRIA Futurs, Ecole Polytechnique, 91128 Palaiseau, France*
[c] *ENSTA, UMA, 32 Bd Victor, 75739 Paris Cedex 15, France*
[d] *CEREMADE, UMR CNRS 7534, Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris Cedex 16, France*

Received 28 March 2008

Available online 30 September 2008

Abstract

In this paper, we consider the fully overdamped Frenkel–Kontorova model. This is an infinite system of coupled first-order ODEs. Each ODE represents the microscopic evolution of one particle interacting with its neighbors and submitted to a fixed periodic potential. After a proper rescaling, a macroscopic model describing the evolution of densities of particles is obtained. We get this homogenization result for a general class of Frenkel–Kontorova models. The proof is based on the construction of suitable hull functions in the framework of viscosity solutions.

MSC: 35B27; 35F20; 45K05; 47G20; 49L25; 35B10

Keywords: Particle systems; Periodic homogenization; Frenkel–Kontorova models; Hamilton–Jacobi equations; Hull function; Cumulative distribution function; Slepčev formulation

1. Introduction

In the present paper we are interested in systems of ODEs describing the motion of particles in interactions with their neighbors and submitted to a periodic potential. An important special case

* Corresponding author.
E-mail address: imbert@ceremade.dauphine.fr (C. Imbert).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.06.034

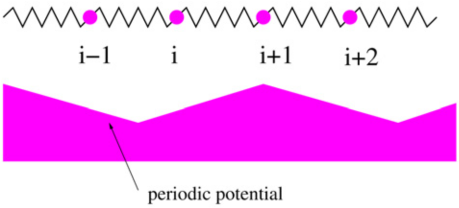

Fig. 1. Sketch of the chain or particles for the classical FK model.

is the classical Frenkel–Kontorova (FK) model in its fully overdamped version. This physical model is a very simple and very important one. For a good overview on the Frenkel–Kontorova model, we refer the reader to the recent book [8] of Braun and Kivshar and the article [13] of Floria and Mazo.

We want to study the limit of the system of ODEs as the number of particles per length unit goes to infinity. As we shall see, this can be understood as a homogenization procedure.

1.1. *The classical fully overdamped Frenkel–Kontorova model*

The classical Frenkel–Kontorova model describes a chain of classical particles evolving in a one-dimensional space, coupled with their neighbors and subjected to a periodic potential. If τ denotes time and $U_i(\tau)$ denotes the position of the particle $i \in \mathbb{Z}$, one of the simplest FK models is given by the following dynamics

$$m\frac{d^2U_i}{d\tau^2} + \gamma\frac{dU_i}{d\tau} = U_{i+1} - 2U_i + U_{i-1} + \sin(2\pi U_i) + L$$

where m denotes the mass of the particle, γ a friction coefficient, L is a constant driving force which can make the whole "train of particles" move and the term $\sin(2\pi U_i)$ describes the force created by a periodic potential whose period is assumed to be 1. Notice that in the previous equation, we set to one physical constants in front of the elastic and the exterior forces. If we assume that $m \ll \gamma = 1$, we can neglect the acceleration term and obtain for $i \in \mathbb{Z}$

$$\frac{dU_i}{d\tau} = U_{i+1} - 2U_i + U_{i-1} + \sin(2\pi U_i) + L \quad \text{for } \tau > 0. \tag{1.1}$$

This is the reason why we say that the dynamics of this model is fully overdamped. It can describe the friction between two materials. Indeed, this model was originally introduced in Kontorova, Frenkel [18] to describe the plasticity at a microscopic level. Such a model is sketched on Fig. 1.

We would like next to give the flavour of the results we obtain in this paper. In order to do so, let us assume that at initial time, particles satisfy

$$U_i(0) = \varepsilon^{-1}u_0(i\varepsilon)$$

for some $\varepsilon > 0$ and some Lipschitz continuous function $u_0(x)$ which satisfies the following assumption:

(A0) *Initial gradient bounded from above and below*

$$0 < 1/K_0 \leqslant (u_0)_x \leqslant K_0 \quad \text{on } \mathbb{R}$$

for some fixed $K_0 > 0$.

Such an assumption can be interpreted by saying that at initial time, the number of particles per length unit at the macroscopic level lies in $(K_0^{-1}\varepsilon^{-1}, K_0\varepsilon^{-1})$.

It is then natural to ask what the macroscopic behaviour of the solution U of (1.1) as ε goes to zero, i.e. as the number of particles per length unit goes to infinity, is. To this end we define the following function that describes the rescaled positions of the particles

$$\overline{u}^\varepsilon(t,x) = \varepsilon U_{\lfloor \varepsilon^{-1}x \rfloor}\big(\varepsilon^{-1}t\big) \tag{1.2}$$

where $\lfloor\cdot\rfloor$ denotes the floor integer part. One of our main results states that the limiting dynamics as ε goes to 0 of (1.1) is determined by a first-order Hamilton–Jacobi equation of the form

$$\begin{cases} u_t^0 = \overline{F}\big(u_x^0\big) & \text{for } (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u^0(0,x) = u_0(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{1.3}$$

where $\overline{F}$ is a continuous function to be determined. More precisely, we have the following homogenization result.

Theorem 1.1 *(Homogenization of the FK model). For all $L \in \mathbb{R}$, there exists a continuous function $\overline{F} : \mathbb{R} \to \mathbb{R}$ such that, under assumption* (A0)*, the function $\overline{u}^\varepsilon$ converges locally uniformly towards the unique viscosity solution u^0 of* (1.3)*.*

1.2. Generalized Frenkel–Kontorova models

In order to present our main results in full generality, we first describe the generalizations of the classical FK model we deal with.

An important remark about (1.1) is that such an ODE system can be embedded into a single PDE. In order to see this, let us first give the following definition. For a given integer $m \in \mathbb{N} \setminus \{0\}$ and for a function $v : \mathbb{R} \to \mathbb{R}$, we define

$$[v]_m(y) = \big(v(y-m), v(y-m+1), \dots, v(y+m)\big).$$

With this notation in hand, we claim that solving (1.1) for the family of initial condition $u_{0,\alpha}(\cdot) = u_0(\cdot + \alpha)$, $\alpha \in [0,1)$, is equivalent to solve

$$\partial_\tau U = F\big(\tau, [U]_1\big) \quad \text{for } (\tau,x) \in (0,+\infty) \times \mathbb{R}$$

submitted to the initial condition $U(0,x) = u_0(x)$ and where

$$F(\tau, V_{-1}, V_0, V_1) = V_{-1} - 2V_0 + V_1 + \sin(2\pi V_0) + L. \tag{1.4}$$

We can then consider generalized FK models with interactions with the mth nearest neighbors. Precisely, we look for solutions $u(\tau, y)$ to the following "finite difference-like" PDE

$$u_\tau = F\big(\tau, \big[u(\tau,\cdot)\big]_m\big) \quad \text{for } (\tau, y) \in (0,+\infty)\times\mathbb{R}. \tag{1.5}$$

In the present paper, we work with viscosity solutions, and even with possibly discontinuous ones (see Definition 2.1). Let us now make precise the assumptions we make on the function $F : \mathbb{R}\times\mathbb{R}^{2m+1} \to \mathbb{R}$ that maps (τ, V) to $F(\tau, V)$.

(A1) *Regularity*

$$\begin{cases} F \text{ is continuous,} \\ F \text{ is Lispchitz continuous in } V \text{ uniformly in } \tau. \end{cases}$$

(A2) *Monotonicity*

$$F(\tau, V_{-m}, \ldots, V_m) \text{ is non-decreasing in } V_i \quad \text{for } i \neq 0.$$

(A3) *Periodicity*

$$\begin{cases} F(\tau, V_{-m}+1, \ldots, V_m+1) = F(\tau, V_{-m}, \ldots, V_m), \\ F(\tau+1, V) = F(\tau, V). \end{cases}$$

Remarks 1.2. 1. When F does not depend on τ, we simply write $F(V)$.

2. We see that these assumptions are in particular satisfied for the classical FK model (1.1) (see Eq. (1.4)).

We next rescale the generalized FK model as we did for the classical one. Precisely, we now consider the following problem satisfied by $u^\varepsilon(t,x)$

$$\begin{cases} u^\varepsilon_t = F\left(\dfrac{t}{\varepsilon}, \left[\dfrac{u^\varepsilon(t,\cdot)}{\varepsilon}\right]^\varepsilon_m\right) & \text{for } (t,x) \in (0,+\infty)\times\mathbb{R}, \\ u^\varepsilon(0,x) = u_0(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{1.6}$$

where for some function $v(x)$ we set

$$[v]^\varepsilon_m(x) = \big(v(x-m\varepsilon), \ldots, v(x+m\varepsilon)\big).$$

We then have the following homogenization result.

Theorem 1.3 *(Homogenization of generalized FK models). Under assumptions* (A0)–(A3)*, there exists a continuous function $\overline{F} : \mathbb{R} \to \mathbb{R}$ such that the solution u^ε to* (1.6) *converges locally uniformly towards the unique viscosity solution u^0 of* (1.3).

We will explain in the next subsection how the so-called effective Hamiltonian $\overline{F}$ is determined. We will see that it has to do with the existence of so-called hull functions (see Theorem 1.5 below). But before giving further details, let us make several comments about this general homogenization result.

We would like first to shed light on an important fact. Assumption (A2) concerning the monotonicity of F is fundamental in our analysis. Indeed, This condition ensures that a comparison principle holds true for the solutions of (1.5) and this allow us to perform the homogenization limit in the framework of viscosity solutions. With this respect, Theorem 1.3 makes part of a huge literature concerning homogenization of Hamilton–Jacobi equations whose pioneering paper is the one of Lions, Papanicolaou, Varadhan [19].

The homogenization of a model with interactions with an infinite number of particles (i.e. the case $m = +\infty$) was studied in Forcadel, Imbert, Monneau [12] for a model describing dislocation dynamics.

Concerning the homogenization of equations with periodic terms in u/ε (which is the case of the models considered in the present paper), only very few results exist. Let us mention the recent result of Imbert, Monneau [15] and the one of Barles [6]. We can also mention the work of Boccardo, Murat [7] about the homogenization of elliptic equations and the one of Bacaër [4].

1.3. Hull functions

In order to study the solutions of (1.5), it is classical to introduce the so-called (dynamical) *hull function*, i.e. a function $h(\tau, z)$ such that $u(\tau, y) = h(\tau, py + \lambda\tau)$ is a solution of (1.5). We refer for instance to the pioneering work of Aubry [1,2], and Aubry, Le Daeron [3] where they studied (among other things) this notion in details.

Definition 1.4 *(Hull function).* Given F satisfying (A1)–(A3), a positive number $p \in (0, +\infty)$ and a real number $\lambda \in \mathbb{R}$, a locally bounded function $h : \mathbb{R}^2 \to \mathbb{R}$ is a *hull function* for (1.5) if it satisfies for all $(\tau, z) \in \mathbb{R}^2$

$$\begin{cases} h_\tau + \lambda h_z = F\big(\tau, h(\tau, z - mp), \ldots, h(\tau, z + mp)\big), \\ h(\tau + 1, z) = h(\tau, z), \\ h(\tau, z + 1) = h(\tau, z) + 1, \\ h_z(\tau, z) \geqslant 0, \\ \big|h(\tau, z + z') - h(\tau, z) - z'\big| \leqslant 1 \quad \text{for all } z' \in \mathbb{R}. \end{cases} \tag{1.7}$$

In the case where F is independent on τ, we require that the hull function h is also independent on τ and we denote it by $h(z)$.

Given $p > 0$, the following theorem explains how the effective Hamiltonian $\overline{F}(p)$ is determined by an existence/non-existence result of hull functions as $\lambda \in \mathbb{R}$ varies.

Theorem 1.5 *(Effective Hamiltonian and hull function). Given F satisfying* (A1)–(A3) *and $p \in (0, +\infty)$, there exists a unique real λ for which there exists a hull function h (depending on p) satisfying* (1.7)*. Moreover the real number λ, seen as a function $\overline{F}$ of p, is continuous on* $(0, +\infty)$.

1.4. Qualitative properties of the effective Hamiltonian

In this subsection, we list important qualitative properties of the effective Hamiltonian defined thanks to Theorem 1.5. Keeping in mind the first FK model we described (1.1), we are in particular interested in the behaviour of $\overline{F}$ when it is computed by replacing F with $F + L$. We will give several results about the function $\overline{F}$ seen as a function of L. Let us state a precise result.

Theorem 1.6 *(Qualitative properties of $\overline{F}$ for general F). Consider a non-linearity F satisfying* (A1)–(A3). *Given $p > 0$ and $L \in \mathbb{R}$, let $\overline{F}(L, p)$ denote the effective Hamiltonian defined thanks to Theorem* 1.5 *where F is replaced with $F + L$.*

Then $\overline{F} : \mathbb{R}^2 \to \mathbb{R}$ is continuous and we have the following properties:

(a1) (Bound) *There exists a constant $C > 0$ such that for all $(L, p) \in \mathbb{R} \times (0, +\infty)$*

$$\left|\overline{F}(L, p) - L\right| \leqslant C(1 + p).$$

(a2) (Monotonicity in L)

$$\overline{F}(L, p) \text{ is non-decreasing in } L.$$

(a3) (Antisymmetry in V) *If for all $(\tau, V) \in \mathbb{R} \times \mathbb{R}^{2m+1}$, $F(\tau, -V) = -F(\tau, V)$, then*

$$\overline{F}(0, p) = 0 \quad \text{for any } p > 0.$$

(a4) (Periodicity in p) *Assume that for all $(\tau, V) \in \mathbb{R} \times \mathbb{R}^{2m+1}$*

$$F(\tau, V_{-m} - m, \ldots, V_m + m) = F(\tau, V_{-m}, \ldots, V_m), \tag{1.8}$$

then

$$\overline{F}(L, p + 1) = \overline{F}(L, p).$$

(a5) (Continuous hull function/no plateau of $L \mapsto \overline{F}(L, p)$) *Assume that for some $(L_0, p) \in \mathbb{R} \times (0, +\infty)$, there exists a continuous hull function $h(\tau, z)$. Then for all $L \neq L_0$*

$$\overline{F}(L, p) \neq \overline{F}(L_0, p).$$

We next say more about property (a5) about the characterization of plateaux of the function $\overline{F}$ seen as a function of L. Let us first consider the following example for $m = 1$

$$F = F(\tau, V_{-1}, V_0, V_1) = \alpha(V_1 - 2V_0 + V_{-1}) + \beta \sin(2\pi V_0) + \gamma \cos(2\pi \tau) \tag{1.9}$$

which satisfies in particular condition (1.8). In particular for this model, the full picture is 1-periodic in p. For some suitable constants $\alpha, \beta, \gamma > 0$, numerical simulations (see Braun and Kivshar [8, p. 334] and the references cited therein), seem to show that the map $L \mapsto \overline{F}(L, p)$ may have many plateaux as illustrated on Fig. 2. See also Hu, Qin, Zheng [14] and Chapter 11 (homeomorphism of the circle) of the book [17] of Katok and Hasselblatt, for an interesting attempt of explanation of this behaviour. From (a5), we deduce in particular that the hull function is not continuous in space at points corresponding to these plateaux.

Here are further results related to this issue in the case where F does not depend on time τ.

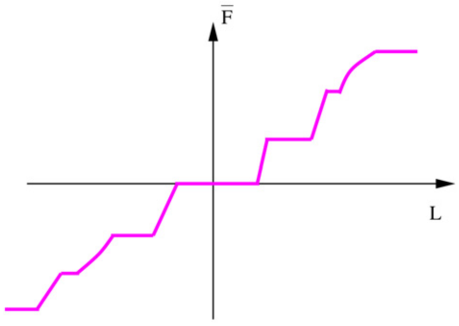

Fig. 2. Sketch of $\overline{F}$ as a function of L when F depends on τ.

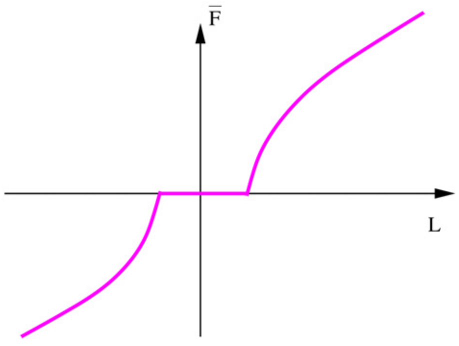

Fig. 3. Sketch of $\overline{F}$ as a function of L for F independent on τ.

Theorem 1.7 *(Further plateau properties when F does not depend on τ). Under the assumptions of Theorem* 1.6*, assume moreover that F does not depend on τ. Then we have the following properties*:

(b1) (No plateau in L if $\overline{F} \neq 0$) *There exists a constant $C > 0$ such that for all $(L, p) \in \mathbb{R} \times (0, +\infty)$, we have, in the distribution sense,*

$$\frac{\partial \overline{F}}{\partial L}(L, p) \geqslant \frac{|\overline{F}(L, p)|}{|L| + C(1 + p)}.$$

(b2) (0-plateau property) *Assume that the map $v \mapsto F(v, \dots, v)$ is not constant and that F satisfies property* (1.8). *Then there exists $L_0 \in \mathbb{R}$ and $\delta > 0$ such that*

$$\overline{F}(L, p) = 0 \quad \textit{for all } (L, p) \textit{ s.t. } L \in (L_0 - \delta, L_0 + \delta),\ p \in \mathbb{N} \setminus \{0\}.$$

In the case where F does not depend on the time τ, we see from (b1) that the map $L \mapsto \overline{F}(L, p)$ has at most a single plateau at the level $\overline{F} = 0$, as illustrated on Fig. 3.

Let us now turn to some comments on the 0-plateau. For model (1.9) with $\alpha = 1$, $\gamma = 0$, more is known when p is a Diophantine number, i.e. satisfies for some $\kappa, \nu > 0$

$$\forall a \in \mathbb{Z},\ \forall b \in \mathbb{Z} \setminus \{0\}, \quad |bp - a| \geqslant \kappa |b|^{-\nu}.$$

These Diophantine numbers (when considered for any $\kappa, \nu > 0$) have full measure. It is proven in De La Llave [10], that if p is Diophantine, then there exists an analytic hull function for $L = 0$ with $\overline{F}(0, p) = 0$ as soon as $|\beta|$ is smaller than a constant depending on κ, ν. In particular, we deduce from (a5), that the map $L \mapsto \overline{F}(L, p)$ has no plateau at all in this case.

As it is well known (see [1]), this result is only valid for β small enough, because the hull function has to satisfy (see (1.7) and recall that $\lambda = \overline{F}(0, p) = 0$)

$$\big(h(z+p) - h(z)\big) - \big(h(z) - h(z-p)\big) = -\beta \sin\big(2\pi h(z)\big).$$

Moreover h satisfies $h(z+1) = h(z) + 1$ and is non-decreasing, which for instance for $p \in (0, 1]$, implies that

$$\big|\beta \sin\big(2\pi h(z)\big)\big| \leqslant 2.$$

Therefore h cannot take the value $1/4$ for $|\beta| > 2$, and then h has to be discontinuous for $|\beta| > 2$. This is the well-known breaking of analyticity. See also [2,16] for some explicit computations of the hull functions for particular potentials.

Even for $|\beta| \neq 0$ arbitrarily small, (b2) shows that the map $L \mapsto \overline{F}(L, p)$ has a 0-plateau for integers $p > 0$. From (a5), this implies in particular the breaking of continuity for the hull function corresponding to such p and L.

As an example, in model (1.9) with $\alpha = 1$, $\gamma = 0$, for any $\beta > 0$ and $L = \beta > 0$, $p = 1$, the following function

$$h(z) = -\frac{1}{4} + \lfloor z \rfloor$$

is a discontinuous hull function with $\overline{F}(L, p) = 0$.

This shows that for the same model, the 0-plateau property of the map $L \mapsto \overline{F}(L, p)$ can be very sensitive to the values of p (and of its irrationality).

1.5. Organization of the article

In Section 2, we recall the notion of viscosity solutions. In Section 3, we prove the homogenization results, namely Theorems 1.1 and 1.3. In Section 4, we prove the ergodicity of the problem; precisely, we prove Theorem 1.5 on the hull function. In Section 5, we build Lipschitz sub- and super-hull function, using an approximate Hamiltonian. In Section 6, we prove Theorems 1.6 and 1.7 on the qualitative properties of the effective Hamiltonian. Finally in Appendix A, we propose a discussion on the relation between the hull function for our problem and the correctors for a "dual" approach of the problem: the so-called Slepčev formulation.

2. Viscosity solutions

This section is devoted to the definition of viscosity solutions for equations such as (1.5), (1.6) and (1.7). In order to construct hull functions when proving Theorem 1.5, we will also need to consider a perturbation of (1.7) with linear plus bounded initial data. For all these reasons, we define a viscosity solution for a generic equation whose Hamiltonian G satisfies proper assumptions.

Before making precise assumptions, definitions and fundamental results we will need later (such as stability, comparison principle, existence), we refer the reader to Barles [5] and the user's guide of Crandall, Ishii, Lions [9] for an introduction to viscosity solutions.

2.1. Main assumptions and definitions

Consider for $0 < T \leqslant +\infty$ the following Cauchy problem

$$\begin{cases} u_\tau = G\Big(\tau, \big[u(\tau,\cdot)\big]_m, \inf_{y'\in\mathbb{R}} \big(u(\tau,y') - py'\big) + py - u(\tau,y), u_y\Big) & \text{for } (\tau,y)\in(0,T)\times\mathbb{R}, \\ u(0,y) = u_0(y) & \text{for } y\in\mathbb{R} \end{cases} \tag{2.10}$$

for a general non-linearity G. The most important example we have in mind is the following

$$G(\tau, V, a, q) = F(\tau, V) + \delta(a_0 + a)q$$

for some constants $\eta, \delta \geqslant 0$, $a_0, a, q \in \mathbb{R}$ and where F appears in (1.5)–(1.7).

We make the following assumptions on G.

(A1′) *Regularity*

$$\begin{cases} G \text{ is continuous,} \\ \text{for all } R>0, \quad G(\tau,V,a,q) \text{ is Lispchitz continuous in } (V,a) \\ \quad \text{uniformly in } (\tau,q)\in\mathbb{R}\times[-R,R]. \end{cases}$$

(A2′) *Monotonicity*

$$G(\tau, V_{-m}, \ldots, V_m, a, q) \text{ is non-decreasing in } a \text{ and } V_i \quad \text{for } i \neq 0.$$

(A3′) *Periodicity*. For all $(\tau, V, a, q) \in \mathbb{R}\times\mathbb{R}^{2m+1}\times\mathbb{R}\times\mathbb{R}$

$$G(\tau, V_{-m}+1, V_{-m+1}+1, \ldots, V_m+1, a, q) = G(\tau, V_{-m}, V_{-m+1}, \ldots, V_m, a, q),$$
$$G(1+\tau, V, a, q) = G(\tau, V, a, q).$$

In view of (2.10), it is clear that, if G effectively depends on the variable a, solutions must be such that the infimum of $u(\tau,y) - p\cdot y$ is finite for all time τ. We will even only consider solutions u satisfying for some $C(T) > 0$: for all $\tau \in [0,T)$ and all $y, y' \in \mathbb{R}$

$$\big|u(\tau, y+y') - u(\tau,y) - py'\big| \leqslant C. \tag{2.11}$$

When $T = +\infty$, we may assume that (2.11) holds true for all time $T_0 > 0$ for a family of constants $C_0 > 0$.

Since we have to solve a Cauchy problem, we have to assume that the initial datum satisfies the assumption

(A0′) (*Initial condition*) u_0 satisfies (A0); it also satisfies (2.11) if G depends on a.

Finally, we recall the definition of the upper and lower semi-continuous envelopes, u^* and u_*, of a locally bounded function u

$$u^*(\tau, y) = \limsup_{(t,x)\to(\tau,y)} u(t,x) \quad \text{and} \quad u_*(\tau, y) = \liminf_{(t,x)\to(\tau,y)} u(t,x).$$

We can now define viscosity solutions for (2.10).

Definition 2.1 *(Viscosity solutions).* Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a continuous function and $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a locally bounded function such that (2.11) holds true if G depends on a.

- The function u is a *subsolution* (resp. a *supersolution*) of (2.10) on an open set $\Omega \subset (0, T) \times \mathbb{R}$ if u is upper semi-continuous (resp. lower semi-continuous) and for all $(\tau, y) \in \Omega$ and all test function $\phi \in C^1(\Omega)$ such that $u - \phi$ attains a strict local maximum (resp. a strict local minimum) at the point (τ, y), then we have

$$\phi_\tau(\tau, y) \leqslant G\Big(\tau, \big[u(\tau, \cdot)\big]_m, \inf_{y'\in\mathbb{R}}\big(u(\tau, y') - py'\big) + py - u(\tau, u), \phi_y(\tau, y)\Big) \quad (\text{resp.} \geqslant). \tag{2.12}$$

- The function u (resp. v) is a *subsolution* (resp. *supersolution*) on $[0, T) \times \mathbb{R}$, if u is a subsolution (resp. v is a supersolution) on $\Omega = (0, T) \times \mathbb{R}$ and if moreover it satisfies for all $y \in \mathbb{R}$

$$u(0, y) \leqslant u_0(y) \quad (\text{resp.} \geqslant).$$

- A function u is a *viscosity solution* of (2.10) if u^* is a subsolution and u_* is a supersolution.

Remark 2.2. A locally bounded function u is also (classically) called a subsolution (resp. supersolution) if its upper semi-continuous envelope (resp. lower semi-continuous envelope) is a subsolution in the sense of the previous definition.

The first main property of this notion of solution is its stability when passing to the limit. More precisely, a family of subsolutions $(u_\varepsilon)_{\varepsilon>0}$ that is uniformly locally bounded from above is stable when passing to the so-called relaxed upper semi-limit $\overline{u}$ defined as follows

$$\overline{u}(\tau, y) = \limsup_{\varepsilon}{}^* u^\varepsilon(\tau, y) = \limsup_{(t,x)\to(\tau,y),\, \varepsilon\to 0} u^\varepsilon(t, x).$$

Such a relaxed upper semi-limit is well defined as soon as the family of functions u_ε is uniformly locally bounded from above. Remark that $\overline{u}$ is upper semi-continuous and if u_ε does not depend on ε ($u_\varepsilon = u$ for all $\varepsilon > 0$), we recover the upper semi-continuous envelope of the function u. In the same way, we can define the relaxed lower semi-limit of a family of lower semi-continuous functions that are uniformly locally bounded from below. The main discontinuous stability result for viscosity solutions is stated as follows.

Proposition 2.3 *(Stability of viscosity solutions). Assume* (A1′), (A2′) *and* $T < +\infty$*. Assume that* $(u^\varepsilon)_\varepsilon$ *is a sequence of subsolutions* (*resp. supersolutions*) *of Eq.* (2.10) *on* $(0, T) \times \mathbb{R}$ *satisfying* (2.11) *with the same constant* $C > 0$*. Then the relaxed upper semi-limit* $\overline{u}$ *is a subsolution* (*resp.* $\underline{u}$ *is a supersolution*) *of* (2.10) *on* $(0, T) \times \mathbb{R}$.

We will also use stability of subsolutions by passing to the supremum. Let us be more specific.

Proposition 2.4 *(Stability of viscosity solutions (II)). Assume* (A1′), (A2′) *and* $T < +\infty$*. Assume that* $(u^\alpha)_{\alpha\in\mathcal{A}}$ *is a family of subsolutions* (*resp. supersolutions*) *of Eq.* (2.10) *on* $(0, T) \times \mathbb{R}$ *satisfying* (2.11) *with the same constant* $C > 0$*. Then* $\sup_{\alpha\in\mathcal{A}} u_\alpha$ *is a subsolution* (*resp.* $\underline{u}$ *is a supersolution*) *of* (2.10) *on* $(0, T) \times \mathbb{R}$.

We skip the proofs of both propositions since they are straightforward adaptations of classical ones (see for instance [5]).

2.2. Comparison principles and existence

This subsection is devoted to state comparison principles that are used throughout the paper and to get the main existence results for the PDEs at stake.

We first state two comparison principles for the generic Hamilton–Jacobi equation (2.10). One is stated on the whole space while the second one is stated on bounded sets.

Proposition 2.5 *(Comparison principle). Assume* (A0′)–(A2′)*. Assume that* u *and* v *are respectively a subsolution and a supersolution of* (2.10) *on* $[0, T) \times \mathbb{R}$*. Then we have* $u \leqslant v$ *on* $[0, T) \times \mathbb{R}$.

For a given point $(\tau_0, y_0) \in (0, T) \times \mathbb{R}$ and for all $r, R > 0$, let us set

$$Q_{r,R} = (\tau_0 - r, \tau_0 + r) \times (y_0 - R, y_0 + R).$$

Proposition 2.6 *(Comparison principle on bounded sets). Assume* (A1′) *and* (A2′) *and that* $G(\tau, V, a, q)$ *does not depend on the variable* a*. Assume that* u *is a subsolution* (*resp.* v *a supersolution*) *of* (2.10) *on the open set* $Q_{r,R} \subset (0, T) \times \mathbb{R}$*. Assume also that*

$$u \leqslant v \quad \text{on } \overline{Q}_{r,R+m} \setminus Q_{r,R}.$$

Then $u \leqslant v$ *on* $Q_{r,R}$.

Remarks 2.7.

- Here we need to increase the domain with a distance m, because the equation is non-local in space (recall that each particle has interactions with its m nearest neighbors on the left and on the right).
- We could ask to have only $u \leqslant v$ on $(\overline{Q}_{r,R+m} \setminus Q_{r,R}) \cap \{\tau < \tau_0 + r\}$.

We now turn to the construction of solution. We recall the celebrated Perron's method.

Proposition 2.8 *(Existence by Perron's method). Assume* (A1′) *and* (A2′). *Assume that u is a subsolution* (*resp. v is a supersolution*) *of* (2.10) *on* $(0,T)\times\mathbb{R}$ *such that*

$$u \leqslant v \quad \text{on } (0,T)\times\mathbb{R}.$$

Let $\mathcal{C}$ be the set of all supersolutions $\tilde{v}$ of (2.10) *on* $(0,T)\times\mathbb{R}$ *satisfying* (2.11) *with C corresponding to u and v and such that $\tilde{v}\geqslant u$. Let*

$$w(\tau,y)=\inf\bigl\{\tilde{v}(\tau,y) \text{ such that } \tilde{v}\in\mathcal{C}\bigr\}.$$

Then w is a (*discontinuous*) *solution of* (2.10) *on* $(0,T)\times\mathbb{R}$ *satisfying $u\leqslant w\leqslant v$ and* (2.11).

We skip the proofs of Propositions 2.5, 2.6 and 2.8 since they are completely classical.

The important corollary of the proposition is the following well-posedness result for (2.10).

Corollary 2.9 *(Existence and uniqueness for the Cauchy problem). Assume* (A0′)–(A3′). *Then there exists a unique solution u of* (2.10) *on* $[0,+\infty)\times\mathbb{R}$. *Moreover u is continuous.*

Remark 2.10. As we will see in the proof, the solution u we construct satisfies (2.11) with a constant C which depends on the initial data u_0. Indeed, this is due to the construction of barriers (see Lemma 2.11).

Proof of Corollary 2.9. In order to apply Proposition 2.8, we need to construct barriers. In view of assumptions (A1′) and (A3′), the constant G_0 defined by

$$G_0=\sup_{\tau\in\mathbb{R},\ |q|\leqslant K_0}\bigl|G(\tau,0,0,q)\bigr| \tag{2.13}$$

is finite. Moreover, using (A1′), let us introduce the constants K_1 and K_2 such that for all $\tau,a,b\in\mathbb{R}$, $V,W\in\mathbb{R}^{2m+1}$, $q\in(-K_0,K_0)$,

$$\bigl|G(\tau,V,a,q)-G(\tau,W,b,q)\bigr|\leqslant K_1|V-W|_\infty+K_2|a-b| \tag{2.14}$$

with $|W|_\infty=\sup_{k=-m,\dots,m}|W_k|$. Then we have the following lemma whose proof is postponed.

Lemma 2.11 *(Existence of barriers). Assume* (A0′)–(A3′). *There exists a constant $C>0$ such that*

$$u^+(\tau,y)=u_0(y)+C\tau \quad \text{and} \quad u^-(\tau,y)=u_0(y)-C\tau$$

are respectively supersolution and subsolution of (2.10) *on* $[0,T)\times\mathbb{R}$ *for any $T>0$.*

Moreover, we can choose

$$C=K_2C_1+C_0(m,K_0,K_1,G_0) \tag{2.15}$$

where K_2, K_1 and G_0 are given respectively in (2.14) *and* (2.13). *Here C_1 is given in* (A0′).

From Lemma 2.11 and Proposition 2.8, we get the existence of a function u which is a solution of (2.10) on $(0,+\infty)\times\mathbb{R}$ and satisfies $u^-\leqslant u\leqslant u^+$. Therefore the initial condition is satisfied. Moreover $u^*(0,\cdot)=u_*(0,\cdot)$ and from the comparison principle (Proposition 2.5), we get that $u^*\leqslant u_*$ for all time which implies that u is continuous. Finally, still from Proposition 2.5, we deduce the uniqueness of the solution of (2.10) on $[0,+\infty)\times\mathbb{R}$. □

We now turn to the proof of the lemma.

Proof of Lemma 2.11. We set $u^\pm(\tau,y)=u_0(y)\pm C\tau$ for some C to be fixed later. We have

$$\begin{aligned}
&\Big|G\Big(\tau,\big[u^\pm(\tau,\cdot)\big]_m(y),\ \inf_{y'\in\mathbb{R}}\big(u^\pm(\tau,y')-py'\big)+py-u^\pm(\tau,y),u^\pm_y(\tau,y)\Big)\Big|\\
&\quad=\Big|G\Big(\tau,\big[u^\pm(\tau,\cdot)-\big\lfloor u^\pm(\tau,y)\big\rfloor\big]_m(y),\ \inf_{y'\in\mathbb{R}}\big(u_0(y')-py'\big)+py-u_0(y),(u_0)_y(y)\Big)\Big|\\
&\quad\leqslant K_2C_1+K_1+\big|G\big(\tau,\big[u^\pm(\tau,\cdot)-u^\pm(\tau,y)\big]_m(y),0,(u_0)_y(y)\big)\big|\\
&\quad\leqslant K_2C_1+K_1+G_0+K_1mK_0=:K_2C_1+C_0
\end{aligned}$$

where we have used the periodicity assumption (A3′) for the second line, assumption (A0′) for the third line, and for the last line, we have used $|u^\pm(\tau,y')-u^\pm(\tau,y)|\leqslant K_0|y'-y|$.

When $G(\tau,V,a,q)$ is independent on a, we can simply choose $K_2=0$. This ends the proof of the lemma. □

3. Convergence

This section is devoted to the proof of the main homogenization result (Theorem 1.3). The proof relies on the existence of hull functions (Theorem 1.5) and qualitative properties of the effective Hamiltonian (Theorem 1.6). As a matter of fact, we will use the existence of Lipschitz sub- and super-hull functions (see Proposition 5.3). All these results are proved in the next sections.

We start with some preliminary results. The following result is a straightforward corollary of Lemma 2.11 by a change of variables:

Lemma 3.1 *(Barriers uniform in ε). Assume* (A0)–(A3). *Then there is a constant $C>0$, such that for all $\varepsilon>0$, the solution u^ε to* (1.6) *satisfies for all $t>0$ and $x\in\mathbb{R}$*

$$\big|u^\varepsilon(t,x)-u_0(x)\big|\leqslant Ct.$$

We have

Lemma 3.2 *(ε-bounds on the gradient). Assume* (A0)–(A3). *Then the solution u^ε of* (1.6) *satisfies for all $t>0$, $x\in\mathbb{R}$, $z>0$*

$$\varepsilon\left\lfloor\frac{z}{\varepsilon K_0}\right\rfloor\leqslant u^\varepsilon(t,x+z)-u^\varepsilon(t,x)\leqslant\varepsilon\left\lceil\frac{zK_0}{\varepsilon}\right\rceil\quad\textit{for all }(t,x)\in[0,+\infty)\times\mathbb{R}.\tag{3.16}$$

Remark 3.3. In particular we find that the solution $u(t,x)$ is non-decreasing in x.

Proof of Lemma 3.2. We prove the bound from below (the proof is similar for the bound from above). We first remark that (A0) implies that the initial condition satisfies

$$u_0(x+z) \geqslant u_0(x) + z/K_0 \geqslant u_0(x) + k\varepsilon \quad \text{with } k = \left\lfloor \frac{z}{\varepsilon K_0} \right\rfloor. \tag{3.17}$$

From (A3), we know that for $\varepsilon = 1$, the equation is invariant by addition of integer to the solutions. After the rescaling, Eq. (1.6) is invariant by addition of constants $k\varepsilon$ with k an integer. For this reason the solution with initial data $u_0 + k\varepsilon$ is $u^\varepsilon + k\varepsilon$. Similarly the equation is invariant by translations. Therefore the solution with initial data $u_0(x+z)$ is $u^\varepsilon(t, x+z)$. Finally, from (3.17) and the comparison principle (Proposition 2.5), we get

$$u^\varepsilon(t, x+z) \geqslant u^\varepsilon(t, x) + k\varepsilon$$

which proves the bound from below. This ends the proof of the lemma. □

We now turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $\overline{u}$ (resp. $\underline{u}$) denote the relaxed upper (resp. lower) semi-limit associated with the family of functions $(u_\varepsilon)_{\varepsilon>0}$. These functions are well defined thanks to Lemma 3.1. We also get from this lemma and Lemma 3.2 that both functions $w = \overline{u}, \underline{u}$ satisfy for all $t > 0$, $x, x' \in \mathbb{R}$

$$\begin{aligned} &\left|w(t,x) - u_0(x)\right| \leqslant Ct, \\ &\qquad K_0^{-1}|x - x'| \leqslant w(t,x) - w(t,x') \leqslant K_0|x - x'|. \end{aligned} \tag{3.18}$$

We are going to prove that $\overline{u}$ is a subsolution of (1.3) on $\mathbb{R}^+ \times \mathbb{R}$. Similarly, we can prove that $\underline{u}$ is a supersolution of the same equation. Therefore, from the comparison principle for (1.3), we get that $u^0 \leqslant \underline{u} \leqslant \overline{u} \leqslant u^0$. And then $\overline{u} = \underline{u} = u^0$, which shows the expected convergence of the full sequence u^ε towards u^0.

We now prove in several steps that $\overline{u}$ is a subsolution of (1.3) on $(0, +\infty) \times \mathbb{R}$. We classically argue by contradiction by assuming that $\overline{u}$ is not a subsolution on $(0, +\infty) \times \mathbb{R}$. Then there exists $(\bar{t}, \bar{x}) \in (0, +\infty) \times \mathbb{R}$ and a test function $\phi \in C^1$ such that

$$\begin{cases} \overline{u}(\bar{t}, \bar{x}) = \phi(\bar{t}, \bar{x}), & \\ \overline{u} \leqslant \phi & \text{on } Q_{r,2r}(\bar{t}, \bar{x}), \text{ with } r > 0, \\ \overline{u} \leqslant \phi - 2\eta & \text{on } Q_{r,2r}(\bar{t}, \bar{x}) \setminus Q_{r,r}(\bar{t}, \bar{x}), \text{ with } \eta > 0, \\ \phi_t(\bar{t}, \bar{x}) = \overline{F}\big(\phi_x(\bar{t}, \bar{x})\big) + \theta, & \text{with } \theta > 0 \end{cases} \tag{3.19}$$

where we recall that $Q_{r,R}(\bar{t}, \bar{x})$ denotes for r, $R > 0$

$$Q_{r,R}(\bar{t}, \bar{x}) = (\bar{t} - r, \bar{t} + r) \times (\bar{x} - R, \bar{x} + R).$$

Let p denote $\phi_x(\bar{t}, \bar{x})$. From (3.18), we get

$$0 < 1/K_0 \leqslant p \leqslant K_0. \tag{3.20}$$

Combining Theorems 1.5 and 1.6 (in particular (a1) and (a2)), we get the existence of a hull function h associated with p such that

$$\lambda = \overline{F}(p) + \frac{\theta}{2} = \overline{F}(\overline{L}, p) \quad \text{with } \overline{L} > 0.$$

Indeed, we know from these results that the effective Hamiltonian is non-decreasing in L, continuous and goes to $\pm\infty$ as $L \to \pm\infty$.

We now apply the perturbed test function method introduced by Evans [11] in terms here of hull functions instead of correctors. Precisely, let us consider the following twisted perturbed test function

$$\phi^\varepsilon(t,x) = \varepsilon h\left(\frac{t}{\varepsilon}, \frac{\phi(t,x)}{\varepsilon}\right).$$

Here the test function is twisted similarly as in [15]. In order to get a contradiction, we first assume that h is smooth and is continuous in z uniformly in $\tau \in \mathbb{R}$. In view of the third line of (1.7), we see that this implies that h is uniformly continuous in z (uniformly in $\tau \in \mathbb{R}$). For simplicity, and since we will construct approximate hull functions with such a regularity, we just assume that h is Lipschitz continuous in z (uniformly in $\tau \in \mathbb{R}$). We will next see how to treat the general case.

Case 1. h is smooth and Lipschitz continuous in z.

Step 1.1: ϕ^ε is a supersolution of (1.6) *on a neighbourhood of* $(\bar{t}, \bar{x})$. When h is smooth enough (i.e. C^1 here), it is sufficient to check directly the supersolution property of ϕ^ε for $(t,x) \in Q_{r,r}(\bar{t}, \bar{x})$. We have, with $\tau = t/\varepsilon$ and $z = \phi(t,x)/\varepsilon$,

$$\begin{aligned}
&\phi^\varepsilon_t(t,x) - F\left(\tau, \left[\frac{\phi^\varepsilon(t,\cdot)}{\varepsilon}\right]^\varepsilon_m(x)\right)\\
&\quad = h_\tau(\tau,z) + \phi_t(t,x)h_z(\tau,z) - F\left(\tau, \left[h\left(\tau, \frac{\phi(t,\cdot)}{\varepsilon}\right)\right]^\varepsilon_m(x)\right)\\
&\quad = \big(\phi_t(t,x) - \lambda\big)h_z(\tau,z) + \overline{L} + F\big(\tau, \big[h(\tau,\cdot)\big]^p_m(z)\big) - F\left(\tau, \left[h\left(\tau, \frac{\phi(t,\cdot)}{\varepsilon}\right)\right]^\varepsilon_m(x)\right)\\
&\quad \geqslant \big(\phi_t(t,x) - \lambda\big)h_z(\tau,z) + \overline{L} - L_F\left|\big[h(\tau,\cdot)\big]^p_m(z) - \left[h\left(\tau, \frac{\phi(t,\cdot)}{\varepsilon}\right)\right]^\varepsilon_m\right|_\infty
\end{aligned} \tag{3.21}$$

where we have used that Eq. (1.7) is satisfied by h to get the third line and (A1) to get the fourth one; here, L_F denotes the Lipschitz constant of F with respect to V for the norm $|\cdot|_\infty$ on $\mathbb{R}^{2m+1}$. Let us next estimate, for $j \in \{-m, \dots, m\}$ and ε such that $m\varepsilon \leqslant r$, the following quantity

$$h(\tau, z + jp) - h\left(\tau, \frac{\phi(t, x + j\varepsilon)}{\varepsilon}\right) = h(\tau, z + jp) - h\big(\tau, z + jp + o_r(1)\big)$$

where $o_r(1)$ only depends on the modulus of continuity of ϕ_x on $Q_{r,r}(\bar{t},\bar{x})$. Hence, if h is Lipschitz continuous with respect to z uniformly in τ, we conclude that we can choose ε small enough so that

$$\overline{L} - L_F \left| \left[h(\tau,\cdot)\right]_m^p(z) - \left[h\left(\tau, \frac{\phi(t,\cdot)}{\varepsilon}\right)\right]_m^\varepsilon \right|_\infty \geqslant 0. \tag{3.22}$$

Combining (3.21) and (3.22), we obtain

$$\begin{aligned}\phi_t^\varepsilon(t,x) - F\left(\tau, \left[\frac{\phi^\varepsilon(t,x)}{\varepsilon}\right]_m^\varepsilon(x)\right) &\geqslant (\phi_t - \lambda) h_z(\tau,z)\\ &= \left(\frac{\theta}{2} + \phi_t(t,x) - \phi_t(\bar{t},\bar{x})\right) h_z(\tau,z)\\ &= \left(\frac{\theta}{2} + o_r(1)\right) h_z(\tau,z) \geqslant 0.\end{aligned}$$

We used the non-negativity of h_z, the fact that $\theta > 0$ and again the fact that ϕ is C^1, to get the result on $Q_{r,r}(\bar{t},\bar{x})$ for $r > 0$ small enough. Therefore, when h is smooth and Lipschitz continuous on z uniformly in τ, ϕ^ε is a viscosity supersolution of (1.6) on $Q_{r,r}(\bar{t},\bar{x})$.

Step 1.2: Getting the contradiction. By construction, we have $\phi^\varepsilon \to \phi$ as $\varepsilon \to 0$, and therefore from (3.19), we get for ε small enough

$$u^\varepsilon \leqslant \phi^\varepsilon - \eta \leqslant \phi^\varepsilon - \varepsilon k_\varepsilon \quad \text{on } \overline{Q}_{r,2r}(\bar{t},\bar{x}) \setminus Q_{r,r}(\bar{t},\bar{x})$$

with the integer

$$k_\varepsilon = \lfloor \eta/\varepsilon \rfloor.$$

Therefore, for $m\varepsilon \leqslant r$, we can apply the comparison principle on bounded sets (Proposition 2.6) to get

$$u^\varepsilon \leqslant \phi^\varepsilon - \varepsilon k_\varepsilon \quad \text{on } Q_{r,r}(\bar{t},\bar{x}). \tag{3.23}$$

Passing to the limit as ε goes to zero, we get

$$\bar{u} \leqslant \phi - \eta \quad \text{on } Q_{r,r}(\bar{t},\bar{x})$$

which gives a contradiction with $\bar{u}(\bar{t},\bar{x}) = \phi(\bar{t},\bar{x})$ in (3.19). Therefore $\bar{u}$ is a subsolution of (1.3) on $(0,+\infty) \times \mathbb{R}$ and this ends the proof of the theorem.

Case 2. General case for h.

In the general case, we cannot check by a direct computation that ϕ^ε is a supersolution on $Q_{r,r}(\bar{t},\bar{x})$. The difficulty is due to the fact that $h(\tau,z)$ may not be Lipschitz continuous in the variable z.

This kind of difficulties was overcomed in [15] by using Lipschitz super-hull functions, i.e. functions satisfying (1.7) with $\geqslant$ instead of $=$ in the first line. Indeed, it is clear from the previous computations that it is enough to conclude. In [15], such regular super-hull functions (as a matter of fact, regular super-correctors) were build as exact solutions of an approximate Hamilton–Jacobi equation. Moreover this Lipschitz hull function is a supersolution for the exact Hamiltonian with a slightly bigger λ.

Here we conclude using a similar result, namely Proposition 5.3. Notice that the fact that h is smooth is not a restriction, the previous argument being completely valid in the viscosity sense since p satisfies (3.20). See [15] for further details. This ends the proof of the theorem. □

We continue with the proof of Theorem 1.1.

Proof of Theorem 1.1. Remark that the initial condition satisfies

$$u_0(y) - \varepsilon\lceil K_0\rceil \leqslant \bar{u}^\varepsilon(0, y) \leqslant u_0(y).$$

Therefore the comparison with the solution u^ε of (1.6) gives

$$u^\varepsilon - \varepsilon\lceil K_0\rceil \leqslant \bar{u}^\varepsilon \leqslant u^\varepsilon \quad \text{on } [0, +\infty) \times \mathbb{R}.$$

Using the convergence of u^ε to u^0 given in Theorem 1.3, we deduce that $\bar{u}^\varepsilon \to u^0$. This ends the proof of the theorem. □

4. Ergodicity and construction of hull functions

In this section, we prove Theorem 1.5 that defines the effective Hamiltonian $\overline{F}$ and states the existence of hull functions.

As we shall see, for given real numbers (L, p), the constant $\overline{F}(L, p)$ is (classically) defined as the "time slope" (in a sense to be made precise, see Proposition 4.1) of the solution of an initial Cauchy problem. This is the reason why the Hamiltonian is said to be *ergodic*.

Since approximate Lipschitz continuous hull functions must be constructed (see the proof of convergence in the preceding section), we work with the general (approximate) Hamiltonian G considered in Section 2. Hence, the Cauchy problem we work with is (2.10).

4.1. Ergodicity

In this subsection, we successively prove two propositions. The first one (Proposition 4.1) asserts that ergodicity holds true for G as soon as we are able to control space oscillations of the solution u of (2.10). The next proposition (Proposition 4.2) asserts that we are indeed able to control space oscillations and that the solution u satisfies additional important properties.

Let us first start with

Proposition 4.1 *(Time oscillations controlled by space oscillations). Assume* (A0′)–(A3′)*, and let u be a solution of* (2.10) *on* $\mathbb{R}^+ \times \mathbb{R}$*. Assume that there exists constants* $p > 0$ *and an integer* $C_1 \geqslant 1$ *such that we have the following control on the space oscillations: for all* $\tau > 0$, $y, y' \in \mathbb{R}$,

$$\left|u(\tau, y + y') - u(\tau, y) - py'\right| \leqslant C_1. \tag{4.24}$$

Then there exists $\lambda \in \mathbb{R}$ *such that for all* $\tau > 0$, $y \in \mathbb{R}$

$$\left|u(\tau, y) - u(0,0) - py - \lambda\tau\right| \leqslant C_2 \quad \text{with } C_2 = 8C_1 + 2M \tag{4.25}$$

where

$$M = \sup\left\{\left|G(\tau, V_{-m}, \ldots, V_m, \pm C_1, p)\right|:\ \tau > 0, V_0 \in \mathbb{R}, V_k = kp \pm C_1 + V_0 \text{ for } k \neq 0\right\}. \tag{4.26}$$

Moreover we have

$$|\lambda| \leqslant M. \tag{4.27}$$

Proof. The proof follows line by line the one given in [15] in a different context. For the reader's convenience, we write all the details below.

In order to control time oscillations, let us introduce the following two continuous functions defined for $T > 0$

$$\lambda_+(T) = \sup_{\tau \geqslant 0} \frac{u(\tau + T, 0) - u(\tau, 0)}{T} \quad \text{and} \quad \lambda_-(T) = \inf_{\tau \geqslant 0} \frac{u(\tau + T, 0) - u(\tau, 0)}{T}$$

which satisfy $-\infty \leqslant \lambda_-(T) \leqslant \lambda_+(T) \leqslant +\infty$.

Step 1. Estimate on the time derivative of the space oscillations.

Let us consider

$$\overline{m}(\tau) = \sup_{y \in \mathbb{R}} \big(u(\tau, y) - py\big) = u\big(\tau, \overline{y}(\tau)\big) - p\overline{y}(\tau) \tag{4.28}$$

if the supremum is reached at some $\overline{y}(\tau) \in \mathbb{R}$. Then we have in the viscosity sense

$$\overline{m}_\tau \leqslant G\big(\tau, \big[u(\tau, \cdot)\big]_m\big(\overline{y}(\tau)\big), C_1, p\big).$$

If the supremum in (4.28) is reached at infinity, we get the same result, up to replace u with $\overline{u}^\infty(\tau, y) = \limsup^*(u(\tau, y + c_n))_{n \geqslant 1}$ for some suitable sequence $(c_n)_{n \geqslant 1}$ going to infinity.

Using moreover that (4.24) implies

$$\left|u\big(\tau, \overline{y}(\tau) + k\big) - u\big(\tau, \overline{y}(\tau)\big) - kp\right| \leqslant C_1,$$

we deduce that

$$\overline{m}_\tau \leqslant G(\tau, V_{-m}, \ldots, V_m, C_1, p) \quad \text{with} \begin{cases} V_0 = u(\tau, \overline{y}(\tau)), \\ V_k = kp + C_1 + V_0 \quad \text{for } k \neq 0. \end{cases}$$

Finally, from the definition (4.26) of M, we get

$$\overline{m}_\tau \leqslant M. \tag{4.29}$$

Similarly we have

$$\underline{m}_\tau \geqslant -M \tag{4.30}$$

with

$$\underline{m}(\tau) = \inf_{y\in\mathbb{R}} \big(u(\tau, y) - py\big).$$

Finally from (4.29), (4.30) and (4.24), we deduce that $\lambda_\pm(T)$ are finite.

Step 2. Estimate on $\lambda_+ - \lambda_-$.

By definition of $\lambda_\pm(T)$, for all $\delta > 0$, there exists $t_\pm \geqslant 0$ such that

$$\left|\lambda^\pm(T) - \frac{u(t_\pm + T, 0) - u(t_\pm, 0)}{T}\right| \leqslant \delta.$$

Let us pick $l \in \mathbb{Z}$ such that

$$0 \leqslant a := t_- + l - t_+ < 1$$

and let us set

$$\tilde{u}(\tau, y) = u(\tau - l, y).$$

Case 1: $T \geqslant 1$. Then we have

$$t_+ \leqslant t_- + l < t_+ + T \leqslant t_- + l + T.$$

Let us define $k \in \mathbb{Z}$ such that $2C_1 < \tilde{u}(t_- + l, 0) + k - u(t_+ + a, 0) \leqslant 3C_1$. Then from (4.24) and the invariance of the equation by addition of integers (see assumption (A3)), we deduce that for all $y \in \mathbb{R}$, we have

$$0 < \tilde{u}(t_- + l, y) + k - u(t_+ + a, y) \leqslant 5C_1. \tag{4.31}$$

Therefore from the comparison principle (Proposition 2.5), we deduce with $T' = T - a$

$$0 \leqslant \tilde{u}(t_- + l + T', y) + k - u(t_+ + a + T', y) \leqslant 5C_1$$

and then from (4.31), we get

$$-5C_1 \leqslant \tilde{u}(t_- + l + T', y) - \tilde{u}(t_- + l, y) - \big(u(t_+ + a + T', y) - u(t_+ + a, y)\big) \leqslant 5C_1. \tag{4.32}$$

Let us consider

$$\overline{\overline{m}}(\tau) = \sup_{y\in\mathbb{R}} \big(\tilde{u}(\tau, y) - py\big)$$

and

$$\underline{\tilde{m}}(\tau) = \inf_{y\in\mathbb{R}}\big(\tilde{u}(\tau, y) - py\big).$$

From (4.29), we deduce that

$$\overline{\tilde{m}}(t_- + l + T) \leqslant \overline{\tilde{m}}(t_- + l + T') + Ma \leqslant C_1 + \underline{\tilde{m}}(t_- + l + T') + Ma$$

which implies that

$$\tilde{u}(t_- + l + T, y) - py \leqslant \tilde{u}(t_- + l + T', y) - py + Ma + C_1,$$

i.e.

$$\tilde{u}(t_- + l + T, y) \leqslant \tilde{u}(t_- + l + T', y) + Ma + C_1.$$

Similarly, using (4.30), we get

$$\tilde{u}(t_- + l + T, y) \geqslant \tilde{u}(t_- + l + T', y) - Ma - C_1$$

and even

$$\big|u(t_+ + a, y) - u(t_+, y)\big| \leqslant Ma + C_1. \tag{4.33}$$

Together with (4.32), we get

$$-7C_1 - 2Ma \leqslant \tilde{u}(t_- + l + T, y) - \tilde{u}(t_- + l, y) - \big(u(t_+ + T, y) - u(t_+, y)\big) \leqslant 7C_1 + 2Ma$$

which implies, for $y = 0$,

$$\big|\lambda_+(T) - \lambda_-(T)\big| \leqslant 2\delta + \frac{7C_1 + 2Ma}{T}.$$

Because $\delta > 0$ is arbitrary small and $a \in [0, 1)$, we deduce that

$$\big|\lambda_+(T) - \lambda_-(T)\big| \leqslant \frac{7C_1 + 2M}{T}. \tag{4.34}$$

Case 2: $T < 1$. Using (4.33) with $a = T$, we deduce that

$$\big|u(t_+ + T, y) - u(t_+, y)\big| \leqslant C_1 + MT.$$

Similarly, we have

$$\big|\tilde{u}(t_- + l + T, y) - \tilde{u}(t_- + l, y)\big| \leqslant C_1 + MT.$$

Therefore

$$\big|\lambda_+(T) - \lambda_-(T)\big| \leqslant 2\delta + \frac{2C_1 + 2MT}{T}.$$

Again, because $\delta > 0$ is arbitrary small, we deduce in particular that (4.34) is still true for $T \in [0, 1)$.

Step 3. $(\lambda_\pm(T))_T$ is a Cauchy sequence.

Let us consider $T_1, T_2 > 0$ such that $T_2/T_1 = P/Q$ with $P, Q \in \mathbb{N} \setminus \{0\}$. Remark that the following inequality holds true

$$\lambda_+(PT_1) = \sup_{\tau \geqslant 0} \sum_{i=1,\dots,P} \frac{u(\tau + iT_1, 0) - u(\tau + (i-1)T_1, 0)}{PT_1} \leqslant \sum_{i=1,\dots,P} \frac{\lambda_+(T_1)}{P} = \lambda_+(T_1).$$

Similarly, we get $\lambda_-(QT_2) \geqslant \lambda_-(T_2)$. Then we have

$$\lambda_+(T_1) \geqslant \lambda_+(PT_1) = \lambda_+(QT_2) \geqslant \lambda_-(QT_2) \geqslant \lambda_-(T_2) \geqslant \lambda_+(T_2) - \frac{7C_1 + 2M}{T_2}.$$

By symmetry, we deduce that

$$\left|\lambda_+(T_2) - \lambda_+(T_1)\right| \leqslant \max\left(\frac{7C_1 + 2M}{T_1}, \frac{7C_1 + 2M}{T_2}\right) \tag{4.35}$$

and similarly

$$\left|\lambda_-(T_2) - \lambda_-(T_1)\right| \leqslant \max\left(\frac{7C_1 + 2M}{T_1}, \frac{7C_1 + 2M}{T_2}\right). \tag{4.36}$$

Since the functions $T \mapsto \lambda_\pm(T)$ are continuous, inequalities (4.35)–(4.36) remain valid in the case $T_2/T_1 \in (0, +\infty)$.

Step 4. Conclusion.

Therefore inequalities (4.35)–(4.36) and (4.34) imply the existence of the following limits

$$\lim_{T \to +\infty} \lambda_+(T) = \lim_{T \to +\infty} \lambda_-(T) = \lambda$$

and we deduce that

$$\left|\lambda_\pm(T) - \lambda\right| \leqslant \frac{7C_1 + 2M}{T}. \tag{4.37}$$

Combining (4.37) with (4.24), we get with $T = \tau$

$$\left|u(\tau, y) - u(0, 0) - py - \lambda\tau\right| \leqslant 8C_1 + 2M.$$

Finally, we deduce easily from (4.29)–(4.30) that $|\lambda| \leqslant M$. This ends the proof of the proposition. □

Proposition 4.2 *(Ergodicity). Assume* (A0′)–(A3′)*, and let u be a solution of* (2.10) *on* $[0, +\infty) \times \mathbb{R}$ *with initial data $u_0(y) = py$ with $p > 0$. Then there exists $\lambda \in \mathbb{R}$ such that*

$$|\lambda| \leqslant M$$

where M is defined in (4.26) *with $C_1 = 1$ and for all $(\tau, y) \in [0, +\infty) \times \mathbb{R}$,*

$$\left|u(\tau, y) - py - \lambda\tau\right| \leqslant C_3 = 2M + 8. \tag{4.38}$$

Moreover we have for all $\tau \geqslant 0$, $y, y' \in \mathbb{R}$,

$$\begin{aligned} u(\tau, y + 1/p) &= u(\tau, y) + 1, \\ u_y(\tau, y) &\geqslant 0, \\ \left|u(\tau, y + y') - u(\tau, y) - py'\right| &\leqslant 1. \end{aligned} \tag{4.39}$$

Proof. We perform the proof in three steps.

Step 1. $u(\tau, y)$ is non-decreasing in y.

First, remark that the equation satisfied by u is invariant by translations in y and for all $b \geqslant 0$, we have

$$u_0(y + b) \geqslant u_0(y).$$

Therefore, from the comparison principle, we get

$$u(\tau, y + b) \geqslant u(\tau, y)$$

which shows that the solution $u(\tau, y)$ is non-decreasing in y.

Step 2. Control of the space-oscillations.

We have

$$u_0(y + 1/p) = u_0(y) + 1.$$

Therefore from the comparison principle and from the integer periodicity (A3′) of G, we get that

$$u(\tau, y + 1/p) = u(\tau, y) + 1.$$

Because $u(\tau, y)$ is non-decreasing in y, we deduce that for all $b \in [0, 1/p]$

$$0 \leqslant u(\tau, b) - u(\tau, 0) \leqslant 1.$$

Let now $y \in \mathbb{R}$, that we write $py = k + a$ with $k \in \mathbb{Z}$ and $a \in [0, 1)$. Then we have

$$u(\tau, y) - u(\tau, 0) = k + u(\tau, a/p) - u(\tau, 0)$$

which implies, for $b \in [0, 1/p)$,

$$u(\tau, y) - u(\tau, 0) - py = -a + u(\tau, b) - u(\tau, 0)$$

and then

$$\left|u(\tau, y) - u(\tau, 0) - py\right| \leqslant 1.$$

Finally, we deduce (4.39) by using the invariance by translations in y of the problem.

Step 3. Control of the time-oscillations.

We can now apply Proposition 4.1 to control the time-oscillations by the space-oscillations. We get the existence of some $\lambda \in \mathbb{R}$ such that

$$\left|u(\tau, y) - u(0, 0) - py - \lambda\tau\right| \leqslant 8 + 2M = C_3.$$

This ends the proof of the proposition. □

4.2. Construction of hull functions for general Hamiltonians

In this subsection, we construct hull functions for the general Hamiltonian G. As we shall see, this is straightforward after we constructed time–space periodic solutions of (4.40) below; see Proposition 4.3 and Corollary 4.4 below. We conclude this subsection by proving that the time slope we constructed in Proposition 4.2 is unique and that the map $p \mapsto \lambda$ is continuous.

Given $p > 0$, we consider the equation in $\mathbb{R} \times \mathbb{R}$

$$u_\tau = G\Big(\tau, \big[u(\tau, \cdot)\big]_m, \inf_{y' \in \mathbb{R}} \big(u(\tau, y') - py'\big) + py - u(\tau, y), u_y\Big). \tag{4.40}$$

Then we have the following result.

Proposition 4.3 *(Existence of time–space periodic solutions of (4.40)). Assume* (A1′)–(A3′) *and consider $p > 0$. Then there exists a function u_∞ solving* (4.40) *on $\mathbb{R} \times \mathbb{R}$ and a real number $\lambda \in \mathbb{R}$ satisfying for all $\tau, y \in \mathbb{R}$,*

$$\begin{aligned} &\left|u_\infty(\tau, y) - py - \lambda\tau\right| \leqslant 2\lceil 2M + 8\rceil, \\ &|\lambda| \leqslant M \end{aligned} \tag{4.41}$$

with M defined by (4.26) *with $C_1 = 1$. Moreover u_∞ satisfies*

$$\begin{cases} u_\infty(\tau, y + 1/p) = u_\infty(\tau, y) + 1, \\ u_\infty(\tau + 1, y) = u_\infty(\tau, y + \lambda/p), \\ (u_\infty)_y(\tau, y) \geqslant 0, \\ \left|u_\infty(\tau, y + y') - u_\infty(\tau, y) - py'\right| \leqslant 1. \end{cases} \tag{4.42}$$

Eventually, when G is independent on τ, we can choose u_∞ independent on τ.

By considering $h(\tau, z) = u_\infty(\tau, (z - \lambda\tau)/p)$, we immediately get the following corollary.

Corollary 4.4 *(Existence of hull functions). Assume* (A1′)–(A3′). *There exists a hull function h for* (2.10) *satisfying*

$$\left|h(\tau, z) - z\right| \leqslant 2\lceil 2M + 8\rceil = 2\lceil C_3\rceil$$

where M is given by (4.26) *with* $C_1 = 1$.

Remark 4.5. The definition of hull function for (2.10) is very similar to Definition 1.4. The only difference is the equation satisfied by h which is replaced here by

$$h_\tau + \lambda h_z = G\Big(\tau, h(\tau, z - mp), \ldots, h(\tau, z + mp), \inf_{z'}\big(h(\tau, z') - z'\big) + z - h(\tau, z), ph_z\Big).$$

Proof of Proposition 4.3. The proof is performed in three steps. In the first one, we construct sub- and supersolutions of (4.40) in $\mathbb{R} \times \mathbb{R}$ with good translation invariance properties (see the first two lines of (4.42)). We next apply Perron's method in order to get a (discontinuous) solution satisfying the same properties. Finally, in Step 3, we prove that if G does not depend on τ, then we can construct such a solution such that it does not depend on τ either.

Step 1. Global sub- and supersolution.

By Proposition 4.2, we know that the solution u of (2.10) with initial data $u_0(y) = py$ satisfies on $[0, +\infty) \times \mathbb{R}$

$$\begin{cases} u_y \geqslant 0, \\ \left|u(\tau, y) - py - \lambda\tau\right| \leqslant 2M + 8 = C_3, \\ \left|u(\tau, y + y') - u(\tau, y) - py'\right| \leqslant 1. \end{cases} \tag{4.43}$$

We first construct a subsolution and a supersolution of (4.40) for $\tau \in \mathbb{R}$ (and not only $\tau \geqslant 0$) that also satisfy the first two lines of (4.42), i.e. satisfy for all $k, l \in \mathbb{Z}$,

$$U(\tau + k, y) = U\left(\tau, y + \lambda\frac{k}{p}\right) \quad \text{and} \quad U\left(\tau, y + \frac{l}{p}\right) = U(\tau, y) + l. \tag{4.44}$$

To do so, we consider the sequence, for $n \in \mathbb{N}$,

$$u_n(\tau, y) = u(\tau + n, y) - \lfloor \lambda n \rfloor$$

and consider

$$\overline{u} = \limsup_{n\to+\infty}{}^* u_n,$$
$$\underline{u} = \liminf_{n\to+\infty}{}_* u_n.$$

Now a way to construct semi-solutions satisfying (4.44) is to consider

$$\overline{u}_\infty(\tau, y) = \sup_{k,l\in\mathbb{Z}} \big(\overline{u}(\tau + k, y - k\lambda/p + l/p) - l\big), \tag{4.45}$$

$$\underline{u}_\infty(\tau, y) = \inf_{k,l\in\mathbb{Z}} \big(\underline{u}(\tau + k, y - k\lambda/p + l/p) - l\big). \tag{4.46}$$

Notice that $\overline{u}_\infty$ and $\underline{u}_\infty$ satisfy moreover (4.43) on $\mathbb{R} \times \mathbb{R}$. Therefore we have in particular

$$\overline{u}_\infty \leqslant \underline{u}_\infty + 2\lceil C_3 \rceil.$$

Step 2. Existence by Perron's method.

Applying Perron's method we see that the lowest supersolution u_∞ above $\overline{u}_\infty$ is a solution of (4.43) on $\mathbb{R} \times \mathbb{R}$ and satisfies

$$\overline{u}_\infty \leqslant u_\infty \leqslant \underline{u}_\infty + 2\lceil C_3 \rceil.$$

We next prove that u_∞ satisfies (4.42).

Moreover let us consider

$$\tilde{u}_\infty(\tau, y) = \inf_{k,l\in\mathbb{Z}} \big(u_\infty(\tau + k, y - k\lambda/p + l/p) - l\big). \tag{4.47}$$

By construction $\tilde{u}_\infty$ is a supersolution and is again above the subsolution $\overline{u}_\infty$. Therefore from the definition of u_∞, we deduce that

$$\tilde{u}_\infty = u_\infty$$

which implies that u_∞ satisfies (4.44), i.e. the first two equalities of (4.42).

Similarly, we can consider

$$\hat{u}_\infty(\tau, y) = \inf_{b\in[0,+\infty)} u_\infty(\tau, y + b)$$

which is again a supersolution above the subsolution $\overline{u}_\infty$. Therefore

$$\hat{u}_\infty = u_\infty$$

which implies that u_∞ is non-decreasing, i.e. the third line of (4.42) is satisfied.

Finally, the function $u_\infty - \lceil C_3 \rceil$ still satisfies (4.42) but also (4.41).

Step 3. Further properties when G is independent on τ.

When G does not depend on τ, we can apply Steps 1 and 2 with $k \in \mathbb{Z}$ in (4.45), (4.46) and (4.47) replaced with $k \in \mathbb{R}$. This implies that the hull function h does not depend on τ. This ends the proof of the proposition. □

Proposition 4.6 *(Definition and continuity of the effective Hamiltonian). Given $p > 0$, and under the assumptions* (A1′)–(A3′)*,*

– *there exists a unique* λ *such that there exists a function* $u_\infty \in L^\infty_{\text{loc}}(\mathbb{R}\times\mathbb{R})$ *solution of* (4.40) *on* $\mathbb{R}\times\mathbb{R}$ *and satisfying*

$$\left|h(0,z)-z\right| \leqslant 1; \tag{4.48}$$

– *if* λ *is seen as a function* $\overline{G}$ *of* p ($\lambda=\overline{G}(p)$)*, then this function* $\overline{G}:(0,+\infty)\to\mathbb{R}$ *is continuous.*

Proof.

Step 1. Uniqueness of λ.

Given some $p\in(0,+\infty)$, assume that there exist $\lambda_1,\lambda_2\in\mathbb{R}$ with their corresponding hull functions h_1,h_2. Then define for $i=1,2$

$$u_i(\tau,y)=h_i(\tau,\lambda_i\tau+py)$$

which are both solutions of Eq. (2.10) on $[0,+\infty)\times\mathbb{R}$. Using the fact that $h_i(\tau,z+1)=h_i(\tau,z)+1$ and the monotonicity of the hull functions in the variable z, we see that for each h_i (up to a substraction of an integer and a translation of h_i in the variable z) we can assume that (4.48) holds true. Then we have

$$u_1(0,y)\leqslant u_2(0,y)+2$$

which implies (from the comparison principle) for all $(\tau,y)\times[0,+\infty)\times\mathbb{R}$

$$u_1(\tau,y)\leqslant u_2(\tau,y)+2.$$

Using the fact that $h_i(\tau+1,z)=h_i(\tau,z)$, we deduce that for $\tau=k\in\mathbb{N}$ and $y=0$ we have

$$h_1(0,\lambda_1 k)\leqslant h_2(0,\lambda_2 k)+2$$

which implies by (4.48)

$$\lambda_1 k\leqslant \lambda_2 k+4.$$

Because this is true for any $k\in\mathbb{N}$, we deduce that

$$\lambda_1\leqslant\lambda_2.$$

The reverse inequality is obtained exchanging h_1 and h_2. We finally deduce that $\lambda_1=\lambda_2$, which proves the uniqueness of the real λ, that we call $\overline{G}(p)$.

Step 2. Continuity of the map $p\mapsto\overline{G}(p)$.

Let us consider a sequence $(p_n)_n$ such that $p_n \to p > 0$. Let $\lambda_n = \overline{G}(p_n)$ and h_n be the corresponding hull functions. From Corollary 4.4, we can choose these hull functions such that

$$\left|h_n(\tau, z) - z\right| \leqslant 2\lceil 2M(p_n) + 8\rceil$$

and we have

$$|\lambda_n| \leqslant M(p_n)$$

where we recall that $M(p)$ is defined in (4.26). We deduce in particular that there exists a constant $C_4 > 0$ such that

$$\left|h_n(\tau, z) - z\right| \leqslant C_4 \quad \text{and} \quad |\lambda_n| \leqslant C_4.$$

Let us consider a limit λ_∞ of $(\lambda_n)_n$, and let us define

$$\overline{h} = \limsup_{n\to+\infty}{}^* h_n.$$

This function $\overline{h}$ is such that

$$\overline{u}(\tau, y) = \overline{h}(\tau, \lambda_\infty \tau + py)$$

is a subsolution of (4.40) on $\mathbb{R} \times \mathbb{R}$. On the other hand, if h denotes the hull function associated with p and $\lambda = \overline{G}(p)$, then

$$u(\tau, y) = h(\tau, \lambda\tau + py)$$

is a solution of (4.40) on $\mathbb{R} \times \mathbb{R}$. Finally, as in Step 1, we conclude that

$$\lambda_\infty \leqslant \lambda.$$

Similarly, considering

$$\underline{h} = \liminf_{n\to+\infty}{}_* h_n$$

we can show that

$$\lambda_\infty \geqslant \lambda.$$

Therefore $\lambda_\infty = \lambda$ and this proves that $\overline{G}(p_n) \to \overline{G}(p)$; the continuity of the map $p \mapsto \overline{G}(p)$ follows and this ends the proof of the proposition. □

Proof of Theorem 1.5. Just apply Proposition 4.6 with $G = F$. □

5. Construction of Lipschitz approximate hull functions

When proving the convergence Theorem 1.3, we explained that, on one hand, it is necessary in order to apply Evans' perturbed test function method, to deal with hull functions $h(\tau, z)$ that are Lipschitz continuous in z (uniformly in τ); on the other hand, given some $p > 0$, we also know some Hamiltonian F, with effective Hamiltonian $\overline{F}(p)$, such that every corresponding hull function h is necessarily discontinuous in z (see the end of the introduction). Recall that a hull function h solves, with $\lambda = \overline{F}(p)$,

$$h_\tau + \lambda h_z = F\big(\tau, \big[h(\tau, \cdot)\big]_m^p(z)\big).$$

We overcome this difficulty as in [15]. As a matter of fact, the argument is simplified here: approximate Hamiltonians are defined in a simpler way.

Let us be more specific now. We show in this section that we can build approximate Hamiltonian G_δ with corresponding effective Hamiltonian $\lambda_\delta = \overline{G}_\delta(p)$, and corresponding hull functions h_δ, such that

$$\begin{cases} h_\delta \text{ is Lipschitz continuous wrt } z \text{ uniformly in } \tau, \\ \overline{G}_\delta(p) \to \overline{F}(p) \quad \text{as } \delta \to 0, \\ h_\delta \text{ is a subsolution (resp. a supersolution) of } (h_\delta)_\tau + \lambda_\delta (h_\delta)_z = F\big(\tau, \big[h_\delta(\tau, \cdot)\big]_m^p(z)\big). \end{cases}$$

We will show that it is enough to choose

$$G_\delta(\tau, V, a, q) = F(\tau, V) + \delta(a_0 + a)q \tag{5.49}$$

with $a_0 \in \mathbb{R}$ (in fact, we will consider $a_0 = \pm 1$).

Using (A1), we know that there exists a constant $K_1 > 0$ such that for all $V, W \in \mathbb{R}^{2m+1}$, $\tau \in \mathbb{R}$,

$$\big|F(\tau, V + W) - F(\tau, V)\big| \leqslant K_1 |W|_\infty \tag{5.50}$$

with $|W|_\infty = \max_{k=-m,\dots,m} |W_k|$.

We have the following regularity result.

Proposition 5.1 *(Bound on the gradient). Assume* (A1)–(A3) *and* $p > 0$. *Then the solution* u *of* (2.10) *with* $G = G_\delta$ *defined by* (5.49) *and* $u_0(y) = py$ *satisfies*

$$0 \leqslant u_y \leqslant p + K_1/\delta \quad \text{on } [0, +\infty) \times \mathbb{R}. \tag{5.51}$$

Proof. For all $\eta \geqslant 0$, we consider the more general equation

$$\begin{cases} u_\tau = G_\delta\Big(\tau, \big[u(\tau, \cdot)\big]_m, \inf_{y' \in \mathbb{R}} \big(u(\tau, y') - py'\big) + py - u(\tau, y), u_y\Big) + \eta u_{yy} \\ \quad \text{on } (0, +\infty) \times \mathbb{R}, \\ u(0, y) = py \quad \text{for } y \in \mathbb{R}. \end{cases} \tag{5.52}$$

Case A. $\eta > 0$ and $F \in C^1$.

For $\eta > 0$, it is possible to show by the classical fixed point method that there exists a unique solution u of (5.52) in $C^{2+\alpha,1+\alpha/2}$ for any $\alpha \in (0,1)$. Moreover u satisfies

$$u(\tau, y + 1/p) = u(\tau, y) + 1.$$

Then, if we define $v = u_y$ we see by derivation with respect to y, that v solves

$$\begin{aligned} v_\tau - \eta v_{yy} = F'_V\big(\tau, \big[u(\tau,\cdot)\big]_m(y)\big) \cdot \big[v(\tau,\cdot)\big]_m(y) - \delta(v-p)v \\ + \delta\Big(a_0 + \inf_{y'\in\mathbb{R}}\big(u(\tau,y') - py'\big) + py - u(\tau,y)\Big)v_y \quad \text{on } (0,+\infty)\times\mathbb{R}, \\ v(0,y) = p \quad \text{for } y \in \mathbb{R}. \end{aligned} \tag{5.53}$$

Again we see that v is in $C^{2+\alpha,1+\alpha/2}$. In particular v is a viscosity solution of (5.53).

Step 1: Bound from below on the gradient. Let us now define

$$\underline{m}(\tau) = \inf_{y\in\mathbb{R}} v(\tau, y).$$

Then we have in the viscosity sense:

$$\begin{cases} \underline{m}_\tau \geqslant K_1 \min(0,\underline{m}) - K_1\underline{m} - \delta(\underline{m} - p)\underline{m}, \\ \underline{m}(0) = p > 0 \end{cases}$$

where we have used the monotonicity assumption (A2) to get the term $K_1 \min(0,\underline{m})$. The fact that 0 is subsolution of the previous equation implies that

$$v \geqslant \underline{m} \geqslant 0.$$

Step 2: Bound from above on the gradient. Similarly we define

$$\overline{m}(\tau) = \sup_{y\in\mathbb{R}} v(\tau, y).$$

Then we have in the viscosity sense

$$\begin{cases} \overline{m}_\tau \leqslant K_1\overline{m} - \delta(\overline{m} - p)\overline{m}, \\ \overline{m}(0) = p > 0 \end{cases}$$

where we have used Step 1 to ensure that $|v| \leqslant \overline{m}$. The fact that $p + K_1/\delta$ is a supersolution of the previous equation implies that

$$v \leqslant \overline{m} \leqslant p + K_1/\delta.$$

Case B. $\eta = 0$ and F general.

We simply consider a C^1 approximation F^η of F and call u^η the solution of (5.52) with F replaced with F^η, for $\eta > 0$. From Case A, we have

$$0 \leqslant \left(u^\eta\right)_y \leqslant p + K_1/\delta + o_\eta(1). \tag{5.54}$$

Then we call

$$\bar{u} = \limsup_{\eta \to 0}{}^* u^\eta,$$
$$\underline{u} = \liminf_{\eta \to 0}{}_* u^\eta.$$

Then $\bar{u}$ and $\underline{u}$ are respectively sub and supersolutions of (5.52) with $\eta = 0$. Therefore

$$\bar{u} \leqslant u \leqslant \underline{u}.$$

But by construction we have $\underline{u} \leqslant \bar{u}$. Therefore

$$u = \bar{u} = \underline{u}$$

and passing to the limit in (5.54), we see that u satisfies (5.51). This ends the proof of the proposition. □

Then we have

Proposition 5.2 *(Existence of Lipschitz approximate hull functions). Assume* (A1)–(A3). *Given $p > 0$, $\delta > 0$ and $a_0 \in \mathbb{R}$, then there exists a Lipschitz hull function $h(\tau, z)$ satisfying*

$$\begin{cases} 0 \leqslant h_z \leqslant 1 + K_1/(p\delta), \\ h(\tau, z+1) = h(\tau, z) + 1, \\ h(\tau+1, z) = h(\tau, z) \end{cases} \tag{5.55}$$

and there exists $\lambda \in \mathbb{R}$ such that

$$h_\tau + \lambda h_z = F\left(\tau, \left[h(\tau, \cdot)\right]_m^p\right) + \delta p \left\{a_0 + \inf_{z' \in \mathbb{R}} \left(h(\tau, z') - z'\right) + z - h(\tau, z)\right\} h_z \tag{5.56}$$

and

$$\left|h(\tau, z') - z' + z - h(\tau, z)\right| \leqslant 1. \tag{5.57}$$

Moreover there exists a constant $M_0 > 0$, only depending on F and $p > 0$, such that

$$|\lambda| \leqslant M_0 + \delta\left(|a_0| + 1\right) p \tag{5.58}$$

and for all $(\tau, z) \in \mathbb{R} \times \mathbb{R}$,

$$\left|h(\tau, z) - z\right| \leqslant M_0 + 4\delta\left(|a_0| + 1\right) p. \tag{5.59}$$

Moreover, when F does not depend on τ, we can choose the hull function h such that it does not depend on τ either.

Proof. This is a simple corollary of Propositions 4.2, 4.3 and 5.1; this leads to an improvement of the statement of Corollary 4.4. This proves in particular the bound on h_z. Lipschitz continuity in time of h follows from the PDE satisfied by h. Indeed, it permits to get a uniform bound on h_τ. This ends the proof of the proposition. □

We finally have

Proposition 5.3 *(Sub- and super-Lipschitz hull functions). For any $\delta > 0$, let $h_\delta^\pm$ be the Lipschitz continuous hull function obtained in Proposition 5.2 for $a_0 = \pm 1$, and $\lambda_\delta^\pm$ the corresponding value of the effective Hamiltonian. Then we have*

$$\big(h_\delta^+\big)_\tau + \lambda_\delta^+\big(h_\delta^+\big)_z \geqslant F\big(\tau, \big[h_\delta^+(\tau,\cdot)\big]_m^p\big) \quad \textit{and} \quad \lambda \leqslant \lambda_\delta^+ \to \lambda \quad \textit{as } \delta \to 0,$$
$$\big(h_\delta^-\big)_\tau + \lambda_\delta^-\big(h_\delta^-\big)_z \leqslant F\big(\tau, \big[h_\delta^-(\tau,\cdot)\big]_m^p\big) \quad \textit{and} \quad \lambda \geqslant \lambda_\delta^- \to \lambda \quad \textit{as } \delta \to 0$$

where $\lambda = \overline{F}(p)$.

Proof. Inequalities $\pm\lambda_\delta^\pm \geqslant \pm\lambda$ follow from the comparison principle. In view of the bounds (5.58) and (5.59) on $\lambda_\delta^\pm$ and $h_\delta^\pm$ we have (in particular they are uniform as δ goes to zero), it is clear that the convergence $\lambda_\delta^\pm \to \lambda$ holds true as $\delta \to 0$. It suffices to adapt Step 2 of the proof of Proposition 4.6. □

6. Qualitative properties of the effective Hamiltonian

Proof of Theorem 1.6: (a1), (a2), (a3), (a4). We recall that we have hull functions h which are solutions of

$$h_\tau + \lambda h_z = L + F\big(\tau, \big[h(\tau,\cdot)\big]_m^p\big)$$

with $\lambda = \overline{F}(L, p)$.

The continuity of the map $(L, p) \mapsto \overline{F}(L, p)$ is easily proved as in Step 2 of the proof of Proposition 4.6.

(a1) *Bound*. This is a straightforward adaptation of Step 1 of the proof of Proposition 4.1.

(a2) *Monotonicity in L*. The monotonicity of the map $L \mapsto \overline{F}(L, p)$ follows from the comparison principle on $u(\tau, y) = h(\tau, \lambda\tau + py)$ where h is the hull function and $\lambda = \overline{F}(L, p)$.

(a3) *Antisymmetry in V*. We just remark that if a hull function h solves

$$h_\tau + \lambda h_z = F\big(\tau, \big[h(\tau,\cdot)\big]_m^p\big)$$

then $\tilde{h}(\tau, z) = -h(\tau, -z)$ satisfies

$$\tilde{h}_\tau - \lambda\tilde{h}_z = -F\big(\tau, -\tilde{h}(\tau, z + mp), \ldots, -\tilde{h}(\tau, z - mp)\big) = F\big(\tau, \big[\tilde{h}(\tau,\cdot)\big]_m^p\big).$$

By the uniqueness of λ, we deduce that $\lambda = -\lambda$ and then $\overline{F}(0, p) = \lambda = 0$.

(a4) *Periodicity in p*. It is sufficient to remark that, given $p > 0$, if h is a hull function for $\lambda = \overline{F}(L, p)$, then h is also a hull function for $p + 1$ with the same λ. □

Remark 6.1 *(Potential with zero mean value).* In Theorem 1.6 point (a3), if we do not assume the antisymmetry in V but that F has the following form:

$$F(V) = -g_0'(V_0) + \sum_{i=1,\dots,m} -g_i'(V_0 - V_{-i}) + g_i'(V_0 - V_i)$$

where $g_0 \in W^{2,\infty}(\mathbb{R};\mathbb{R})$ is a 1-periodic function and $g_i \in W^{2,\infty}(\mathbb{R};\mathbb{R})$ for $i = 1,\dots,m$ are convex functions, then it is expected as in [12, Theorem 2.6] that $\overline{F}(0, p) = 0$, but it is not proved here.

Before to prove the point (a5) of Theorem 1.6, let us prove the following easier result, which also shows that the Lipschitz constant of the hull function is related to the inverse of the bound from below of the gradient in L of the effective Hamiltonian.

Proposition 6.2 *(Lipschitz continuous hull function/bound from below on $\frac{\partial \overline{F}}{\partial L}$). Given $(L_0, p) \in \mathbb{R} \times (0, +\infty)$, assume that there exists a corresponding hull function h which satisfies for some $K_3 \geqslant 1$*

$$0 \leqslant h(\tau, z + a) - h(\tau, z) \leqslant K_3 a \quad \textit{for any } (a, z) \in [0, +\infty) \times \mathbb{R}.$$

Then we have for all $L \in \mathbb{R}$

$$\left|\overline{F}(L + L_0, p) - \overline{F}(L_0, p)\right| \geqslant \frac{|L|}{K_3}.$$

Proof. Up to redefine F, we can assume that $L_0 = 0$. Then we have with $\lambda = \overline{F}(0, p)$:

$$h_\tau + \lambda h_z = F\left(\tau, \left[h(\tau,\cdot)\right]_m^p(z)\right).$$

This implies

$$h_\tau + \tilde{\lambda} h_z \leqslant L + F\left(\tau, \left[h(\tau,\cdot)\right]_m^p(z)\right)$$

with $L = (\tilde{\lambda} - \lambda)K_3$. From the comparison principle, we deduce that $\tilde{\lambda} \leqslant \overline{F}(L, p)$, i.e.

$$L/K_3 \leqslant \overline{F}(L, p) - \overline{F}(0, p)$$

which gives the result for positive L. We get similarly the corresponding inequality for negative L. This ends the proof of the proposition. □

Proof of Theorem 1.6: (a5). *Continuous hull function/no plateau of $L \mapsto \overline{F}(L, p)$.* Up to redefine F, we can assume that $L_0 = 0$. We assume that h is continuous with the following space-modulus of continuity ω: for all $\tau \geqslant 0$, $z' \geqslant 0$, $z \in \mathbb{R}$,

$$0 \leqslant h(\tau, z + z') - h(\tau, z) \leqslant \omega(z') \tag{6.60}$$

and solves, for $\lambda = \overline{F}(0, p)$,

$$h_\tau + \lambda h_z = F\big(\tau, \big[h(\tau, \cdot)\big]^p_m(z)\big). \tag{6.61}$$

Then we define for $\alpha > 0$ the sup-convolution (in space only)

$$h^\alpha(\tau, z) = \sup_{y \in \mathbb{R}} \left(h(\tau, y) - \frac{|z - y|^2}{2\alpha} \right).$$

We (classically) show that h^α is a Lipschitz continuous subsolution of Eq. (6.61) perturbed by some error term.

Step 1. The basic viscosity inequality satisfied by h^α.

More precisely, let $\varphi \in C^1(\mathbb{R}^2)$ such that

$$h^\alpha \leqslant \varphi \quad \text{with equality at } (\tau_0, z_0)$$

and let $y_0 \in \mathbb{R}$ be such that

$$h(\tau_0, z_0) \leqslant h^\alpha(\tau_0, z_0) = h(\tau_0, y_0) - \frac{|z_0 - y_0|^2}{2\alpha}. \tag{6.62}$$

Then we have

$$h(\tau, y) \leqslant \varphi(\tau, z_0) + \frac{|z_0 - y|^2}{2\alpha} =: \tilde{\varphi}(\tau, y) \quad \text{with equality at } (\tau_0, y_0).$$

This implies

$$\tilde{\varphi}_\tau + \lambda \tilde{\varphi}_y \leqslant F\big(\tau_0, \big[h(\tau_0, \cdot)\big]^p_m(y_0)\big) \quad \text{at } (\tau_0, y_0).$$

By definition of h^α, we have

$$h^\alpha(\tau_0, z_0 + kp) \geqslant h(\tau_0, y_0 + kp) - \frac{|z_0 - y_0|^2}{2\alpha}.$$

We deduce that

$$\varphi_\tau(\tau_0, z_0) + \lambda \left(\frac{y_0 - z_0}{\alpha} \right) \leqslant F\left(\tau_0, \left[\frac{|z_0 - y_0|^2}{2\alpha} + h^\alpha(\tau_0, \cdot) \right]^p_m (z_0) \right)$$

where we have used (6.62) and the monotonicity assumption (A2) on F. We classically have $\varphi_z(\tau_0, z_0) = (y_0 - z_0)/\alpha \geqslant 0$ (recall that h is non-decreasing). This gives the basic viscosity inequality satisfied by h^α

$$\varphi_\tau(\tau_0, z_0) + \lambda \varphi_z(\tau_0, z_0) \leqslant F\left(\tau_0, \left[\frac{|z_0 - y_0|^2}{2\alpha} + h^\alpha(\tau_0, \cdot) \right]^p_m (z_0) \right). \tag{6.63}$$

Step 2. Getting a bound from below on the effective Hamiltonian.

Using the Lipschitz constant $K_1 > 0$ defined in (5.50), we get, from (6.63),

$$\varphi_\tau(\tau_0, z_0) + \lambda\varphi_z(\tau_0, z_0) \leqslant K_1 \frac{|z_0 - y_0|^2}{2\alpha} + F\left(\tau_0, \left[h^\alpha(\tau_0, \cdot)\right]_m^p(z_0)\right).$$

This implies

$$\varphi_\tau(\tau_0, z_0) + \tilde{\lambda}\varphi_z(\tau_0, z_0) \leqslant L + F\left(\tau_0, \left[h^\alpha(\tau_0, \cdot)\right]_m^p(z_0)\right)$$

for any $(\tilde{\lambda}, L)$ such that

$$L \geqslant K_1 \frac{|z_0 - y_0|^2}{2\alpha} + (\tilde{\lambda} - \lambda)\left(\frac{y_0 - z_0}{\alpha}\right). \tag{6.64}$$

Now using (5.57) and (6.62), we get

$$\frac{|z_0 - y_0|^2}{2\alpha} \leqslant h(\tau_0, y_0) - h(\tau_0, z_0) \leqslant \omega(y_0 - z_0) \leqslant 1 + |y_0 - z_0|$$

which implies $|y_0 - z_0| \leqslant 4\sqrt{\alpha}$ for $\alpha \leqslant 2$. Consider now $L > 0$ and $\tilde{\lambda}$ such that (6.64) holds true and

$$\tilde{\lambda} \geqslant \lambda^\alpha := \lambda + \frac{\sqrt{\alpha}}{4}\left(L - K_1\omega(4\sqrt{\alpha})\right).$$

We then have, in the viscosity sense, for all $(\tau, z) \in \mathbb{R}^2$,

$$h_\tau^\alpha + \lambda^\alpha h_z^\alpha \leqslant L + F\left(\tau, \left[h^\alpha(\tau, \cdot)\right]_m^p(z)\right).$$

Therefore, for any $L > 0$, we have

$$\overline{F}(L, p) \geqslant \lambda^\alpha > \lambda = \overline{F}(0, p)$$

for α small enough.

Step 3. The bound from above on the effective Hamiltonian.

Proceeding by inf-convolution, we get similarly the expected result for negative L. This ends the proof of the theorem. □

Remark 6.3. We can also get explicit estimates to bound $|\overline{F}(L, p) - \overline{F}(0, p)|$ from below, using the modulus of continuity $\omega(\cdot)$.

Proof of Theorem 1.7. (b1) *No plateau in L if $\overline{F} \neq 0$.* Consider $L_2 > L_1$ and the corresponding hull functions $h_i(z)$ independent on time and satisfying

$$\lambda_i (h_i)_z = L_i + F\big(\big[h_i(\cdot)\big]_m^p(z)\big), \quad i = 1, 2,$$

for the corresponding $\lambda_i = \overline{F}(L_i, p)$. We assume that $\lambda_1 > 0$ and we already know that $\lambda_2 \geqslant \lambda_1 > 0$. Let us define

$$F_0 = \sup_{V_0 \in \mathbb{R}} \big|F(V_0, \ldots, V_0)\big|.$$

Remark now that (5.57) implies

$$\big|h_i(z + kp) - h_i(z) - kp\big| \leqslant 1$$

and then

$$\big|F\big(\big[h_i(\cdot)\big]_m^p(z)\big)\big| \leqslant F_0 + K_1(mp + 1).$$

Therefore

$$0 \leqslant (h_1)_z \leqslant \lambda_1^{-1}\big(|L_1| + F_0 + K_1(mp + 1)\big).$$

Hence

$$\big(\lambda_1 + \delta(L_2 - L_1)\big)(h_1)_z \leqslant L_2 + F\big(\big[h_1(\cdot)\big]_m^p(z)\big)$$

for $\delta \leqslant \lambda_1(|L_1| + F_0 + K_1(mp + 1))^{-1}$. This implies that $\lambda_2 \geqslant \lambda_1 + \delta(L_2 - L_1)$, i.e.

$$\frac{\lambda_2 - \lambda_1}{L_2 - L_1} \geqslant \lambda_1\big(|L_1| + F_0 + K_1(mp + 1)\big)^{-1}.$$

This implies the result for $\overline{F} > 0$. We get a similar result for $\overline{F} < 0$.

(b2) *0-plateau property*. Because $V_0 \mapsto F(V_0, \ldots, V_0)$ is assumed not constant, we see that there exists $L_0 \in \mathbb{R}$ such that

$$\inf_{V_0 \in \mathbb{R}} F(V_0, \ldots, V_0) < -L_0 < \sup_{V_0 \in \mathbb{R}} F(V_0, \ldots, V_0).$$

Up to redefine F, we can assume that $L_0 = 0$ to simplify. Recall also that for (L, p), the (possibly discontinuous) hull function h satisfies

$$\lambda h_z = L + F\big(\big[h(\cdot)\big]_m^p(z)\big).$$

Now for $p \in \mathbb{N} \setminus \{0\}$, and using property (1.8), we deduce that

$$\lambda h_z = L + F\big(h(z), \ldots, h(z)\big).$$

Consider $\lambda \neq 0$. Assume for instance that $\lambda > 0$. Then h is Lipschitz continuous. Moreover, h is non-decreasing. Then using a test function ϕ which touches h at z in a region where $F(h(z), \dots, h(z)) < 0$, we get a contradiction for $|L|$ small enough. This shows that $\lambda \leqslant 0$. Similarly, we show that $\lambda \geqslant 0$. Therefore $\overline{F}(L, p) = \lambda = 0$ for L small enough. This ends the proof of the theorem. □

We have moreover the following result.

Proposition 6.4 *(Uniqueness of the continuous hull functions). Assume* (A1)–(A3). *Assume also that there exist* $\delta_0 > 0$ *and* $k_0 \in \{-m, \dots, m\} \setminus \{0\}$ *such that*

$$\frac{\partial F}{\partial V_{k_0}}(\tau, V) \geqslant \delta_0 > 0 \quad \textit{for any } V = (V_{-m}, \dots, V_m) \in \mathbb{R}^{2m+1} \tag{6.65}$$

and we consider hull functions for some fixed irrational $p > 0$.

If there exists a continuous hull function $h(\tau, z)$*, then every hull function is continuous and is equal to h, up to a fixed translation in z. In that case, the hull function is moreover strictly monotone in z, i.e. satisfies*

$$h(\tau, z') > h(\tau, z) \quad \textit{if } z' > z.$$

Remark 6.5. We do not know if Proposition 6.4 is still true without assuming the continuity of the hull function, but only assuming that p is irrational.

Remark 6.6. The classical FK model (1.4) gives an example of non-uniqueness of hull functions which can be discontinuous for $\overline{F}(p) = 0$. Indeed for $f = 0$ and $p = 1$, the following functions (for any $a \in (0, 1)$)

$$h_1(z) = \lfloor z \rfloor \quad \text{and} \quad h_2(z) = \begin{cases} \lfloor z \rfloor & \text{if } 0 \leqslant z - \lfloor z \rfloor < a, \\ \frac{1}{2} + \lfloor z \rfloor & \text{if } a \leqslant z - \lfloor z \rfloor < 1 \end{cases}$$

are two admissible discontinuous hull functions.

Proof of Proposition 6.4. (i) *Uniqueness of the hull function.* Assume that h_1 and h_2 are two hull functions, with h_2 continuous. We can slide $h_2(\tau, z + a)$ above $h_1^*(\tau, z)$ for a large enough. Then we decrease a until some a^* to get a contact between $h_2(\tau, z + a^*)$ and $h_1^*(\tau, z)$ at some point (τ_0, z_0). Up to redefine h_2, we can assume that $a^* = 0$.

Step 1: Strong maximum principle at the contact point. Let us consider

$$b(\tau) = \inf_{z \in \mathbb{R}} \big(h_2(\tau, z) - h_1^*(\tau, z)\big) = h_2\big(\tau, \bar{z}(\tau)\big) - h_1^*\big(\tau, \bar{z}(\tau)\big)$$

for some $\bar{z}(\tau) \in \mathbb{R}$. Recall that we have in the viscosity sense

$$(h_i)_\tau + \lambda (h_i)_z = F\big(\tau, \big[h_i(\tau, \cdot)\big]_m^p(z)\big).$$

Then up to a dedoubling of variable in time and in space, we can identify the space derivatives at $(\tau, \bar{z}(\tau))$ of h_2 and h_1 which implies (this is a routine exercise to justify this in the viscosity framework):

$$
\begin{aligned}
\frac{d}{d\tau}b(\tau) &\geqslant F\big(\tau, \big[h_2(\tau,\cdot)\big]_m^p\big(\bar{z}(\tau)\big)\big) - F\big(\tau, \big[h_1^*(\tau,\cdot)\big]_m^p\big(\bar{z}(\tau)\big)\big)\\
&\geqslant \delta_0\big(h_2\big(\tau, \bar{z}(\tau)+k_0p\big) - h_1^*\big(\tau, \bar{z}(\tau)+k_0p\big)\big)\\
&\geqslant \delta_0 b(\tau).
\end{aligned}
$$

In particular, from the fact that $b(\tau_0)=0$, we deduce that

$$b(\tau)=0 \quad \text{for } \tau \leqslant \tau_0.$$

Moreover, we deduce that the function $g(\tau,z)=h_2(\tau,z)-h_1^*(\tau,z)$ satisfies

$$g\big(\tau, \bar{z}(\tau)+k_0p\big)=0 \quad \text{for } \tau \leqslant \tau_0.$$

Step 2: Conclusion. We can now reapply Step 1 iteratively to $\bar{z}(\tau)+k_0pl$ for $l=1,2,\ldots.$ We deduce that for all $l\in\mathbb{N}$ and for $\tau\leqslant\tau_0$,

$$g\big(\tau, \bar{z}(\tau)+k_0pl\big)=0.$$

Because p is irrational, we deduce that h_1^* is equal to the continuous function h_2 on a set which is dense in $(-\infty,\tau_0]\times\mathbb{R}$. Therefore h_1^* is continuous on $(-\infty,\tau_0]\times\mathbb{R}$. But recall that $u_1(\tau,y)=h_1(\tau,\lambda\tau+py)$ solves

$$(u_1)_\tau = F\big(\tau, \big[u_1(\tau,\cdot)_m(z)\big]\big).$$

Because the right-hand side is bounded, this implies that u_1 is Lipschitz in time. On the other hand, we have $u_1(\tau,y)$ is non-decreasing in y, so $u_1\neq u_1^*$ only if u_1 has a jump in space at some point (τ_1,y_1). This would imply that u_1^* has also a jump at the same point. This is impossible, because u_1^* is continuous as a consequence of the continuity of h_1^*. Therefore u_1 and h_1 are continuous. Hence $h_1=h_1^*=h_2$ on $(-\infty,\tau_0]\times\mathbb{R}$ and then on $\mathbb{R}\times\mathbb{R}$, using the periodicity in time of the hull functions.

(ii) *Strict monotonicity of the hull function.* We simply apply (i) with $h_1(\tau,z)=h(\tau,z)$ and $h_2(\tau,z)=h(\tau,z+a_0)\geqslant h_1(\tau,z)$ for some $a_0>0$. Assume by contradiction, the existence of a contact point between h_1 and h_2. Point (i) implies that $h_1=h_2$, i.e. $h(\tau,z+a_0)=h(\tau,z)$. This implies that

$$h(\tau,z+ka_0)=h(\tau,z) \quad \text{for any } k\in\mathbb{Z}$$

which is impossible. Therefore, we have

$$h(\tau,z+a_0)>h(\tau,z) \quad \text{for } a_0>0$$

which ends the proof of the proposition. □

Acknowledgments

This work was supported by the ACI "Dislocations" (2003–2007) and the ANR MICA (2006–2009).

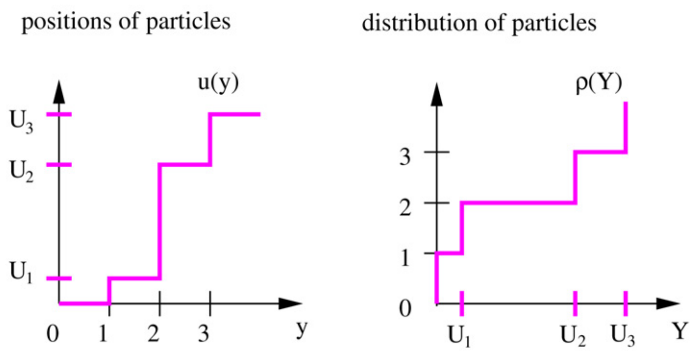

Fig. 4. The function ρ as the inverse of u.

Appendix A. The hull function versus Slepčev formulation

In this section we present a kind of "dual formulation" of the equations, called the Slepčev formulation and satisfied by the inverse in space of the functions. This presentation is done formally, but can be made rigorous.

A.1. The classical FK model

Let us start with the solution $U_i(\tau)$ of (1.1). Then we can define the "cumulative distribution of particles"

$$\rho(\tau, Y) = \sum_{i \geqslant 0} H\big(Y - U_i(\tau)\big) + \sum_{i<0} \big(-1 + H\big(Y - U_i(\tau)\big)\big)$$

where H is the Heaviside function defined by

$$H(x) = \begin{cases} 1 & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Here $\rho(\tau, \cdot)$ is nothing else than the inverse (in space) of the function

$$y \mapsto U_{\lfloor y \rfloor}(\tau).$$

Then we can check that the discontinuous function ρ solves the following equation

$$\rho_\tau = |\nabla \rho| \big\{ M\big[\rho(\tau, \cdot)\big](Y) - \sin(2\pi Y) - f \big\} \tag{A1.66}$$

where the non-local operator M is defined for $v(Y)$ by

$$M[v](Y) = \lim_{a \to +\infty} M^a[v](Y)$$

where for any $a > 0$ we set

$$M^a[v](Y) = \int_{[-a,a]} dZ\, E_{-1,1}\big(v(Y+Z) - v(Y)\big)$$

with

$$E_{-1,1}(x) = \begin{cases} -\frac{3}{2} & \text{if } x < -1, \\ -\frac{1}{2} & \text{if } -1 \leqslant x < 0, \\ \frac{1}{2} & \text{if } 0 \leqslant x < 1, \\ \frac{3}{2} & \text{if } 1 \leqslant x. \end{cases}$$

Remark that $M_a[v](Y)$ is independent on a for any a sufficiently large (depending on v and Y).

Eq. (A1.66) has to be understood in the sense of Slepčev viscosity solutions as in Forcadel, Imbert, Monneau [12].

More generally, if a continuous function ρ solves Eq. (A1.66) and satisfies for some $\delta > 0$

$$\rho_Y \geqslant \delta > 0$$

then the sequence $(U_i(\tau))_i$ defined by

$$\rho\big(\tau, U_i(\tau)\big) = i$$

solves (1.1) (see Fig. 4).

Another approach to the homogenization of system (1.1) consists in doing the homogenization of Eq. (A1.66) following the lines of [12]. Consider the rescaled function

$$\rho^\varepsilon(t, X) = \varepsilon\rho\big(\varepsilon^{-1}t, \varepsilon^{-1}X\big)$$

where $\rho^\varepsilon(t,\cdot)$ appears to be the inverse (in space) of $u^\varepsilon(t,\cdot)$ defined in (1.2). Under suitable assumptions, it is possible to show that ρ^ε converges to ρ^0 which solves the following equation:

$$\rho_t^0 = \overline{H}\big(\rho_X^0\big). \tag{A1.67}$$

Here $\rho^0(t,\cdot)$ is the inverse (in space) of the function $u^0(t,\cdot)$ which solves (1.3). Taking the derivatives of the identity:

$$\rho^0\big(t, u^0(t,x)\big) = x,$$

a simple computation shows that

$$\overline{H}(q) = -q\overline{F}(1/q). \tag{A1.68}$$

Moreover the quantity $\theta = \rho_X^0$ can be interpreted as the density of particles and satisfies the following conservation law (the derivative of (A1.67)):

$$\theta_t = \big(\overline{H}(\theta)\big)_X.$$

The cell equation corresponding to Eq. (A1.66) is found setting $\rho(\tau, Y) = \mu\tau + qY + v(Y)$. We see that the corrector v satisfies

$$\mu = |q + v_Y|\big(M[v](Y) - \sin(2\pi Y) - f\big)$$

with $\mu = \overline{H}(q)$ and v is 1-periodic. Therefore, if we set

$$w(Y) = Y + \frac{v(Y)}{q}$$

and if w satisfies for some $\delta > 0$:

$$0 < \delta \leqslant w_Y \leqslant 1/\delta$$

then we see (from (1.7)) that the hull function h is nothing else than the inverse of w, i.e.

$$h\big(w(Y)\big) = Y \tag{A1.69}$$

and $-\mu/q = \lambda$ with $p = 1/q$ which again is exactly the relation (A1.68).

If both w and h are monotone, then a discontinuity of the hull function corresponds to a zero gradient of w and a discontinuity of w corresponds to a zero gradient of h. But in general, we do not know how to exclude the possibility for h and for w to be non-monotone and then (A1.69) could be no longer true.

Let us also remark that, while the case $p \to +\infty$ seems difficult to deal with in the "hull function approach," this corresponds to a density $q = 1/p$ going to zero with a corresponding effective Hamiltonian $\overline{H}(0) = 0$ (because $\overline{F}$ is bounded). Therefore this case is well-posed for the formulation in ρ^ε and could be proven naturally using directly the "Slepčev formulation." Another proof should be possible working in the "hull function approach" with initial data in BUC_{loc} with gradient bounded from below. Using the relation (A1.68), it should be possible to show that u^ε converges to u^0 whose the inverse is a solution of (A1.67). The case of infinite gradient should be treated by an approximation argument by comparison with functions with large, but finite, gradient.

Similarly the case $p \to 0$ could be treated following the lines of Imbert, Monneau [15] in the hull function approach. This could also be treated in the "Slepčev formulation" dealing with solutions with initial data in BUC_{loc}, rather than Lipschitz initial data.

A.2. The generalized FK model

We define for any $k \in \mathbb{Z} \setminus \{0\}$

$$E_k(x) = H(x) + H(x-k) - 1$$

and

$$E_0 = 0$$

and the operator for $v(Y)$

$$M_k[v](Y) = \lim_{a \to +\infty} M_k^a[v](Y)$$

where for any $a > 0$ we set

$$M_k^a[v](Y) = \int_{[-a,a]} dZ\, E_k\big(v(Y+Z) - v(Y)\big).$$

Then we see that if $u(\tau, y)$ solves (1.5), then its inverse (in space) $\rho(\tau, Y)$ solves the following non-local and non-linear equation

$$\rho_\tau = -|\rho_Y| F\big(\tau, Y - M_{-m}\big[\rho(\tau,\cdot)\big](Y), \ldots, Y - M_m\big[\rho(\tau,\cdot)\big](Y)\big). \tag{A2.70}$$

This equation is still monotone in ρ and could be treated directly with a suitable "Slepčev formulation."

References

[1] S. Aubry, The twist map, the extended Frenkel–Kontorova model and the devil's staircase, Phys. D 7 (1983) 240–258.

[2] S. Aubry, Exact models with a complete Devil's staircase, J. Phys. C 16 (1983) 2497–2508.

[3] S. Aubry, P.Y. Le Daeron, The discrete Frenkel–Kontorova model and its extensions, Phys. D 8 (1983) 381–422.

[4] N. Bacaër, Convergence of numerical methods and parameter dependence of min-plus eigenvalue problems, Frenkel–Kontorova models and homogenization of Hamilton–Jacobi equations, M2AN Math. Model. Numer. Anal. 35 (2001) 1185–1195.

[5] G. Barles, Solutions de viscosité des équations de Hamilton–Jacobi, Math. Appl. (Berlin), vol. 17, Springer-Verlag, Paris, 1994.

[6] G. Barles, Some homogenization results for non-coercive Hamilton–Jacobi equations, Calc. Var. Partial Differential Equations 30 (4) (2007) 449–466.

[7] L. Boccardo, F. Murat, Remarques sur l'homogénéisation de certains problèmes quasi-linéaires, Port. Math. 41 (1–4) (1984) 535–562.

[8] O.M. Braun, Y.S. Kivshar, The Frenkel–Kontorova Model, Concepts, Methods and Applications, Springer-Verlag, 2004.

[9] M.G. Crandall, H. Ishii, P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.) 27 (1) (1992) 1–67.

[10] R. De La Llave, KAM theory for equilibrium states in 1-D statistical mechanics models, preprint 2005-310, University of Texas at Austin, 2005.

[11] L.C. Evans, The perturbed test function method for viscosity solutions of nonlinear PDE, Proc. Roy. Soc. Edinburgh Sect. A 111 (3–4) (1989) 359–375.

[12] N. Forcadel, C. Imbert, R. Monneau, Homogenization of some systems of particles with two-body interactions and dislocation dynamics, Discrete Contin. Dyn. Syst. Ser. A, in press.

[13] L.M. Floria, J.J. Mazo, Dissipative dynamics of the Frenkel–Kontorova model, Adv. in Phys. 45 (6) (1996) 505–598.

[14] B. Hu, W.-X. Qin, Z. Zheng, Rotation number of the overdamped Frenkel–Kontorova model with ac-driving, Phys. D 208 (2005) 172–190.

[15] C. Imbert, R. Monneau, Homogenization of first order equations with (u/ε)-periodic Hamiltonians. Part I: Local equations, Arch. Ration. Mech. Anal. 187 (1) (2008) 49–89.

[16] H.-C. Kao, S.-C. Lee, W.-J. Tzeng, Exact solution of Frenkel–Kontorova models with a complete devil's staircase in higher dimensions, preprint.

[17] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge Univ. Press, Cambridge, 1995.

[18] T. Kontorova, Y.I. Frenkel, On the theory of plastic deformation and doubling, Zh. Eksp. Teor. Fiz. 8 (1938) 89–, 1340–, 1349– (in Russian).

[19] P.-L. Lions, G.C. Papanicolaou, S.R.S. Varadhan, Homogeneization of Hamilton–Jacobi equations, unpublished preprint, 1986.