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A non-local regularization of first order Hamilton–Jacobi equations

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Abstract

In this paper, we investigate the regularizing effect of a non-local operator on first-order Hamilton–Jacobi equations. We prove that there exists a unique solution that is C^2 in space and C^1 in time. In order to do so, we combine viscosity solution techniques and Green's function techniques. Viscosity solution theory provides the existence of a $W^{1,\infty}$ solution as well as uniqueness and stability results. A Duhamel's integral representation of the equation involving the Green's function permits to prove further regularity. We also state the existence of C^∞ solutions (in space and time) under suitable assumptions on the Hamiltonian. We finally give an error estimate in L^∞ norm between the viscosity solution of the pure Hamilton–Jacobi equation and the solution of the integro-differential equation with a vanishing non-local part.

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0. Introduction

The present paper is concerned with the non-local first-order Hamilton–Jacobi equation:

$$\partial_t u + H(t, x, u, \nabla u) + g[u] = 0 \quad \text{in } [0, +\infty) \times \mathbb{R}^N, \tag{1}$$

$$u(0, x) = u_0(x) \quad \text{for all } x \in \mathbb{R}^N, \tag{2}$$

with $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, where ∇u denotes the gradient w.r.t. x and $g[u]$ denotes the pseudodifferential operator defined by the symbol $|\xi|^\lambda$, $1 < \lambda < 2$. More precisely, if $\mathcal{S}(\mathbb{R}^N)$ denotes the space of Schwartz functions, $g[v](x)$ is defined by

$$g[v](x) = \mathcal{F}^{-1}(|\cdot|^\lambda \mathcal{F} v(\cdot))(x),$$

where $\mathcal{F}$ denotes the Fourier transform. If $1 < \lambda < 2$, as far as Hamilton–Jacobi equations are concerned, the following equivalent form of $g[v]$ is needed:

$$g[v](x) = -\int_{\mathbb{R}^N \setminus \{0\}} (v(x+z) - v(x) - \nabla v(x) \cdot z)\, d\mu(z), \tag{3}$$

where μ denotes the measure whose derivative w.r.t. the Lebesgue measure is $v_0 |z|^{-(N+\lambda)}$ (v_0 is a positive constant, see Lemma 1).

We were first motivated by a paper by Droniou et al. [14] in which the existence of a global smooth solution of a scalar conservation law with the non-local term $g[u]$ is proved. Those non-local conservation laws (sometimes called fractional conservation laws) appear in many applications, in particular in the context of pattern formation in detonation waves [11]. More generally, Lévy processes appear in many areas of physical sciences; in particular Hamilton–Jacobi equations of the form of (1) appear in few models [24, Section 5]. Lévy operators also appear in the context of optimal control of jump diffusion processes. Eq. (1) can be interpreted as the Bellman–Isaacs equation of such an optimal control problem if there is no control on the jumps; otherwise the integro-partial differential equation (integro-pde for short) is no more linear w.r.t. to $g[u]$. Viscosity solution theory provides a good framework to solve these equations and there is a important literature about it, from mathematical finance [1,7–9,2] to systems of integro-pde's [6]. As far as stability, comparison results and existence of viscosity solutions are concerned, results were obtained by Sayah [22] in the stationary case by using first order equation techniques.

Jakobsen and Karlsen [19] developed a general theory for second order parabolic nonlinear integro-pdes. In particular, they establish comparison results and continuous dependance estimates. These later results rely on a "maximum principle for integro-pde's" [20]. Because of the dependance of H on the Hessian of u, their arguments are more technical. In our case, classical techniques work with minor modifications. We construct a viscosity solution by Perron's method and show that the "bump" construction needed to conclude (see [12]) can be adapted. We also point out that we give an existence result in $[0;+\infty)\times\mathbb{R}^N$ (Theorem 3) but one can construct solutions in $[0,T)\times\mathbb{R}^N$ under slightly weaker assumption on the dependance of H on u (compare (A1) and (A1′)); the remaining results (regularity and error estimate) still hold true.

Our main result is Theorem 3. It asserts that there exists a solution of (1) with bounded Lipschitz continuous initial condition that is twice continuously differentiable in x and continuously differentiable in t; in the following, we will say that the solution is regular. If $\lambda = 2$, the classical parabolic theory applies (see [17] for assumptions comparable to ours). In our case, we first use the viscosity solution theory to give a notion of merely continuous solution of (1) and to construct a bounded Lipschitz continuous one; secondly, using Duhamel's integral representation of (1), we construct an "integral" solution that is C^2 in x by a fixed point method (Lemma 4); next, we prove that the "integral" solution is C^1 in t (Lemma 5) and it finally turns out to be a viscosity solution of (1) (with classical derivatives);[1] the comparison result (which implies uniqueness) permits to conclude. We also prove that higher regularity (in fact C^∞ regularity in (t,x)) can be obtained if the assumptions on H are strengthened. See Theorem 5. Even for $\lambda = 2$, this method for proving regularity results is new.

In the last section, thinking of the vanishing viscosity method [13,21], we consider a vanishing Lévy operator:

$$\partial_t u^\varepsilon + H(t,x,u^\varepsilon,\nabla u^\varepsilon) + \varepsilon g[u^\varepsilon] = 0 \quad \text{in } [0;+\infty)\times\mathbb{R}^N. \tag{4}$$

Such an equation appears in [19] and the authors ask if the solution is regular. Our main result answers this question. Moreover, we give an error estimate between the solution u^ε of (4) and the solution u of the pure Hamilton–Jacobi equation:

$$\partial_t u + H(t,x,u,\nabla u) = 0 \quad \text{in } [0;+\infty)\times\mathbb{R}^N. \tag{5}$$

We prove that $\|u^\varepsilon - u\|_{L^\infty([0,T)\times\mathbb{R}^N)}$ is of order $\varepsilon^{1/\lambda}$. In the case $\lambda = 2$, such a result appears first in [16,21]; both proofs rely on probabilistic arguments. In [23], the proof relies on continuous dependance estimates for first-order Hamilton–Jacobi equations. An error estimate of order $\varepsilon^{1/2}$ is obtained in [19], also as a by-product of continuous dependance estimates. Their rate of convergence is less precise than ours since they consider a singular measure such that $|z|^2\mu(z)$ is bounded on the unit ball B; ours is such that $|z|^\delta\mu(z)$ is bounded on B for any $\delta > \lambda$.

We conclude this introduction by mentioning that the techniques and results of this paper only rely on the properties of the kernel K associated with the Lévy operator. Hence, one can adapt them to a different non-local operator if the associated kernel enjoys properties similar to (7)–(10).

The paper is organized as follows. In Section 1, we recall the assumptions needed on the Hamiltonian in order to ensure uniqueness for (5) (and (1)), we recall the

[1] We will see in Section 1 that viscosity solutions are not only used to give a generalized sense to derivatives but also to give a weak sense to the non-local operator via (1).

notion of viscosity solution for such an integro-pde and we list the properties of the kernel associated with the non-local operator that we use in the following. In Section 2, stability, existence and comparison results of viscosity solutions of (1) are proved. Section 3 is devoted to our main result, the regularizing effect of the Lévy operator. In Section 4, we state and prove an error estimate in L^∞ norm between the solution of (4) and the solution of (5). As a conclusion, we give in appendix a non-probabilistic proof of the equivalent form (3) of $g[\cdot]$.

1. Preliminaries

Throughout the paper, we assume that $1 < \lambda < 2$. Here are the assumptions we make about the Hamiltonian H. For any $T > 0$,

(A0) The function $H : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous.
(A1) For any $R > 0$, there exists $\gamma_R \in \mathbb{R}$ such that for all $x \in \mathbb{R}^N$, $u, v \in [-R, R]$, $u < v$, $p \in \mathbb{R}^N$, $t \in [0, T)$,

$$H(t, x, v, p) - H(t, x, u, p) \geqslant \gamma_R(v - u).$$

(A2) For any $R > 0$, there exists $C_R > 0$ such that for all $x \in \mathbb{R}^N$, $u \in [-R, R]$, $p \in \mathbb{R}^N$, $t \in [0, T)$,

$$|H(t, x, u, p) - H(t, y, u, p)| \leqslant C_R(|p| + 1)|x - y|.$$

(A3) For any $R > 0$, there exists $C_R > 0$ such that for all $x \in \mathbb{R}^N$, $u, v \in [-R, R]$, $p, q \in B_R$, $t \in [0, T)$,

$$|H(t, x, u, p) - H(t, x, v, q)| \leqslant C_R(|u - v| + |p - q|).$$

(A4) $\sup_{t\in[0,T),x\in\mathbb{R}^N} |H(t, x, 0, 0)| \leqslant C_0$.

We assume (A0) throughout the paper and we do not mention it in the following.

1.1. Viscosity solutions for (1)

In order to construct first $W^{1,\infty}$ solutions of (1), we need to consider viscosity solutions (see [12] and references therein for an introduction to this theory). This is the reason why we need the equivalent form (3) of the non-local operator g.

Lemma 1. *Let $1 < \lambda < 2$. For any $v \in \mathcal{S}(\mathbb{R}^N)$, (3) holds with μ, the positive measure whose derivative w.r.t. the Lebesgue measure is $v_0|z|^{-(N+\lambda)}$ and v_0, a positive constant.*

Remark 1. This lemma is perhaps classical but we did not find any reference for it. We provide a non-probabilistic proof of it in appendix.

We now turn to the definition of viscosity solution of (1). It relies on the notion of subgradients.

Definition 1. Let $u : [0, +\infty[\times \mathbb{R}^N \to \mathbb{R}$ be bounded and lower semicontinuous (lsc for short). Then $(\alpha, p) \in \mathbb{R} \times \mathbb{R}^N$ is a *subgradient* of u at (t, x) if there exists $r > 0$ and $\sigma > 0$ such that for all $y \in B(x, r)$:

$$u(s, y) \geqslant u(t, x) + \alpha(s - t) + p \cdot (y - x) - \sigma(|y - x|^2) + o(|s - t|), \tag{6}$$

where $o(\cdot)$ is such that $o(l) \to 0$ as $l \to 0$.

In the following, $\partial_P u(t, x)$ denotes the set of all subgradients of u at (t, x) and it is referred to as the subdifferential of u at (t, x). If u is upper semicontinuous (usc for short), we then define supergradients and superdifferentials by $\partial^P u(t, x) = -\partial_P(-u)(t, x)$. Remark that $\partial_P u(t, x)$ is the projection on $\mathbb{R} \times \mathbb{R}^N$ of the parabolic subjet of u (see [12] for the definition of semi-jets). It also can be seen as a "parabolic" version of the proximal subdifferential introduced by Clarke (see [10] for a definition). We can now define viscosity solutions of (1).

Definition 2. 1. A lsc function $u : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$ is a *viscosity supersolution* of (1) if it is bounded and if for any $(t, x) \in (0, +\infty) \times \mathbb{R}^N$ and any $(\alpha, p) \in \partial_P u(t, x)$,

$$\alpha + H(t, x, u(t, x), p) + \int_{B_r\setminus\{0\}} \sigma|z|^2\, d\mu(z) - \int_{B_r^c} (u(t, x + z) - u(t, x) - p \cdot z)\, d\mu(z) \geqslant 0,$$

where r and σ denote constants introduced in Definition 1.

2. A usc function $u : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$ is a *viscosity subsolution* of (1) if it is bounded and if for any $(t, x) \in (0, +\infty) \times \mathbb{R}^N$ and any $(\alpha, p) \in \partial^P u(t, x)$,

$$\alpha + H(t, x, u(t, x), p) - \int_{B_r\setminus\{0\}} \sigma|z|^2 d\mu(z) - \int_{B_r^c} (u(t, x + z) - u(t, x) - p \cdot z)\, d\mu(z) \leqslant 0.$$

3. A *viscosity solution* of (1) is a bounded and continuous function that is both a viscosity subsolution and a viscosity supersolution of (1).

Remarks 2. 1. Note that both integrals are well defined since $\min(|z|^2, |z|)$ is μ-integrable. Moreover, one can replace r by any $s \in]0, r[$ (it is a consequence of the definition of subgradients).

2. Note that one can even take $r = 0$ because of the particular form of the equation. Indeed, the function $u(t, x+z)-u(t, x)-p\cdot z$ is μ-integrable far away from 0 and

is bounded from above by the μ-integrable function $\sigma|z|^2$ in the neighbourhood of 0. This implies that it is μ-quasi-integrable. The equation permits to see that it is in fact μ-integrable.
3. It is not hard to prove that this definition is equivalent to the one given in [22].
4. The definition still makes sense for sublinear functions but we will not use this notion of unbounded solution in the following.

Throughout the paper and unless otherwise stated, subsolution (resp. supersolution and solution) refers to viscosity subsolution (resp. viscosity supersolution and viscosity solution).

1.2. The kernel associated with the non-local operator

The semi-group generated by g is formally given by the convolution with the kernel (defined for $t > 0$ and $x \in \mathbb{R}^N$),

$$K(t,x) = \mathcal{F}(e^{-t|\cdot|^\lambda})(x).$$

Let us recall the main properties of K (see [14]).

$$K \in C^\infty((0,+\infty) \times \mathbb{R}^N) \quad \text{and} \quad K \geqslant 0, \tag{7}$$

$$\forall (t,x) \in (0,+\infty) \times \mathbb{R}^N, K(t,x) = t^{-N/\lambda} K(1, t^{-1/\lambda} x) \tag{8}$$

for all $m \geqslant 0$ and all multi-index α, $|\alpha| = m$, there exists B_m such that

$$\forall\, (t,x) \in (0,+\infty) \times \mathbb{R}^N, |\partial_x^\alpha K(t,x) \leqslant t^{-(N+m)/\lambda} \frac{B_m}{(1 + t^{-(N+1)/\lambda}|x|^{N+1})}, \tag{9}$$

$$\|K(t)\|_{L^1(\mathbb{R}^N)} = 1 \quad \text{and} \quad \|\nabla K(t)\|_{L^1(\mathbb{R}^N)} = \mathcal{K}_1 t^{-1/\lambda}. \tag{10}$$

An easy consequence of the main result of [14] is the fact that K is the kernel of the semi-group generated by g for bounded continuous (even L^∞) data.

Proposition 1 (*Droniou et al. [14]*). *Consider $u_0 \in C_b(\mathbb{R}^N)$. Then $K(t,\cdot) * u_0(\cdot)$ is a C^∞ (in (t,x)) solution of $\partial_t u + g[u] = 0$ submitted to the initial condition $u(t,\cdot) = u_0(\cdot)$.*

2. Stability, uniqueness and existence of continuous solutions

This section is devoted to stability, uniqueness and existence results. In the stationary case, similar results were established in [22]. Nevertheless, our stability results are more general and the proof of uniqueness is simpler. The techniques used here are classical.

For the sake of completeness, we state and prove a general result of discontinuous stability for subsolutions of (1). Recall that the upper semi-limit of $(u_n)_{n\geqslant 1}$, a uniformly bounded from above sequence of usc functions is defined by:

$$\limsup{}^* u_n(t,x) = \limsup_{(s,y)\to(t,x), n\to+\infty} u_n(s,y).$$

This function is usc. If the sequence is constant ($u_n = u$ for any n), $\limsup^* u_n$ (resp. $\liminf_* u_n$) is the upper usc (resp. lower lsc) envelope of u and is denoted by u^* (resp. u_*). See [4,5,12] for more details about semi-limits, semi-continuous envelopes and their use in the viscosity solution theory.

Theorem 1 (*Stability*). *Suppose that H is continuous and $(u_n)_{n\geqslant 1}$ is a sequence of viscosity subsolutions of* (1) *that is locally uniformly bounded from above. Then* $\limsup^* u_n$ *is a viscosity subsolution of* (1).

Remark 3. An analogous result for supersolution can easily be stated and proved. Hence one can pass to the limit in (1) w.r.t. the local uniform convergence.

Proof. Let u denotes $\limsup^* u_n$ and $(\alpha, p) \in \partial^P u(t,x)$. This means that $(\alpha, p, 2\sigma I) \in \mathcal{P}^+ u(t,x)$ and it is well-known (see for instance [12]) that there then exists $(t_n, x_n) \to (t,x)$ and $(k_n)_{n\geqslant 1}$ such that $u(t,x) = \lim_n u_{k_n}(t_n, x_n)$ and $(\alpha_n, p_n, \sigma_n) \to (\alpha, p, \sigma)$ such that $(\alpha_n, p_n, 2\sigma_n I) \in \mathcal{P}^+ u_{k_n}(t_n, x_n)$. In particular $(\alpha_n, p_n) \in \partial^P u_{k_n}(t_n, x_n)$ and since u_{k_n} is a subsolution of (1), we get,

$$\alpha_n + H(t_n, x_n, u_{k_n}(t_n, x_n), p_n) - \int_{\mathbb{R}^N\setminus\{0\}} (u_{k_n}(x_n+z) - u_{k_n}(x_n) - p_n\cdot z)\, d\mu(z) \leqslant 0.$$

We therefore must pass to the upper limit in the integral to conclude. This is an easy consequence of Fatou's lemma. □

We next state stability of subsolutions w.r.t. the "sup" operation; this property is used when constructing a solution by Perron's method. The proof is analogous to the proof of Theorem 1 and is classical; we omit it.

Proposition 2. *Consider $(u_\alpha)_{\alpha\in A}$, a family of viscosity subsolutions of* (1) *that is locally uniformly bounded from above. Then $u = (\sup\{u_\alpha : \alpha \in A\})^*$ is a viscosity subsolution of* (1).

We now turn to strong uniqueness results. It permits to compare sub- and supersolutions of (1).

Theorem 2 (*Comparison principle*). *Assume* (A1)–(A3). *Let $T > 0$ and u_0 be a bounded uniformly continuous function. Suppose that u is a bounded subsolution of* (1) *on $[0,T)\times\mathbb{R}^N$ and v is a bounded supersolution of* (1) *on $[0,T)\times\mathbb{R}^N$. If $u(0,x)\leqslant u_0(x)$ and $v(0,x)\geqslant u_0(x)$ then $u\leqslant v$ on $[0;T)\times\mathbb{R}^N$.*

Remark 4. A comparison result for unbounded sub- and supersolution can be proven in the class of sublinear functions. Since we will not use such an extension, we do not prove it.

Proof. First, we make a classical change of variables so that the Hamiltonian is nondecreasing w.r.t. u. Set $\lambda_1 = (\gamma_{R_0})^- + 1$ where γ_{R_0} is given by (A1) and $R_0 = \|u\|_\infty + \|v\|_\infty$. The functions $U(t,x) = e^{-\lambda_1 t} u(t,x)$ and $V(t,x) = e^{-\lambda_1 t} v(t,x)$ are, respectively, sub- and supersolution of:

$$\partial_t W + \lambda_1 W + e^{-\lambda_1 t} H(t, x, e^{\lambda_1 t} W, e^{\lambda_1 t} \nabla W) + g[W] = 0. \tag{11}$$

It suffices to prove a comparison result for this equation.

Let $M = \sup_{[0,T)\times\mathbb{R}^N} (U - V)$. We must prove that $M \leqslant 0$. Suppose that $M > 0$ and let us exhibit a contradiction. Consider a function $\phi \in C^2(\mathbb{R}^N)$ such that:

$$|\nabla\varphi| + |D^2\varphi| \leqslant C \quad \text{and} \quad \lim_{|x|\to+\infty} \varphi(x) = +\infty$$

and four parameters $\varepsilon, \nu, \alpha, \gamma > 0$. Define

$$M_{\varepsilon,\alpha,\nu} = \sup_{[0,T)\times[0,T)\times\mathbb{R}^N\times\mathbb{R}^N} \left\{ U(t,x) - V(s,y) - \frac{|x-y|^2}{2\varepsilon} - \frac{(s-t)^2}{2\nu} - \alpha\varphi(x) - \frac{\gamma}{T-t} \right\}.$$

There exists $(\bar{t}, \bar{s}, \bar{x}, \bar{y}) \in [0,T) \times [0,T) \times \mathbb{R}^N \times \mathbb{R}^N$ where the supremum is attained. We remark that

$$M_{\varepsilon,\alpha,\nu} \geqslant \sup_{[0,T)\times\mathbb{R}^N} \left\{ U(t,x) - V(t,x) - \alpha\varphi(x) - \frac{\gamma}{T-t} \right\} > 0$$

for α and γ small enough (we use here that $M > 0$). In the following $\alpha, \varepsilon \leqslant 1$.

First case: Suppose that there exists $\varepsilon_n \to 0$, $\alpha_p \to 0$ et $\nu_q \to 0$ such that $M_{\varepsilon_n,\alpha_p,\nu_q}$ is attained at $\bar{t} = 0$ or $\bar{s} = 0$. Then we claim that $M_{\varepsilon_n,\alpha_q} - \gamma/T = \lim_{q\to+\infty} M_{\varepsilon_n,\alpha_p,\nu_q} \geqslant 0$ where

$$M_{\varepsilon,\alpha} := \sup_{x,y\in\mathbb{R}^N} \left\{ U(0,x) - V(0,y) - \frac{|x-y|^2}{2\varepsilon} - \alpha\varphi(x) \right\}$$
$$\leqslant \sup_{x,y\in\mathbb{R}^N} \left\{ u_0(x) - u_0(y) - \frac{|x-y|^2}{2\varepsilon} \right\}.$$

Since u_0 is bounded and uniformly continuous, the right-hand side tends to 0 as $\varepsilon \to 0$. Hence $-\gamma/T \geqslant 0$ is a contradiction.

Second case: Suppose that for any $\varepsilon, \alpha, \nu > 0$ small enough, the supremum is attained at $\bar{t} > 0$ and $\bar{s} > 0$. We first get that,

$$\lim_{\varepsilon\to 0}\lim_{\alpha\to 0}\lim_{\nu\to 0} U(\bar{t},\bar{x}) - V(\bar{s},\bar{y}) \geqslant \frac{\gamma}{T} > 0, \tag{12}$$

$$\lim_{\varepsilon\to 0}\lim_{\alpha\to 0}\lim_{\nu\to 0} \frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} + \frac{(\bar{s}-\bar{t})^2}{2\nu} + \alpha\varphi(\bar{x}) = 0, \tag{13}$$

$$\frac{|\bar{x}-\bar{y}|^2}{\varepsilon} \leqslant 2(\|u\|_\infty + \|v\|_\infty) = 2R_0. \tag{14}$$

Then,

$$\left(\frac{\gamma}{(T-\bar{t})^2} + \frac{\bar{t}-\bar{s}}{\nu}, \bar{p} + \alpha\nabla\varphi(\bar{x})\right) \in \partial^P U(\bar{t},\bar{x}) \quad \text{and} \quad \left(\frac{\bar{t}-\bar{s}}{\nu}, \bar{p}\right) \in \partial_P V(\bar{s},\bar{y}),$$

where $\bar{p} = \frac{\bar{x}-\bar{y}}{\varepsilon}$. Since U is a subsolution and V is a supersolution of (11), we get,

$$\begin{aligned}
&\frac{\gamma}{(T-\bar{t})^2} + \frac{\bar{t}-\bar{s}}{\nu} + \lambda_1 U(\bar{t},\bar{x}) + e^{-\lambda_1\bar{t}} H(\bar{t},\bar{x},e^{\lambda_1\bar{t}}U(\bar{t},\bar{x}),\bar{p}+\alpha\nabla\varphi(\bar{x}))\\
&\quad + g[U](\bar{t},\bar{x},\bar{p}+\nabla\varphi(\bar{x})) \leqslant 0,\\
&\frac{\bar{t}-\bar{s}}{\varepsilon} + \lambda_1 V(\bar{s},\bar{y}) + e^{-\lambda_1\bar{s}} H(\bar{s},\bar{y},e^{\lambda_1\bar{s}}V(\bar{s},\bar{y}),\bar{p}) + g[V](\bar{s},\bar{y},\bar{p}) \geqslant 0.
\end{aligned}$$

Substracting the two inequalities, using (12), (A1) and the definition of λ_1, it comes,

$$\begin{aligned}
\frac{\gamma}{T^2} \leqslant\ & e^{-\lambda_1\bar{t}} H(\bar{t},\bar{x},e^{\lambda_1\bar{t}}V(\bar{s},\bar{y}),\bar{p}+\alpha\nabla\varphi(\bar{x})) - e^{-\lambda_1\bar{s}} H(\bar{s},\bar{y},e^{\lambda_1\bar{s}}V(\bar{s},\bar{y}),\bar{p})\\
&+ g[V](\bar{s},\bar{y},\bar{p}) - g[U](\bar{t},\bar{x},\bar{p}+\nabla\varphi(\bar{x})).
\end{aligned} \tag{15}$$

Using the fact that $U(\bar{t},\bar{x}+z) - V(\bar{t},\bar{y}+z) - \alpha\varphi(\bar{x}+z) \leqslant U(\bar{t},\bar{x}) - V(\bar{t},\bar{y}) - \alpha\varphi(\bar{x})$, we get,

$$g[V](\bar{s},\bar{y},\bar{p}) - g[U](\bar{t},\bar{x},\bar{p}+\nabla\varphi(\bar{x})) \leqslant -\alpha g[\phi](\bar{x}). \tag{16}$$

Combining (15) and (16), we obtain

$$\begin{aligned}
0 < \frac{\gamma}{T^2} \leqslant\ & e^{-\lambda_1\bar{t}} H(\bar{t},\bar{x},e^{\lambda_1\bar{t}}V(\bar{s},\bar{y}),\bar{p}+\alpha\nabla\varphi(\bar{x})) - e^{-\lambda_1\bar{s}} H(\bar{s},\bar{y},e^{\lambda_1\bar{s}}V(\bar{s},\bar{y}),\bar{p})\\
&-\alpha g[\phi](\bar{x}).
\end{aligned}$$

Now let $\nu \to 0$ and use (A2) with R_0 and (A3) with $R_\varepsilon = \sqrt{\frac{2R_0}{\varepsilon}} + C$ (use (14)):

$$\frac{\gamma}{2T} \leqslant C_{R_0}(1 + |\overline{p}| + C\alpha)|\overline{x} - \overline{y}| + C_{R_\varepsilon}C\alpha - \alpha g[\phi](\overline{x}) = C_{R_0}|\overline{x} - \overline{y}| + C_{R_0}\frac{|\overline{x} - \overline{y}|^2}{\varepsilon}$$
$$+ C_{R_0}C\alpha|\overline{x} - \overline{y}| + C_{R_\varepsilon}C\alpha.$$

Using (13), we see that the right-hand side tends to 0 as $\alpha \to 0$ and $\varepsilon \to 0$ successively; we therefore get the desired contradiction. □

In order to prove the existence of a solution of (1) in $[0, +\infty) \times \mathbb{R}^N$, we must strenghten assumption (A1). We suppose that either γ_R is positive (that is H is nondecreasing w.r.t. u) or that it does not depend on R (that is H is Lipschitz continuous w.r.t. u uniformly in (x, p)). With classical change of variables, the second case reduces to first one:
(A1′) H is nondecreasing w.r.t. u.
We use Perron's method to prove the following result.

Theorem 3 (*Existence*). *Assume* (A1′)–(A4). *For any* $u_0 : \mathbb{R}^N \to \mathbb{R}$ *bounded and uniformly continuous, there exists a* (*unique*) *viscosity solution of* (1) *in* $[0, +\infty) \times \mathbb{R}^N$ *such that* $u(0, x) = u_0(x)$.

Proof. Suppose we already constructed solutions for initial conditions that are C_b^2. Then if u_0 is bounded and uniformly continuous, there exists $(u_n^0)_{n\geqslant 1}$ that converges uniformly to u_0. Let u_n be the associated solution of (1). One can easily see that $v_q^{\pm} = u_q \pm e^{\gamma t}\|u_0^p - u_0^q\|_\infty$ are, respectively, a super- and a subsolution of (1) and $v_q^+(0, x) \geqslant u_0^p(x) \geqslant v_q^-(0, x)$. Using the comparison principle, we then conclude that $\|u_p - u_q\|_\infty \leqslant e^{\gamma t}\|u_0^p - u_0^q\|_\infty$ so that the sequence $(u_n)_{n\geqslant 1}$ satisfies Cauchy criterion and thus it converges uniformly to a bounded continuous function u. Using the stability of solutions, we conclude that u is a solution of (1).

Let us construct a solution for a C_b^2 initial condition. Define $u^{\pm}(t, x) = u_0(x) \pm Ct$ with C such that:

$$C \geqslant C_0 + C_{R_0}R_0 + 2\|D^2 u_0\|_\infty \int_{B\setminus\{0\}} |z|^2\, d\mu(z) + 2R_0 \int_{B^c} |z|\, d\mu(z)$$
$$\geqslant |H(x, u_0(x), \nabla u_0)| + |g[u_0]|,$$

where C_0 is given by (A4), $R_0 = \|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$ and C_{R_0} is given by (A3). The functions u^+ and u^- are, respectively, a super- and a subsolution of (1). Moreover, both u^+ and u^- satisfy the initial condition in a strong sense:

$$(u^-)_*(0, x) = u^-(0, x) = (u^+)^*(0, x) = u^+(0, x) = u_0(x).$$

Consider now the set

$$\mathcal{S} = \{w : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}, \text{ subsolution of (1)}, w \leqslant u^+\}$$

and define $u = (\sup\{w : w \in \mathcal{S}\})^*$. By Proposition 2, u is a subsolution of (1). Using the barriers u^- and u^+, we also get that u satisfies the initial condition. Consider now u_*. We remark that $u_*(0, x) \le (u^-)_*(0, x) = u_0(x)$. Thus if we prove that u_* is a supersolution of (1), the comparison principle yields $u_* \geqslant u$ and we conclude that u is continuous, that it is a solution of (1) and that it satisfies the initial condition.

It remains to prove that u_* is a supersolution of (1). Suppose that it is false and let us construct a subsolution $U \in \mathcal{S}$ such that $U > u$ at least at one point. This will contradict the definition of u. Thus, suppose that there exists $(t, x) \in (0, +\infty) \times \mathbb{R}^N$ and $(\alpha, p) \in \partial_P u_*(t, x)$ such that,

$$\begin{aligned} \alpha + H(t, x, u_*(t, x), p) - \int_{\mathbb{R}^N \setminus \{0\}} (u_*(t, x+z) - u_*(t, x) - p \cdot z)\, d\mu(z) \\ \leqslant -\theta < 0 \end{aligned} \tag{17}$$

and for all $z \in B_{r_0}$,

$$u_*(t+\tau, x+z) - u_*(t, x) - p \cdot z \geqslant \alpha\tau - \sigma|z|^2 + o(|\tau|).$$

Note that in (17), the integral can be $+\infty$. Define on $(t-\varepsilon, t+\varepsilon) \times B_r(x)$:

$$Q(s, y) = u_*(t, x) + \alpha(s-t) + p \cdot (y-x) - \sigma|y-x|^2 + \delta - \gamma(|y-x|^2 + |s-t|),$$

where $\varepsilon, \delta, \gamma$ are constants to be fixed later and $r \leqslant r_0$. Thus,

$$\begin{aligned} u(s, y) &\geqslant u_*(s, y) \geqslant u_*(t, x) + \alpha(s-t) + p \cdot (y-x) - \sigma|y-x|^2 + o(|s-t|) \\ &\geqslant Q(s, y) - \delta + \gamma|y-x|^2 + (\gamma|s-t| + o(|s-t|)). \end{aligned}$$

We can choose ε small enough such that for all $(s, y) \in (t-\varepsilon, t+\varepsilon) \times B_r(x)$:

$$u(s, y) \geqslant Q(s, y) - \delta/2 + \gamma|y-x|^2.$$

Choose next $\delta = \gamma r^2/4$ so that for $(s, y) \in (t-\varepsilon, t+\varepsilon) \times (B_r(x) \setminus B_{r/2}(x))$,

$$u(s, y) \geqslant Q(s, y) - \gamma r^2/8 + \gamma r^2/4 = Q(s, y) + \gamma r^2/8 > Q(s, y).$$

Now define a function U by

$$U = \begin{cases} \max(u, Q) & \text{in } (t-\varepsilon, t+\varepsilon) \times B_r(x), \\ u & \text{elsewhere.} \end{cases}$$

Let us prove that U is a subsolution of (1). Consider $(s, y) \in (0, +\infty) \times \mathbb{R}^N$ and $(\beta, q) \in \partial^P U(s, y)$.

First case: Suppose that $U(s, y) = u(s, y)$. Then $(\beta, q) \in \partial^P u(s, y)$. Since u is a subsolution of (1), we get,

$$\beta + H(s, y, U(s, y), q) - \int_{\mathbb{R}^N \setminus \{0\}} (U(s, y+z) - U(s, y) - q \cdot z)\, d\mu(z)$$
$$\leqslant \beta + H(s, y, u(s, y), q) - \int_{\mathbb{R}^N \setminus \{0\}} (u(s, y+z) - u(s, y) - q \cdot z)\, d\mu(z) \leqslant 0.$$

Second case: Suppose that $U(s, y) = Q(s, y) > u(s, y)$. Then $(s, y) \in (t-\varepsilon, t+\varepsilon) \times B_r(x)$ and $(\beta, q) \in \partial^P Q(s, y)$; in particular, $\beta = \alpha - \gamma e$ with $|e| \leqslant 1$, $q = p - 2(\sigma + \gamma)(y - x)$. We claim that if $\varepsilon = r^2$, then

$$\liminf_{r \to 0} \int_{\mathbb{R}^N \setminus \{0\}} (U(s, y+z) - U(s, y) - q \cdot z)\, d\mu(z)$$
$$\geqslant \int_{\mathbb{R}^N \setminus \{0\}} (u_*(t, x+z) - u_*(t, x) - p \cdot z))\, d\mu(z). \tag{18}$$

To see this, write $\int_{\mathbb{R}^N \setminus \{0\}} = \int_{B_r \setminus \{0\}} + \int_{B_r^c}$ and study each term:

$$\int_{B_r \setminus \{0\}} (U(s, y+z) - U(s, y) - q \cdot z)\, d\mu(z)$$
$$= \int_{B_r \setminus \{0\}} (\frac{1}{2} D^2 Q z \cdot z)\, d\mu(z) = (\sigma + \gamma) \int_{B_r \setminus \{0\}} |z|^2\, d\mu(z) \to 0$$

as $r \to 0$.

$$\int_{B_r^c} (U(s, y+z) - U(s, y) - q \cdot z)\, d\mu(z)$$
$$\geqslant \int_{B_r^c} (u_*(s, y+z) - Q(s, y) - q \cdot z)\, d\mu(z)$$
$$\geqslant \int_{B_r^c} (u_*(s, y+z) - u_*(t, x) - \alpha(s-t) - p \cdot (y-x) - \delta - q \cdot z)\, d\mu(z).$$

The integrand of the right-hand side converges to $[u_*(t,x+z)-u_*(t,x)-p\cdot z]1_{\mathbb{R}^N\setminus\{0\}}(z)$. Hence, it suffices to exhibit a lower bound independant of r and integrable to conclude by using Fatou's lemma. On $B^c_{r_0}$, we choose $C(1+|z|)$ for C large enough. On $B_{r_0}\setminus B_r$, we have

$$\begin{aligned} &u_*(s,y+z)-u_*(t,x)-\alpha(s-t)-p\cdot(y-x)-\delta-q\cdot z \\ &\quad\geqslant -\sigma|z+y-x|^2-Cr^2-Cr|z|\geqslant -C(r^2+|z|^2)\geqslant -C|z|^2 \end{aligned}$$

for C large enough and we are done.

Suppose first that $\int_{\mathbb{R}^N\setminus\{0\}}(u_*(t,x+z)-u_*(t,x)-p\cdot z))\,d\mu(z)=+\infty$. Then for r small enough, we have:

$$\beta+H(s,y,U(s,y),q)-\int_{\mathbb{R}^N\setminus\{0\}}(U(s,y+z)-U(s,y)-q\cdot z)\,d\mu(z)\leqslant 0.$$

If now $\int_{\mathbb{R}^N\setminus\{0\}}(u_*(t,x+z)-u_*(t,x)-p\cdot z))\,d\mu(z)<+\infty$, then,

$$\begin{aligned} &\beta+H(s,y,U(s,y),q)-\int_{\mathbb{R}^N\setminus\{0\}}(U(s,y+z)-U(s,y)-q\cdot z)\,d\mu(z) \\ &\leqslant -\theta+\gamma-H(t,x,u_*(t,x),p)+H(s,y,U(s,y),q) \\ &\quad+\int_{\mathbb{R}^N\setminus\{0\}}(u_*(t,x+z)-u_*(t,x)-p\cdot z))\,d\mu(z) \\ &\quad-\int_{\mathbb{R}^N\setminus\{0\}}(U(s,y+z)-U(s,y)-q\cdot z)\,d\mu(z). \end{aligned}$$

Choosing $\gamma=\theta/2$ and r small enough permits to conclude that U is a subsolution of (1).

By the comparison principle, since $U(0,x)=u(0,x)$, we have $U\leqslant u^+$. Thus $U\in\mathcal{S}$. Moreover, if (t_n,x_n) is a sequence such that $u_*(t,x)=\lim_n u(t_n,x_n)$, we get,

$$\limsup_{n\to\infty} U(t_n,x_n)\geqslant \lim_{n\to\infty} Q(t_n,x_n)-u_*(t,x)=\delta>0.$$

There then exists (s,y) such that $U(s,y)>u(s,y)$ which is a contradiction. The proof is now complete. $\square$

3. Regularizing effect

In this section, if the natural assumptions that ensure the existence and the uniqueness of a continuous (viscosity) solution of (1) are slightly strengthened the solution is in fact C^2 in x and C^1 in t. We also show that C^∞ regularity is obtained if assumptions are further strenghtened (Theorem 5). We use techniques and ideas introduced in (A3′) For any $R > 0$, there exists $C_R > 0$ s.t. $\partial_u H, \nabla_p H, \nabla^2_{p,x} H, \nabla_p \partial_u H$ and $\nabla^2_{p,p} H$ are bounded by C_R on $[0, T) \times \mathbb{R}^N \times [-R, R] \times B_R$. [14].

Theorem 4 ($C^{1,2}$ *regularity*). *Assume that H satisfies* (A1′)–(A2)–(A3′)–(A4) *and consider an initial condition* $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. *Then the* (*unique*) *viscosity solution of* (1) *is* C^2 *in the space variable and* C^1 *in the time variable in* $]0; +\infty[\times \mathbb{R}^N$.

Proof. We first remark that the viscosity solution u of (1) remains (globally) Lipschitz continuous at any time $t > 0$. This fact is well-known for local equations (see [13,21,18,3]) and the classical proof can be adapted to our situation; this is the reason why we omit details.

Lemma 2. *For any* $t \in [0, T)$, $\|u(t, \cdot)\|_{W^{1,\infty}(\mathbb{R}^N)} \leqslant M_T$ *with* M_T *that only depends on* $\|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$, C_0 *and T*.

Proof (*Sketch*)**.** The comparison principle gives immediately: $\|u\|_\infty \leqslant \|u_0\|_\infty + C_0 T$ where C_0 denotes the constant in (A4). Next, define: $u^\varepsilon(t, x) = \sup_{y \in \mathbb{R}^N} \{u(t, y) - e^{Kt} \frac{|x-y|^2}{2\varepsilon}\}$ with $K = 4C_{\|u\|_\infty}$ from (A2) and verify that it is a viscosity subsolution of:

$$\partial_t u^\varepsilon + H(t, x, u^\varepsilon, \nabla u^\varepsilon) + g[u^\varepsilon](t, x) \leqslant \frac{K\varepsilon}{16}.$$

The non-local term makes no trouble since

$$u^\varepsilon(t, x+z) - u^\varepsilon(t, x) - p \cdot z \geqslant u(t, x_\varepsilon + z) - u(t, x_\varepsilon) - p \cdot z,$$

where x_ε denotes a point such that $u^\varepsilon(t, x) = u(t, x_\varepsilon) - e^{Kt} \frac{|x-x_\varepsilon|^2}{2\varepsilon}$. The comparison principle yields:

$$u^\varepsilon(t, x) \leqslant u(t, x) + \frac{K\varepsilon}{16} t + \sup_{x \in \mathbb{R}^N} \{u_0^\varepsilon(x) - u_0(x)\}.$$

Using the definition of u^ε and the fact that u_0 is Lipchitz continuous, we get

$$u(t, y) \leqslant u(t, x) + (Kt/16 + \|\nabla u_0\|^2_\infty/2)\varepsilon + e^{Kt} \frac{|y-x|^2}{2\varepsilon}.$$

Optimizing w.r.t. ε, we finally obtain

$$u(t, y) \leqslant u(t, x) + e^{Kt/2}(K/8 + \|\nabla u_0\|^2_\infty)^{1/2} |y - x|. \qquad \square$$

We next construct a solution using Duhamel's integral representation of (1). More precisely, we look for functions satisfying:

$$v(t,x) = K(t,\cdot) * v_0(\cdot)(x) - \int_0^t K(t-s,\cdot) * H(s,x,v(s,\cdot),\nabla v(s,\cdot))(x)\,ds. \tag{19}$$

Lemma 3. *Let $v_0 \in W^{1,\infty}(\mathbb{R}^N)$. There exists $T_1 > 0$, that depends only on λ, N and $\|v_0\|_{W^{1,\infty}(\mathbb{R}^N)}$, and $v \in C_b(]0,T_1[\times\mathbb{R}^N)$ such that $\nabla v \in C_b(]0,T_1[\times\mathbb{R}^N)$ and (19) holds true.*

Remark 5. If C_R in (A2) and (A3) does not depend on R ($C_R = \overline{C}$), then T_1 in Lemma 3 only depends on λ, $\mathcal{K}_1$ and $\overline{C}$. Hence we can construct classical solutions of (1) in $[0,+\infty)\times\mathbb{R}^N$ without using viscosity solutions (time regularity is studied below).

Proof of Lemma 3. We use a contracting fixed point theorem. Consider the space

$$E_1 = \{v \in C_b(]0,T_1[\times\mathbb{R}^N), \nabla v \in C_b(]0,T_1[\times\mathbb{R}^N)\}$$

endowed with its natural norm $\|v\|_{E_1} = \|v\|_{C_b(]0,T_1[\times\mathbb{R}^N)} + \|\nabla v\|_{C_b(]0,T_1[\times\mathbb{R}^N)}$. We define

$$\psi_1(v)(t,x) = K(t,\cdot) * u_0(\cdot)(x) - \int_0^t K(t-s,\cdot) * H(s,x,v(s,\cdot),\nabla v(s,\cdot))(x)\,ds. \tag{20}$$

Let us first show that ψ_1 maps E_1 into E_1. Consider $v \in E_1$ such that $\|v\|_{E_1} \leqslant R_1$. By Proposition 1, $K(t,\cdot) * u_0(\cdot)$ is C^1 in space and $K(t,\cdot) * u_0(\cdot)$ and its gradient are continuous in (t,x). Let $\Phi(v)(t,x) = \int_0^t (K(t-s,\cdot) * H(x,v(s,\cdot),\nabla v(s,\cdot))(x)\,ds$. Then defining

$$\mathcal{H}(s,x) = H(s,x,v(s,x),\nabla v(s,x))\mathbf{1}_{]0,T_1[}(s),$$
$$\mathcal{K}(s,x) = K(s,x)\mathbf{1}_{]0,T_1[}(s),$$

we have: $\Phi(v) = \mathcal{H} * \mathcal{K}$ where the convolution is computed w.r.t. (t,x). The function $\mathcal{K}$ is continuous in (t,x) in $]0,T_1[\times\mathbb{R}^N$ and, using (A3)–(A4),

$$|\mathcal{H}(s,x)\,\mathcal{K}(t-s,x-y)| \leqslant (C_0 + C_{R_1}R_1)\mathcal{K}(t-s,x-y)$$

and the right-hand side is integrable since $\int_{\mathbb{R}\times\mathbb{R}^N} \mathcal{K}(t,x)\,dt\,dx = T_1$ (see estimate (10)). The theorem of continuity under the integral sign ensures that $\Phi(v)$ is continuous in

$]0, T_1[\times\mathbb{R}^N$. We also have the following upper bound:

$$|\psi_1(v)(t,x)| \leqslant \|u_0\|_\infty + (C_0 + C_{R_1}R_1)T_1.$$

Since $K(t,x)$ is continuously differentiable and

$$|H(s,x,v(s,x),\nabla v(s,x))\nabla K(t-s,x-y)| \leqslant (C_0 + C_{R_1}R_1)|\nabla K(t-s,x-y)|$$

and $|\nabla K(t-s,x-y)|$ is integrable with $\|\nabla K(t-s,x-y)\|_{L^1(]0,t[\times\mathbb{R}^N)} = \frac{\lambda\mathcal{K}_1}{\lambda-1} t^{(\lambda-1)/\lambda}$ (see estimate (10)), we see that $\psi_1(v)$ is continuously differentiable in x and

$$\nabla\psi_1(v)(t,x) = K(t) * \nabla v_0(x) - \int_0^t ((\nabla K)(t-s) * H(s,x,v(s,\cdot),\nabla v(s,\cdot))(x)\, ds,$$

$$|\nabla\psi_1(v)(t,x)| \leqslant \|\nabla u_0\|_\infty + (C_0 + C_{R_1}R_1)\mathcal{K}_1 \frac{\lambda}{\lambda-1} T_1^{(\lambda-1)/\lambda}.$$

We conclude that $\psi_1(v) \in E_1$ and

$$\|\psi_1(v)\|_{E_1} \leqslant R_0 + (C_0 + C_{R_1}R_1)\left(T_1 + \mathcal{K}_1 \frac{\lambda}{\lambda-1} T_1^{(\lambda-1)/\lambda}\right)$$

if $\|v_0\|_{W^{1,\infty}(\mathbb{R}^N)} \leqslant R_0$. Choose $R_1 = 2R_0$ and T_1 such that

$$(C_0 + C_{R_1}R_1)\left(T_1 + \mathcal{K}_1 \frac{\lambda}{\lambda-1} T_1^{(\lambda-1)/\lambda}\right) \leqslant R_0.$$

This implies that ψ_1 maps B_{R_1}, the closed ball of E_1 of radius R_1, into itself. Moreover, this condition ensures that ψ_1 is a contraction:

$$\begin{aligned}\|\psi_1(v) - \psi_1(w)\|_{E_1} &\leqslant C_{R_1}\left(T_1 + \mathcal{K}_1 \frac{\lambda}{\lambda-1} T_1^{(\lambda-1)/\lambda}\right)\|u-v\|_{E_1} \\ &\leqslant \frac{R_0}{R_1}\|u-v\|_{E_1} = \frac{1}{2}\|u-v\|_{E_1}.\end{aligned}$$

By the Banach fixed point theorem, there then exists a unique fixed point $v \in B_{R_1}$.□

Let us turn to second order regularity in x.

Lemma 4. *The function v constructed in Lemma 3 is continuously twice differentiable in x in $]0, T_2[\times\mathbb{R}^N$, with $T_2 \leqslant T_1$ that only depends on λ, N and $\|v_0\|_{W^{1,\infty}(\mathbb{R}^N)}$. Moreover $t^{1/\lambda}D^2 v$ is bounded in $]0, T_2[\times\mathbb{R}^N$.*

Proof. Remark that $\overline{w} = \nabla v$ verifies:

$$\overline{w} = K(t,\cdot) * w_0(\cdot) - \int_0^t \nabla K(t-s,\cdot) * H(s,\cdot,v(s,\cdot),\overline{w}(s,\cdot))(x)\,ds \quad \text{and}$$
$$\|\overline{w}\|_{C_b(]0,T[\times\mathbb{R}^N)} \leqslant R_1 \tag{21}$$

with $w_0 = \nabla v_0$. Consider the space

$$E_2 = \{w \in C_b(]0,T_2[\times\mathbb{R}^N,\mathbb{R}^N),\, t^{1/\lambda} Dw \in C_b(]0,T_2[\times\mathbb{R}^N)\}$$

endowed with its natural norm $\|w\|_{E_2} = \|w\|_{C_b(]0,T_2[\times\mathbb{R}^N,\mathbb{R}^N)} + \|t^{1/\lambda} Dw\|_{C_b(]0,T_2[\times\mathbb{R}^N)}$. We consider the map ψ_2 defined by

$$\psi_2(w)(t,x) = K(t,\cdot) * w_0(\cdot)(x) - \int_0^t \nabla K(t-s,\cdot) * H(s,\cdot,v(s,\cdot),w(s,\cdot))(x)\,ds$$

with $w_0 = \nabla v_0$. Choose w such that $\|w\|_{E_2} \leqslant R_2$ with $R_2 \geqslant R_1$. Remark first that

$$|H(s,x,v(s,x),w(s,x))| \leqslant C_0 + 2C_{R_2} R_2.$$

Moreover, $x \mapsto H(s,x,v(s,x),w(s,x))$ is differentiable on $]0,T_2[\times\mathbb{R}^N$ and:

$$\begin{aligned}
\nabla(H(s,x,v(s,x),w(s,x))) &= \nabla_x H(s,x,v(s,x),w(s,x)) \\
&\quad + \partial_u H(s,x,v(s,x),w(s,x)) \nabla v(s,x) \\
&\quad + Dw(s,x) \nabla_p H(s,x,v(s,x),w(s,x)) \\
|\nabla(H(s,x,v(s,x),w(s,x)))| &\leqslant C_{R_2}(1+2R_2) + C_{R_2} R_2 s^{-1/\lambda}
\end{aligned}$$

if $\|w\|_{E_2} \leqslant R_2$ (we used $R_2 \geqslant R_1 \geqslant \|\nabla v\|_\infty$). Using the theorem of continuity and differentiability under the integral sign, we conclude that ψ_2 maps E_2 into E_2 and

$$\begin{aligned}
D\psi_2(w)(t,x) &= w_0(\cdot) *_\otimes \nabla K(t,\cdot)(x) \\
&\quad - \int_0^t \nabla K(t-s,\cdot) *_\otimes \nabla(H(s,\cdot,v(s,\cdot),w(s,\cdot)))(x)\,ds,
\end{aligned}$$

where $*_\otimes$ is defined as follows: if $F, G : \mathbb{R}^N \to \mathbb{R}^N$, $F *_\otimes G(x) = \int F(y) \otimes G(x-y)\,dy$. Recall that $\otimes$ denote the tensor product.

We also have the following estimates:

$$\begin{aligned}
|\psi_2(w)(t,x)| &\leqslant R_0 + (C_0 + 2C_{R_2} R_2)\mathcal{K}_1 \frac{\lambda}{\lambda-1} T_2^{(\lambda-1)/\lambda} \\
|t^{1/\lambda} D\psi_2(w)(t,x)| &\leqslant \mathcal{K}_1 R_0 + C_{R_2}\left((1+2R_2)\frac{\lambda}{\lambda-1}\mathcal{K}_1 T_2 + R_2 \gamma_\lambda \mathcal{K}_1 T_2^{(\lambda-1)/\lambda} \right),
\end{aligned}$$

with $\gamma_\lambda = \frac{\lambda}{\lambda-1}\int_0^1 s^{-1/\lambda}(1-s)^{-1/\lambda}\,ds$. We therefore have

$$\|\psi_2(w)\|_{E_2} \leqslant (1+\mathcal{K}_1)R_0 + (C_0 + 2C_{R_2}R_2)\mathcal{K}_1 \frac{\lambda}{\lambda-1} T_2^{(\lambda-1)/\lambda} + C_{R_2}\left((1+2R_2)\frac{\lambda}{\lambda-1}\mathcal{K}_1 T_2 + R_2\gamma_\lambda \mathcal{K}_1 T_2^{(\lambda-1)/\lambda}\right).$$

We now choose $\max(2(1\mathcal{K}_1)R_0, 1) = 2(1+\mathcal{K}_1)R_0 \geqslant R_1$ and $T_2 \leqslant T_1$ such that

$$(C_0 + 2C_{R_2}R_2)\mathcal{K}_1 \frac{\lambda}{\lambda-1} T_2^{(\lambda-1)/\lambda} + C_{R_2}\left((1+2R_2)\frac{\lambda}{\lambda-1}\mathcal{K}_1 T_2 + R_2\gamma_\lambda \mathcal{K}_1 T_2^{(\lambda-1)/\lambda}\right) \leqslant \min\left(\tfrac{1}{2}, (1+\mathcal{K}_1)R_0\right). \tag{22}$$

This condition thus ensures that ψ_2 maps B_{R_2}, the closed ball of E_2 of radius R_2, into itself and that it is a contraction for the norm E_2. Hence, there is a unique fixed point w. Moreover if w_1, w_2 ly in D_{R_2}, the closed ball of $C_b(]0,T[\times\mathbb{R}^N)$ of radius $R_2 \geqslant R_1$, (22) implies that

$$\|\psi_2(w_1) - \psi_2(w_2)\|_{C_b(]0,T_2[\times\mathbb{R}^N)} \leqslant \frac{1}{4}\|w_1 - w_2\|_{C_b(]0,T_2[\times\mathbb{R}^N)}$$

and ψ_2 is also a contraction in $D_{R_2} \subset C_b(]0,T[\times\mathbb{R}^N)$. Using (21), we conclude that the fixed point we just constructed coincide with $\overline{w}$. The proof is now complete. □

We next prove that the function v constructed in Lemma 3 is C^1 in the time variable t and that it satisfies (1). This lemma is adapted from [14, p. 512].

Lemma 5. *Suppose that* $w \in C_b(]0,T_2[\times\mathbb{R}^N)$ *is* C^2 *in* x *such that* $\nabla w, D^2 w \in C_b(]0,T_2[\times\mathbb{R}^N)$. *Then* $\Phi(w)(t,x) = \int_0^t K(t-s,\cdot) * w(s,\cdot)(x)\,ds$ *is* C^1 *w.r.t.* $t \in]0,T_2[$ *and* $\partial_t \Phi(w)(t,x) = w(t,x) - g[\Phi(w)](t,x)$.

Proof. It is enough to prove the result for $t \in]\delta_0, T_2-\delta_0[$ for any $\delta_0 \in]0, T_2/2[$. Fix such a δ_0, consider $\delta \in]0,\delta_0[$ and define $\Phi_\delta(w)(t,x) = \int_0^{t-\delta} K(t-s,\cdot) * w(s,\cdot)(x)\,ds$ in $]\delta_0, T_2-\delta_0[\times\mathbb{R}^N$. It is easy to see that $\Phi_\delta(w)$ converges uniformly to $\Phi(w)$ in $]\delta_0, T_2-\delta_0[\times\mathbb{R}^N$. We next prove that $\Phi_\delta(w)$ is continuously differentiable in $]\delta_0, T_2 - \delta_0[\times\mathbb{R}^N$ and we compute its time derivative. To do so, consider $\phi : \{(t,s,x) :]\delta, T_2 - \delta[\times]0, T_2-\delta[\times\mathbb{R}^N : s \leqslant t - \delta_2\} \to \mathbb{R}$ defined by $\phi(t,s,x) = K(t-s,\cdot) * w(s,\cdot)(x)$. It is enough to prove that ϕ and $\partial_t\phi$ are bounded and continuous to get that $t \mapsto \int_0^{t-\delta}\phi(t,s,x)\,ds$ is continuously differentiable and its time derivative equals

$$\phi(t, t-\delta, x) + \int_0^{t-\delta} \partial_t \phi(t,s,x)\,ds.$$

The function ϕ satisfies $\|\phi\|_\infty \leqslant \|w\|_\infty$ and its continuity is a consequence of the theorem of continuity under the integral sign. Using Proposition 1, we can assert that ϕ is differentiable in time and $\partial_t \phi(t,s,x) = -g[\phi(t,s,\cdot)](x)$. The space derivatives $\nabla\phi, D^2\phi$ are bounded since $\nabla w, D^2 w$ are bounded. It follows that $g[\phi]$ is bounded. We conclude that $\Phi_\delta(w)$ is differentiable in time and, using Fubini's theorem:

$$\partial_t \Phi_\delta(w)(t,x) = \partial_t \Phi_\delta(w)(t,x) = K(\delta,\cdot) * w(t-\delta,\cdot)(x) - g[\Phi_\delta(w)](t,x).$$

It is now easy to see that $\partial_t \Phi_\delta(w)$ converges to the continuous function $w(t,x) - g[\Phi(w)](t,x)$ as $\delta \to 0$. Since $\Phi_\delta(w)$ converges uniformly to $\Phi(w)$ on $]\delta_0, T_2 - \delta_0[\times\mathbb{R}^N$ and remains bounded, it also converges in the distribution sense. We conclude that $\partial_t \Phi(w) = w(t,x) - g[\Phi(w)](t,x)$ and the proof is complete. $\square$

Apply Lemma 5 to the continuous and bounded function $w = H(x, v(t,x), \nabla v(t,x))$:

$$\begin{aligned} \partial_t v(t,x) &= -g[K(t,\cdot) * v_0(\cdot)](x) - H(x, v(t,x), \nabla v(t,x)) \\ &\quad + g\left[\int_0^t K(t-s,\cdot) * H(x, v(s,\cdot), \nabla v(s,\cdot))\right] \\ &= -H(t,x,v(t,x),\nabla v(t,x)) - g[v(t,\cdot)](x). \end{aligned}$$

Hence v is the viscosity solution of (1) in $]0,T_2[\times\mathbb{R}^N$ and its Fréchet derivatives $\partial_t v, \nabla v, D^2 v$ exist.

Consider now the viscosity solution u of (1) in $(0,+\infty)\times\mathbb{R}^N$ and fix $T > 0$. Lemma 2 implies that for any $t \in [0,T]$, $\|u(t,\cdot)\|_{W^{1,\infty}(\mathbb{R}^N)} \leqslant M_T$. For any $T_0 \in [0,T]$, $v(t,x) = u(T_0 + t, x)$ is a viscosity solution of (1) in $[0,+\infty[\times\mathbb{R}^N$ with initial data $v_0(t,x) = u(T_0,x) \in W^{1,\infty}(\mathbb{R}^N)$. By Lemmas 3–5, there exists $T_2 > 0$ that depends only on λ, N and M_T such that v is C^2 in x and C^1 in t in $]0,T_2[\times\mathbb{R}^N$; this implies that u has the same regularity in $]T_0, T_0+T_2[\times\mathbb{R}^N$. Since T_0 and T are arbitrary, the proof is complete. $\square$

We conclude this section with the following regularity result which asserts the existence of a solution of (1) that is infinitely differentiable in time and space.

Theorem 5 (C^∞ *regularity*). *Let $H \in C^\infty(\mathbb{R}^N)$. The unique viscosity solution of*

$$\partial_t u + H(\nabla u) + g[u] = 0$$

with initial data $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ is C^∞ in both time and space variables in $]0;+\infty[\times\mathbb{R}^N$.

Remarks 6. 1. If $N = 1$, the result is an immediate consequence of the integral representation of $\partial_x u$, (21), and of the main result of [14].

2. An analogous result with an Hamiltonian H depending on t, x and u can be stated and proved under suitable assumptions. The ideas are exactely the same as the ones

presented here. We choose to restrict ourselves to $H(\nabla u)$ so that technical difficulties do not hide the key points of the proof.

Proof. We first prove that u is C^∞ with respect to x.

Space regularity. We already proved that the (unique) viscosity solution u of (1) is C^2 in x and C^1 in t and that $t^{1/\lambda}D^2u$ is bounded in $]0, T_2[\times\mathbb{R}^N$. Then construct an "integral" solution on $]T_2/2, 3T_2/2[$ with Lemmas 3 and 4. It coincides with u in $]T_2/2, 3T_2/2[\times\mathbb{R}^N$ and ∇u, D^2u are bounded in $]T_2, 3T_2/2[$ by a constant C only depending on λ, N and $\|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$. Iterating this process, we conclude that D^2u is bounded in $]t_0, +\infty[\times\mathbb{R}^N$ by a constant only depending on λ, N, $\|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$ and t_0.

We now prove by induction that u is C^k in the space variable in $(0, +\infty) \times \mathbb{R}^N$ and that $D^k u$ is bounded on $]t_0, +\infty[\times\mathbb{R}^N$ by a constant only depending on λ, N, $\|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$ and t_0. We proved this assertion at rank $k = 2$. Suppose it is true at any rank i for $2 \leqslant i \leqslant k+1$ and let us prove it at rank $k+2$. Let us fix $t_0 > 0$. Then $\overline{W}(t, x) = \nabla u(t_0 + t, x)$ satisfies for any $t > 0$:

$$\overline{W}(t, x) = K(t, \cdot) * W_0(x) - \int_0^t \nabla K(t-s, \cdot) * H(\overline{W}(s, \cdot))(x)\, ds,$$

where $W_0(\cdot) = \overline{W}(0, \cdot)(= \nabla u(t_0, \cdot))$. By assumption, we know that $\overline{W}$ is C^k in space and its k first derivatives are bounded in $(0, +\infty) \times \mathbb{R}^N$.

Remark that if v is sufficiently regular and $j \in \{1, \dots, k+1\}$:

$$\begin{aligned} D^j(\nabla K(t-s) * H(v(s)) \;&=\; \nabla K(t-s) *_\otimes (D^j v \odot \nabla H(v))(s) \\ &\qquad + \nabla K(t-s) *_\otimes G_j(v, Dv, \dots, D^{j-1}v)(s), \end{aligned}$$

where G_j is C^∞ and $\odot$ denotes the contraction product of tensors. Consider the space

$$\begin{aligned} E_{k+1} = \{&v \in C_b(]0, T_{k+1}[\times\mathbb{R}^N), \nabla v, \dots, D^k v \in C_b(]0, T_{k+1}[\times\mathbb{R}^N), \\ &t^{1/\lambda} D^{k+1} v \in C_b(]0, T_{k+1}[\times\mathbb{R}^N)\} \end{aligned}$$

endowed with its natural norm $\|v\|_{E_{k+1}} = \|v\|_0 + \|v\|_k + \|t^{1/\lambda} D^{k+1} v\|_0$ where

$$\|v\|_0 = \|v\|_{C_b(]0, T_{k+1}[\times\mathbb{R}^N)} \quad \text{and} \quad \|v\|_k = \|v\|_0 + \sum_{i=1}^k \|D^i v\|_0.$$

We consider ψ_2 defined in the proof of Lemma 4:

$$\psi_2(W)(t, x) \;=\; K(t, \cdot) * W_0(\cdot)(x) - \int_0^t \nabla K(t-s, \cdot) * H(W(s, \cdot))(x)\, ds$$

$$
\begin{aligned}
D^i\psi_2(W)(t,x) &= K(t,\cdot)*D^iW_0(\cdot)(x)\\
&\quad -\int_0^t \nabla K(t-s,\cdot)*_{\otimes}\Big(D^iW(s,\cdot)\odot\nabla H(W(s,\cdot))\Big)(x)\,ds\\
&\quad -\int_0^t \nabla K(t-s,\cdot)*_{\otimes} G_i(W(s,\cdot),DW(s,\cdot),\dots,D^{i-1}W(s,\cdot))(x)\,ds\\
D^{k+1}\psi_2(W)(t,x) &= D^kW_0*_{\otimes}\nabla K(t)(x)\\
&\quad -\int_0^t \nabla K(t-s,\cdot)*_{\otimes}\Big(D^{k+1}W(s,\cdot)\odot\nabla H(W(s,\cdot))\Big)(x)\,ds\\
&\quad -\int_0^t \nabla K(t-s,\cdot)*_{\otimes} G_{k+1}(W(s,\cdot),DW(s,\cdot),\dots,D^kW(s,\cdot))(x)\,ds,
\end{aligned}
$$

where $i\in\{1,\dots,k\}$. Now estimate each term:

$$
\begin{aligned}
|\psi_2(W)(t,x)| &\leqslant \|\overline{W}\|_0+\frac{\lambda\mathcal{K}_1}{\lambda-1}\,T_{k+1}^{(\lambda-1)/\lambda}(C_0+C_{\|W\|_0}\|W\|_0)\\
|D^i\psi_2(W)(t,x)| &\leqslant \|D^i\overline{W}\|_0+\frac{\lambda\mathcal{K}_1}{\lambda-1}T_{k+1}^{(\lambda-1)/\lambda}(C_{\|W\|_0}\|D^iW\|_0+D_{\|W\|_k}\|W\|_i)\\
|t^{1/\lambda}D^{k+1}\psi_2(W)(t,x)| &\leqslant \mathcal{K}_1\|D^k\overline{W}\|_0+\frac{\lambda\mathcal{K}_1}{\lambda-1}T_{k+1}D_{\|W\|_k}\|W\|_k\\
&\quad +\mathcal{K}_1\gamma_\lambda T_{k+1}^{(\lambda-1)/\lambda}C_{\|W\|_0}\|t^{1/\lambda}D^kW\|_0,
\end{aligned}
$$

where $D_{\|W\|_k}$ only depends on $\|W\|_k$. If W is such that $\|W\|_{E_{k+1}}\leqslant R_{k+1}$, then:

$$
\begin{aligned}
\|\psi_2(W)\|_{E_{k+1}} &\leqslant (1+\mathcal{K}_1)\|\overline{W}\|_k+\frac{\lambda\mathcal{K}_1}{\lambda-1}T_{k+1}^{(\lambda-1)/\lambda}(C_0+C_{R_{k+1}}R_{k+1})\\
&\quad +\frac{\lambda\mathcal{K}_1}{\lambda-1}\,T_{k+1}D_{R_{k+1}}R_{k+1}+\mathcal{K}_1\gamma_\lambda T_{k+1}^{(\lambda-1)/\lambda}C_{R_{k+1}}R_{k+1}.
\end{aligned}
$$

If now one chooses $R_{k+1}=2(1+\mathcal{K}_1)\|\overline{W}\|_k$ and T_{k+1} such that:

$$
\begin{aligned}
&\frac{\lambda\mathcal{K}_1}{\lambda-1}\,T_{k+1}^{(\lambda-1)/\lambda}(C_0+C_{R_{k+1}}R_{k+1})+\frac{\lambda\mathcal{K}_1}{\lambda-1}\,T_{k+1}D_{R_{k+1}}R_{k+1}\\
&+\mathcal{K}_1\gamma_\lambda T_{k+1}^{(\lambda-1)/\lambda}C_{R_{k+1}}R_{k+1}\leqslant(1+\mathcal{K}_1)\|\overline{W}\|_k,
\end{aligned}
$$

we ensure that ψ_2 maps $B_{R_{k+1}}$ into itself (in the space E_{k+1}). There then exists a fixed point $W_{k+1}\in F_{k+1}$. Moreover one can check that it is a contraction map in the subspace $F_k\subset E_{k+1}$ defined by

$$
F_k=\{v\in C_b(]0,T_{k+1}[\times\mathbb{R}^N),\nabla v,\dots,D^kv\in C_b(]0,T_{k+1}[\times\mathbb{R}^N)\}
$$

endowed with its natural norm $\|v\|_k$; since $\|W_{k+1}\|_{F_k} \leqslant \|W_{k+1}\|_{E_{k+1}} \leqslant R_{k+1}$ and $\|\overline{W}\|_{F_k} \leqslant R_{k+1}$, we conclude that $W_{k+1} = \overline{W}$. Finally, since T_{k+1} only depends on λ, N, t_0 and $\|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$, by arguing as at the beginning of this proof, we conclude that u is C^{k+2} on $]t_0, +\infty[\times\mathbb{R}^N$ and that $D^{k+2}u$ is bounded in $]2t_0, +\infty[\times\mathbb{R}^N$ by a constant that only depends on λ, N, t_0 and $\|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$. Since t_0 is arbitrary, this achieves the proof of space regularity. We now turn to time regularity.

Time regularity. We first prove that ∇u is C^1 in time. In order to do so, we represent ∇u in the following way:

$$\nabla u(t,x) = K(t,\cdot) * \nabla u_0(\cdot)(x) - \int_0^t K(t-s,\cdot) * D^2 u(s,\cdot)\nabla H(\nabla u(s,\cdot))(x)\, ds$$

and we apply Lemma 5 to prove that the second term of the right-hand side is C^1 in t (we already know that the first one is C^1 in time). Next, since $\partial_t u(t,x) = -g[u](t,x) - H(\nabla u(t,x))$, we see that $\partial_t u$ is bounded in $]t_0, +\infty[\times\mathbb{R}^N$; the theorem of differentiability under the integral sign ensures that $\partial_t u$ has second-order spacial derivatives that they are bounded in $]t_0, +\infty[\times\mathbb{R}^N$. Hence, $\partial_t u$ is differentiable w.r.t. t and

$$\partial_t^2 u(t,x) = -g[\partial_t u](t,x) - \nabla H(\nabla u(t,x)) \cdot \partial_t(\nabla u)(t,x).$$

This process can be iterated to conclude. □

4. An error estimate

In this section, we compare the solution of the Hamilton–Jacobi equation with a vanishing Lévy operator (4) with the solution of the pure Hamilton–Jacobi equation (5) (we impose the same initial condition (2) to both equations).

Theorem 6. *Assume* (A0)–(A4) *and consider* $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. *There then exists a constant* $C > 0$ *only depending on* H *and* u_0 *and* T *such that, if* u^ε *and* u, *respectively, denote the solutions of* (4) *and* (5) *such that* $u^\varepsilon(0,\cdot) = u(0,\cdot) = u_0(\cdot)$, *then for all* $t \in [0,T]$:

$$\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \leqslant C\varepsilon^{1/\lambda}\sqrt{t}.$$

Remarks 7. 1. Using the fact that u^ε is C^2 in x and C^1 in t and the bound on $t^{1/\lambda}D^2u^\varepsilon$, we get an error estimate of the form $C\varepsilon^{1/\lambda}t^{1-1/\lambda}$, which is less precise than the one of Theorem 6.

2. About the optimality of the estimate, the power in ε cannot be improved: choosing $H = 0$, $u_0(z) = \min(|z|, 1)$ and $x = 0$, we get $u^\varepsilon(t,0) - u(t,0) = C\varepsilon^{1/\lambda}(t^{1/\lambda} + o_t(1))$. We do not know if one can do better about the power in t.

Proof. Let us define

$$M = \sup_{t\in[0,T),x,y\in\mathbb{R}^N} \left\{ u(t,x) - u^\varepsilon(t,y) - \frac{|x-y|^2}{2\alpha} - \frac{\beta}{2}|x|^2 - \eta t - \frac{\gamma}{T-t} \right\}.$$

Since u and u^ε are bounded, this supremum is attained. We now prove that if one chooses η, γ and β properly, the supremum cannot be achieved at $t = 0$.

Consider:

$$M_\nu = \sup_{t,s\in[0,T),x,y\in\mathbb{R}^N} \left\{ u(t,x) - u^\varepsilon(s,y) - \frac{|x-y|^2}{2\alpha} - \frac{(s-t)^2}{2\nu} - \frac{\beta}{2}|x|^2 - \eta t - \frac{\gamma}{T-t} \right\}.$$

It is classical to prove that M_ν tends to M as $\nu \to 0$. Let $(t_\nu, s_\nu, x_\nu, y_\nu)$ denote a point where the supremum is attained. We have:

$$q_\nu = \frac{x_\nu - y_\nu}{\alpha} \quad \text{and} \quad \left(\eta + \frac{\gamma}{(T-t_\nu)^2} + \frac{t_\nu - s_\nu}{\nu}, q_\nu + \beta x_\nu \right) \in \partial^P u(t_\nu, x_\nu)$$

$$\left(\frac{t_\nu - s_\nu}{\nu}, q_\nu \right) \partial_P u^\varepsilon(s_\nu, y_\nu).$$

Since u^ε is regular, its subgradient is the set $(\partial_t u^\varepsilon, \nabla u_\varepsilon)$; using Remark 2.2, we can take $r = 0$ in the viscosity formulation of (1). Since u is a viscosity solution of (5) and u^ε is viscosity (classical) solution of (4), we get:

$$\eta + \frac{\gamma}{(T-t_\nu)^2} + \frac{t_\nu - s_\nu}{\nu} + H(t_\nu, x_\nu, u(t_\nu, x_\nu), q_\nu + \beta x_\nu) \leqslant 0$$

$$\frac{t_\nu - s_\nu}{\nu} + H(s_\nu, y_\nu, u^\varepsilon(s_\nu, y_\nu), q_\nu) - \varepsilon \int_{\mathbb{R}^N\setminus\{0\}} [u^\varepsilon(s_\nu, y_\nu + z) - u^\varepsilon(s_\nu, y_\nu) - q_\nu \cdot z]\, d\mu(z)$$
$$\geqslant 0.$$

Substracting these two inequalities yields:

$$\eta + \frac{\gamma}{(T-t_\nu)^2} + H(t_\nu, x_\nu, u(t_\nu, x_\nu), q_\nu + \beta x_\nu) - H(s_\nu, y_\nu, u^\varepsilon(s_\nu, y_\nu), q_\nu)$$
$$+\varepsilon \int_{\mathbb{R}^N\setminus\{0\}} [u^\varepsilon(s_\nu, y_\nu + z) - u^\varepsilon(s_\nu, y_\nu) - q_\nu \cdot z] d\mu(z) \leqslant 0.$$

Now let $\nu \to 0$. We can ensure that $(t_\nu, s_\nu, x_\nu, y_\nu) \to (\bar{t}, \bar{t}, \bar{x}, \bar{y})$ such that M is achieved at $(\bar{t}, \bar{x}, \bar{y})$. We can pass to the limit in the integral thanks to Fatou's lemma. We obtain:

$$\eta + \frac{\gamma}{(T-\bar{t})^2} + H(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), \bar{q} + \beta\bar{x}) - H(\bar{t}, \bar{y}, u^\varepsilon(\bar{t}, \bar{y}), \bar{q})$$
$$+\varepsilon \int_{\mathbb{R}^N\setminus\{0\}} [u^\varepsilon(\bar{t}, \bar{y} + z) - u^\varepsilon(\bar{t}, \bar{y}) - \bar{q} \cdot z]\, d\mu(z) \leqslant 0.$$

Notice that $u(\bar t,\bar x)-u^\varepsilon(\bar s,\bar y)-\frac{\gamma}{T-\bar t}\geqslant -\frac{\gamma}{T}$ which implies that $u(\bar t,\bar x)\geqslant u^\varepsilon(\bar s,\bar y)$. Since u is Lipschitz continuous, we now that $|\bar q|\leqslant \|\nabla u\|_\infty\leqslant C$ and $|\bar x-\bar y|\leqslant C\alpha$. We easily get $\beta|\bar x|^2\leqslant C$ and thus $\beta\bar x\leqslant C\sqrt{\beta}$. Using (A1′), (A2) and (A3) we therefore get for $\alpha\leqslant 1$:

$$\eta+\frac{\gamma}{(T-\bar t)^2}-C\alpha-C\sqrt{\beta}$$
$$+\varepsilon\int_{\mathbb{R}^N\setminus\{0\}}[u^\varepsilon(\bar t,\bar y+z)-u^\varepsilon(\bar t,\bar y)-\bar q\cdot z]\,d\mu(z)\leqslant 0. \tag{23}$$

We now make a change of variables $r=\varepsilon^{-1/\lambda}z$ in the remaining integral:

$$\varepsilon\int_{\mathbb{R}^N\setminus\{0\}}[u^\varepsilon(\bar t,\bar y+z)-u^\varepsilon(\bar t,\bar y)-\bar q\cdot z]\,d\mu(z)$$
$$=\mu_0\varepsilon\int_{\mathbb{R}^N\setminus\{0\}}[u^\varepsilon(\bar t,\bar y+\varepsilon^{1/\lambda}r)-u^\varepsilon(\bar t,\bar y)-\varepsilon^{1/\lambda}\bar q\cdot r]|\varepsilon^{1/\lambda}r|^{-N-\lambda}\varepsilon^{N/\lambda}\,dr$$
$$=\int_{\mathbb{R}^N\setminus\{0\}}[u^\varepsilon(\bar t,\bar y+\varepsilon^{1/\lambda}r)-u^\varepsilon(\bar t,\bar y)-\varepsilon^{1/\lambda}\bar q\cdot r]\,d\mu(r)=\int_{B\setminus\{0\}}\{\ldots\}+\int_{B^c}\{\ldots\},$$

where B denotes the unit ball. Using the fact that $u^\varepsilon(\bar t,\bar y+z)\geqslant u^\varepsilon(\bar t,\bar y)+\langle\bar q,z\rangle-\frac{1}{2\alpha}|z|^2$, we get:

$$\left|\int_{B^c}\{\ldots\}\right|\leqslant\|\nabla u^\varepsilon\|_\infty\varepsilon^{1/\lambda}\int_{B^c}|r|d\mu(r)\leqslant C\varepsilon^{1/\lambda}$$
$$\int_{B\setminus\{0\}}\{\ldots\}\geqslant-\frac{1}{2\alpha}\varepsilon^{2/\lambda}\int_{B\setminus\{0\}}|r|^2d\mu(r)\geqslant-\frac{C\varepsilon^{2/\lambda}}{2\alpha}$$

(the fact that u^ε is Lipschitz continuous and its Lipschitz constant is bounded independently of ε can be proven as we did when $\varepsilon=1$). Rewriting (23) yields,

$$\eta+\frac{\gamma}{(T-\bar t)^2}-C\alpha-C\sqrt{\beta}-C\varepsilon^{1/\lambda}-\frac{C\varepsilon^{2/\lambda}}{2\alpha}\leqslant 0.$$

Now choosing $\eta=C(\alpha+\varepsilon^{1/\lambda}+\varepsilon^{2/\lambda}/\alpha)$ and $\gamma=CT^2\sqrt{\beta}$ yields, $\frac{\gamma}{(T-\bar t)^2}-\frac{\gamma}{T^2}\leqslant 0$ which contradicts the fact that $\bar t>0$.

We conclude that

$$u(t,x)-u^\varepsilon(t,y)-\frac{|x-y|^2}{2\alpha}-\frac{\beta}{2}|x|^2-\eta t-\frac{CT^2\sqrt{\beta}}{T-t}$$
$$\leqslant\sup_{x,y\in\mathbb{R}^N}\{u_0(x)-u_0(y)-\frac{|x-y|^2}{2\alpha}\}\leqslant\alpha\frac{\|\nabla u_0\|^2}{2}$$

and letting $\beta \to 0$,

$$u(t,x) \leqslant u^\varepsilon(t,y) + \frac{|x-y|^2}{2\alpha} + C(\alpha + \varepsilon^{1/\lambda} + \varepsilon^{2/\lambda}/\alpha)t + \alpha \frac{\|\nabla u_0\|^2}{2}.$$

Choosing $x = y$ and $\alpha = \varepsilon^{1/\lambda}\sqrt{t}$, we finally get,

$$u(t,x) \leqslant u^\varepsilon(t,x) + C\varepsilon^{1/\lambda}\sqrt{t}.$$

We can argue similarly to get the other inequality. The proof is now complete. □

Appendix A. Proof of Lemma 1

To prove Lemma 1, we use Lemma 5.1 from [15, p.17].

Lemma A.1 (*Droniou [15]*). *There exists $\mu_0 \in \mathbb{R}$ such that*

$$g[u](x) = -\mu_0 |\cdot|^{-(N+\lambda)+2} * \Delta u,$$

where $*$ denotes the convolution.

It is not proven that μ_0 is positive. To see this, let us fix $u \in \mathcal{S}(\mathbb{R}^N)$ and write $\mu_0(\lambda)$ to enhance the fact that it is a function of λ. Since it never vanishes and it is continuous w.r.t. λ (use the theorem of continuity under the integral sign), it suffices to prove that $\lim_{\lambda\to 2} \mu_0(\lambda) > 0$ to conclude. We know that $g_\lambda[u] \to \frac{1}{-4\pi}\Delta u$ as $\lambda \to 2$ and $g_\lambda[u] = \mu_0(\lambda)D(\lambda)(D(\lambda)^{-1}|\cdot|^{-N+2-\lambda}) * \Delta u$ where $D(\lambda) = \||\cdot|^{-N+2-\lambda}\|_{L^1(B)}$. Since the limit of $D(\lambda)^{-1}|\cdot|^{-N+2-\lambda}$ as $\lambda \to 2$, in the distribution sense, is the Dirac mass at the origin, we conclude that μ_0 is positive.

Let α denote $-(N+\lambda)+2$. We first remark that if x is fixed and if one defines $\tilde{u}(y) = u(y) - u(x) - \nabla u(x)\cdot y$, then $\Delta\tilde{u}(y) = \Delta u(y)$. Combining this fact with Lemma A.1 yields:

$$\begin{aligned}\frac{1}{\mu_0} g[u](x) &= \lim_{\varepsilon\to 0+} \int_{\varepsilon \leqslant |z| \leqslant 1/\varepsilon} |z|^\alpha \Delta\tilde{u}(x+z) = \lim_{\varepsilon\to 0+} \int_{\varepsilon \leqslant |z| \leqslant 1/\varepsilon} \Delta(|z|^\alpha)\tilde{u}(x+z) \\ &\quad + \int_{|z|=\varepsilon \text{ or } |z|=1/\varepsilon} \left(|z|^\alpha \frac{\partial \tilde{u}}{\partial n}(x+z) - \tilde{u}(x+z)\frac{\partial |z|^\alpha}{\partial n} \right).\end{aligned}$$

Easy computation gives $\Delta|z|^\alpha = \alpha(N+\alpha-2)|z|^{\alpha-2} = (N+\lambda-2)\lambda|z|^{-(N+\lambda)}$. Let us set $v_0 = \mu_0(N+\lambda-2)\lambda > 0$. Thus, it remains to prove that the second term of the right-hand side goes to 0 as $\varepsilon \to 0$. We use the fact that $\tilde{u}$ is sublinear and $\frac{\partial\tilde{u}}{\partial n}$ is bounded.

If $|z| = \varepsilon$: $|z|^\alpha| \frac{\partial \tilde{u}}{\partial n}(x+z)| \leqslant C\varepsilon^\alpha \varepsilon$ and $|\tilde{u}(x+z)\frac{\partial |z|^\alpha}{\partial n}| \leqslant C\varepsilon^2\varepsilon^{\alpha-1}$. Moreover, $|\{z : |z| = \varepsilon\}| = C\varepsilon^{N-1}$. We conclude that

$$\left| \int_{|z|=\varepsilon} \left(|z|^\alpha \frac{\partial \tilde{u}}{\partial n}(x+z) - \tilde{u}(x+z) \frac{\partial |z|^\alpha}{\partial n} \right) \right| \leqslant C\varepsilon^{N+\alpha} = C\varepsilon^{2-\lambda} \to 0$$

as $\varepsilon \to 0$.

If $|z| = 1/\varepsilon$: $|z|^\alpha| \frac{\partial \tilde{u}}{\partial n}(x+z)| \leqslant C(1/\varepsilon)^\alpha C\varepsilon^{N+\lambda-2}$ and $|\tilde{u}(x+z)\frac{\partial |z|^\alpha}{\partial n}| \leqslant C\varepsilon^{-\alpha} = C\varepsilon^{N+\lambda-2}$. Moreover, $|\{z : |z| = 1/\varepsilon\}| = C\varepsilon^{-N+1}$. We conclude that

$$\left| \int_{|z|=\varepsilon} \left(|z|^\alpha \frac{\partial \tilde{u}}{\partial n}(x+z) - \tilde{u}(x+z) \frac{\partial |z|^\alpha}{\partial n} \right) \right| \leqslant C\varepsilon^{\lambda-1} \to 0$$

as $\varepsilon \to 0$. The proof is now complete.

A.1. Details dans la preuve du Lemme 2

Notons x_ε un point tel que $u^\varepsilon(t,x) = u(t,x_\varepsilon) - e^{Kt}\,\frac{|x-x_\varepsilon|^2}{2\varepsilon}$. On prouve alors le

Lemma A.2. *Si* $(\alpha, p) \in D^{1,+}u^\varepsilon(t,x)$, *alors* $\left(\alpha + Ke^{Kt}\,\frac{|x-x_\varepsilon|^2}{2\varepsilon}\right) \in D^{1,+}u(t,x_\varepsilon)$ *et* $p = e^{Kt}\,\frac{x-x_\varepsilon}{\varepsilon}$.

Proof. On commence par écrire la définition du sur-différentiel. Pour (s,y) proche de (t,x).

$$\begin{aligned} \alpha(s-t) + p\cdot(y-x) &\geqslant u^\varepsilon(s,y) - u^\varepsilon(t,x) + o(|s-t|) - \sigma|y-x|^2 \\ &\geqslant u(s,z) - e^{Ks}\,\frac{|y-z|^2}{2\varepsilon} - u(t,x_\varepsilon) + e^{Kt}\frac{|x-x_\varepsilon|^2}{2\varepsilon} \\ &\quad +o(|s-t|) - \sigma|y-x|^2. \end{aligned}$$

En prenant alors $z = x_\varepsilon$ et $y = x + d$, on obtient que $p = e^{Kt}\frac{x-x_\varepsilon}{\varepsilon}$. Puis en choisissant y tel que $y - x = z - x_\varepsilon$, on obtient:

$$\begin{aligned} \alpha(s-t) + p\cdot(y-x) &\geqslant u(s,z) - u(t,x_\varepsilon) - e^{Ks}\,\frac{|x-x_\varepsilon|^2}{2\varepsilon} + e^{Kt}\,\frac{|x-x_\varepsilon|^2}{2\varepsilon} \\ &\quad +o(|s-t|) - \sigma|y-x|^2. \end{aligned}$$

Il suffit alors de voir cela comme une fonction-test en temps pour conclure. □

Ensuite, on estime:

$$\begin{aligned} H(t,x,u^\varepsilon(t,x),p) &\leqslant H(t,x_\varepsilon,u^\varepsilon(t,x),p) + C_{\|u\|_\infty}(1+|p|)|x_\varepsilon - x| \\ &\leqslant H\left(t,x^\varepsilon,u(t,x_\varepsilon) - e^{Kt}\frac{|x-x_\varepsilon|^2}{2\varepsilon},p\right) + (1+|p|)|x_\varepsilon - x| \\ &\leqslant H(t,x^\varepsilon,u(t,x_\varepsilon,p) + C_{\|u\|_\infty}\left(1+e^{Kt}\,\frac{|x-x_\varepsilon|}{\varepsilon}\right)|x-x_\varepsilon|. \end{aligned}$$

Comme u est une solution de (1), on a:

$$\alpha + Ke^{Kt}\,\frac{|x-x_\varepsilon|^2}{2\varepsilon} + H(t,x_\varepsilon,u(t,x_\varepsilon),p) + g^+[u](t,x_\varepsilon,p) \leqslant 0$$

et donc:

$$\begin{aligned} &\alpha + H(t,x,u^\varepsilon(t,x),p) + g^+[u^\varepsilon](t,x) \\ &\quad\leqslant -Ke^{Kt}\,\frac{|x-x_\varepsilon|^2}{2\varepsilon} + C_{\|u\|_\infty}\left(1+e^{Kt}\,\frac{|x-x_\varepsilon|}{\varepsilon}\right)|x-x_\varepsilon|. \end{aligned}$$

On choisit alors $K = 4C_{\|u\|_\infty}$ et on obtient:

$$\begin{aligned} \alpha + H(t,x,u^\varepsilon(t,x),p) + g^+[u^\varepsilon](t,x) &\leqslant C_{\|u\|_\infty}\sup_r\,(r - e^{Kt}r^2/\varepsilon) = \frac{C_{\|u\|_\infty}\varepsilon}{4}\,e^{-Kt} \\ &\leqslant (K/16)\varepsilon. \end{aligned}$$

A.2. Remarque supplémentaire après le théorème 5

Notice that it is a alternative way to prove that the derivative of the viscosity solution of (5) is the entropy solution of the associated scalar conservation law (results of [14] are needed).

A.3. Dans la preuve de l'estimation d'erreur

On a pour tout s,t,x,y,

$$\begin{aligned} &u(t_\nu,x_\nu) - u^\varepsilon(s_\nu,y_\nu) - \frac{|x_\nu - y_\nu|^2}{2\alpha} - \frac{(s_\nu - t_\nu)^2}{2\alpha} - \frac{\beta}{2}|x_\nu|^2 - \eta t_\nu - \frac{\gamma}{T-t_\nu} \\ &\geqslant u(t,x) - u^\varepsilon(s,y) - \frac{|x-y|^2}{2\alpha} - \frac{(s-t)^2}{2\alpha} - \frac{\beta}{2}|x|^2 - \eta t - \frac{\gamma}{T-t} \end{aligned}$$

donc en particulier pour tout y

$$
\begin{aligned}
u^\varepsilon(s_\nu, y) &\geqslant u^\varepsilon(s_\nu, y_\nu) + \frac{|x_\nu - y_\nu|^2}{2\alpha} - \frac{|x_\nu - y|^2}{2\alpha} \\
&= u^\varepsilon(s_\nu, y_\nu) + \langle \frac{x_\nu - y_\nu}{\alpha}, y - y_\nu \rangle - \frac{1}{2\alpha} |y - y_\nu|^2.
\end{aligned}
$$

(on peut même prendre n'importe quel r).

A.4. Optimalité de l'estimation

Si u est solution de $\partial_t u + \varepsilon g[u] = 0$, alors $v(t,x) = u(t, \varepsilon^{1/\lambda} x)$ est solution de $\partial_t v + g[v] = 0$. Ainsi, $v = K(t) \star v_0$ et donc $u(t,x) = v(t, \varepsilon^{-1/\lambda} x) = \int K(t,y) v_0(\varepsilon^{-1/\lambda} x - y) = \int K(t,y) u_0(x - \varepsilon^{1/\lambda} y)$. Donc pour $u_0(z) = \min(|z|, 1)$ et $x = 0$, on trouve: $v(t,0) = \int_B K(t,y) \varepsilon^{1/\lambda} |y| dy + \int_{B^c} \cdots = \varepsilon^{1/\lambda} (t^{1/\lambda} \int_{B_{t^{-1/\lambda}}} K(1,y)|y|\, dy + \int_{B^c_{t^{-1/\lambda}}} K(1,y)\, dy)$.

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