A non-local regularization of first order Hamilton–Jacobi equations

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Received May 7 2004; revised June 2 2004

Abstract

In this paper, we investigate the regularizing effect of a non-local operator on first-order Hamilton–Jacobi equations. We prove that there exists a unique solution that is $C^2$ in space and $C^1$ in time. In order to do so, we combine viscosity solution techniques and Green’s function techniques. Viscosity solution theory provides the existence of a $W^{1,\infty}$ solution as well as uniqueness and stability results. A Duhamel’s integral representation of the equation involving the Green’s function permits to prove further regularity. We also state the existence of $C^\infty$ solutions (in space and time) under suitable assumptions on the Hamiltonian. We finally give an error estimate in $L^\infty$ norm between the viscosity solution of the pure Hamilton–Jacobi equation and the solution of the integro-differential equation with a vanishing non-local part.

MSC: 35B65; 35B05; 35G25; 35K55

Keywords: Integro-differential Hamilton–Jacobi equation; Non-local regularization; Lévy operator; Viscosity solution; Error estimate

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doi:10.1016/j.jde.2004.06.001
0. Introduction

The present paper is concerned with the non-local first-order Hamilton–Jacobi equation:

\[ \partial_t u + H(t, x, u, \nabla u) + g(u) = 0 \text{ in } [0, +\infty) \times \mathbb{R}^N, \]

\[ u(0, x) = u_0(x) \text{ for all } x \in \mathbb{R}^N, \]

with \( u_0 \in W^{1,\infty}(\mathbb{R}^N) \), where \( \nabla u \) denotes the gradient w.r.t. \( x \) and \( g[u] \) denotes the pseudodifferential operator defined by the symbol \( |\xi|^{\lambda} \), \( 1 < \lambda < 2 \). More precisely, if \( S(\mathbb{R}^N) \) denotes the space of Schwartz functions, \( g[v](x) \) is defined by

\[ g[v](x) = F^{-1}(|\cdot|^{\lambda} F v(\cdot))(x), \]

where \( F \) denotes the Fourier transform. If \( 1 < \lambda < 2 \), as far as Hamilton–Jacobi equations are concerned, the following equivalent form of \( g[v] \) is needed:

\[ g[v](x) = -\int_{\mathbb{R}^N \setminus \{0\}} \left( v(x + z) - v(x) - \nabla v(x) \cdot z \right) d\mu(z), \]

where \( \mu \) denotes the measure whose derivative w.r.t. the Lebesgue measure is \( \nu_0 |z|^{-(N+\lambda)} \) (\( \nu_0 \) is a positive constant, see Lemma 1).

We were first motivated by a paper by Droniou et al. [14] in which the existence of a global smooth solution of a scalar conservation law with the non-local term \( g[u] \) is proved. Those non-local conservation laws (sometimes called fractional conservation laws) appear in many applications, in particular in the context of pattern formation in detonation waves [11]. More generally, Lévy processes appear in many areas of physical sciences; in particular Hamilton–Jacobi equations of the form of (1) appear in few models [24, Section 5]. Lévy operators also appear in the context of optimal control of jump diffusion processes. Eq. (1) can be interpreted as the Bellman–Isaacs equation of such an optimal control problem if there is no control on the jumps; otherwise the integro-partial differential equation (integro-pde for short) is no more linear w.r.t. to \( g[u] \). Viscosity solution theory provides a good framework to solve these equations and there is an important literature about it, from mathematical finance [1,7–9,2] to systems of integro-pde’s [6]. As far as stability, comparison results and existence of viscosity solutions are concerned, results were obtained by Sayah [22] in the stationary case by using first order equation techniques.

Jakobsen and Karlsen [19] developed a general theory for second order parabolic nonlinear integro-pdes. In particular, they establish comparison results and continuous dependance estimates. These later results rely on a “maximum principle for integro-pde’s” [20]. Because of the dependance of \( H \) on the Hessian of \( u \), their arguments are more technical. In our case, classical techniques work with minor modifications. We
construct a viscosity solution by Perron’s method and show that the “bump” construction needed to conclude (see [12]) can be adapted. We also point out that we give an existence result in \([0; +\infty) \times \mathbb{R}^N\) (Theorem 3) but one can construct solutions in \([0, T) \times \mathbb{R}^N\) under slightly weaker assumption on the dependance of \(H\) on \(u\) (compare (A1) and (A1')); the remaining results (regularity and error estimate) still hold true.

Our main result is Theorem 3. It asserts that there exists a solution of (1) with bounded Lipschitz continuous initial condition that is twice continuously differentiable in \(x\) and continuously differentiable in \(t\); in the following, we will say that the solution is regular. If \(\lambda = 2\), the classical parabolic theory applies (see [17] for assumptions comparable to ours). In our case, we first use the viscosity solution theory to give a notion of merely continuous solution of (1) and to construct a bounded Lipschitz continuous one; secondly, using Duhamel’s integral representation of (1), we construct an “integral” solution that is \(C^2\) in \(x\) by a fixed point method (Lemma 4); next, we prove that the “integral” solution is \(C^1\) in \(t\) (Lemma 5) and it finally turns out to be a viscosity solution of (1) (with classical derivatives);\(^1\) the comparison result (which implies uniqueness) permits to conclude. We also prove that higher regularity (in fact \(C^\infty\) regularity in \((t, x)\)) can be obtained if the assumptions on \(H\) are strengthened. See Theorem 5. Even for \(\lambda = 2\), this method for proving regularity results is new.

In the last section, thinking of the vanishing viscosity method [13,21], we consider a vanishing Lévy operator:

\[
\partial_t u^\varepsilon + H(t,x,u^\varepsilon, \nabla u^\varepsilon) + \varepsilon g[u^\varepsilon] = 0 \quad \text{in} \quad [0; +\infty) \times \mathbb{R}^N.
\] (4)

Such an equation appears in [19] and the authors ask if the solution is regular. Our main result answers this question. Moreover, we give an error estimate between the solution \(u^\varepsilon\) of (4) and the solution \(u\) of the pure Hamilton–Jacobi equation:

\[
\partial_t u + H(t,x,u, \nabla u) = 0 \quad \text{in} \quad [0; +\infty) \times \mathbb{R}^N.
\] (5)

We prove that \(\|u^\varepsilon - u\|_{L^\infty([0, T) \times \mathbb{R}^N)}\) is of order \(\varepsilon^{1/\lambda}\). In the case \(\lambda = 2\), such a result appears first in [16,21]; both proofs rely on probabilistic arguments. In [23], the proof relies on continuous dependance estimates for first-order Hamilton–Jacobi equations. An error estimate of order \(\varepsilon^{1/2}\) is obtained in [19], also as a by-product of continuous dependance estimates. Their rate of convergence is less precise than ours since they consider a singular measure such that \(|z|^2 \mu(z)\) is bounded on the unit ball \(B\); ours is such that \(|z|^\delta \mu(z)\) is bounded on \(B\) for any \(\delta > \lambda\).

We conclude this introduction by mentioning that the techniques and results of this paper only rely on the properties of the kernel \(K\) associated with the Lévy operator. Hence, one can adapt them to a different non-local operator if the associated kernel enjoys properties similar to (7)–(10).

The paper is organized as follows. In Section 1, we recall the assumptions needed on the Hamiltonian in order to ensure uniqueness for (5) (and (1)), we recall the

\(^1\)We will see in Section 1 that viscosity solutions are not only used to give a generalized sense to derivatives but also to give a weak sense to the non-local operator via (1).
notion of viscosity solution for such an integro-pde and we list the properties of the kernel associated with the non-local operator that we use in the following. In Section 2, stability, existence and comparison results of viscosity solutions of (1) are proved. Section 3 is devoted to our main result, the regularizing effect of the Lévy operator. In Section 4, we state and prove an error estimate in $L^\infty$ norm between the solution of (4) and the solution of (5). As a conclusion, we give in appendix a non-probabilistic proof of the equivalent form (3) of $g[\cdot]$.

1. Preliminaries

Throughout the paper, we assume that $1 < \lambda < 2$. Here are the assumptions we make about the Hamiltonian $H$. For any $T > 0$,

(A0) The function $H : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is continuous.
(A1) For any $R > 0$, there exists $\gamma_R \in \mathbb{R}$ such that for all $x \in \mathbb{R}^N$, $u, v \in [-R, R], u < v$, $p \in \mathbb{R}^N$, $t \in [0, T)$,

\[ H(t, x, v, p) - H(t, x, u, p) \geq \gamma_R (v - u). \]

(A2) For any $R > 0$, there exists $C_R > 0$ such that for all $x \in \mathbb{R}^N$, $u \in [-R, R]$, $p \in \mathbb{R}^N$, $t \in [0, T)$,

\[ |H(t, x, u, p) - H(t, y, u, p)| \leq C_R (|p| + |x - y|). \]

(A3) For any $R > 0$, there exists $C_R > 0$ such that for all $x \in \mathbb{R}^N$, $u, v \in [-R, R]$, $p, q \in B_R$, $t \in [0, T)$,

\[ |H(t, x, u, p) - H(t, x, v, q)| \leq C_R (|u - v| + |p - q|). \]

(A4) $\sup_{t \in [0, T), x \in \mathbb{R}^N} |H(t, x, 0, 0)| \leq C_0.$

We assume (A0) throughout the paper and we do not mention it in the following.

1.1. Viscosity solutions for (1)

In order to construct first $W^{1,\infty}$ solutions of (1), we need to consider viscosity solutions (see [12] and references therein for an introduction to this theory). This is the reason why we need the equivalent form (3) of the non-local operator $g$.

**Lemma 1.** Let $1 < \lambda < 2$. For any $v \in \mathcal{S}(\mathbb{R}^N)$, (3) holds with $\mu$, the positive measure whose derivative w.r.t. the Lebesgue measure is $\nu_0|z|^{-(N+\lambda)}$ and $\nu_0$, a positive constant.

**Remark 1.** This lemma is perhaps classical but we did not find any reference for it. We provide a non-probabilistic proof of it in appendix.
We now turn to the definition of viscosity solution of (1). It relies on the notion of subgradients.

**Definition 1.** Let \( u : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R} \) be bounded and lower semicontinuous (lsc for short). Then \((x, p) \in \mathbb{R} \times \mathbb{R}^N\) is a subgradient of \( u \) at \((t, x)\) if there exists \( r > 0 \) and \( \sigma > 0 \) such that for all \( y \in B(x, r)\):

\[
u(s, y) \geq u(t, x) + x(s - t) + p \cdot (y - x) - \sigma(|y - x|^2) + o(|s - t|),
\]

where \( o(\cdot) \) is such that \( o(l) \to 0 \) as \( l \to 0 \).

In the following, \( \partial_P u(t, x) \) denotes the set of all subgradients of \( u \) at \((t, x)\) and it is referred to as the subdifferential of \( u \) at \((t, x)\). If \( u \) is upper semicontinuous (usc for short), we then define supergradients and superdifferentials by \( \partial_P u(t, x) = -\partial_P (-u)(t, x) \). Remark that \( \partial_P u(t, x) \) is the projection on \( \mathbb{R} \times \mathbb{R}^N \) of the parabolic subjett of \( u \) (see [12] for the definition of semi-jets). It also can be seen as a “parabolic” version of the proximal subdifferential introduced by Clarke (see [10] for a definition). We can now define viscosity solutions of (1).

**Definition 2.** 1. A lsc function \( u : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R} \) is a viscosity supersolution of (1) if it is bounded and if for any \((t, x) \in (0, +\infty) \times \mathbb{R}^N\) and any \((x, p) \in \partial_P u(t, x),\)

\[
x + H(t, x, u(t, x), p) + \int_{B_r \setminus \{0\}} \sigma |z|^2 d\mu(z) - \int_{B_{r'}^c} (u(t, x + z) - u(t, x) - p \cdot z) d\mu(z)
\]

\[\geq 0,\]

where \( r \) and \( \sigma \) denote constants introduced in Definition 1.

2. A usc function \( u : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R} \) is a viscosity subsolution of (1) if it is bounded and if for any \((t, x) \in (0, +\infty) \times \mathbb{R}^N\) and any \((x, p) \in \partial_P u(t, x),\)

\[
x + H(t, x, u(t, x), p) - \int_{B_r \setminus \{0\}} \sigma |z|^2 d\mu(z) - \int_{B_{r'}^c} (u(t, x + z) - u(t, x) - p \cdot z) d\mu(z)
\]

\[\leq 0.\]

3. A viscosity solution of (1) is a bounded and continuous function that is both a viscosity subsolution and a viscosity supersolution of (1).

**Remarks 2.** 1. Note that both integrals are well defined since \( \min(|z|^2, |z|) \) is \( \mu \)-integrable. Moreover, one can replace \( r \) by any \( s \in ]0, r[ \) (it is a consequence of the definition of subgradients).

2. Note that one can even take \( r = 0 \) because of the particular form of the equation. Indeed, the function \( u(t, x + z) - u(t, x) - p \cdot z \) is \( \mu \)-integrable far away from 0 and
is bounded from above by the $\mu$-integrable function $\sigma|z|^2$ in the neighbourhood of 0. This implies that it is $\mu$-quasi-integrable. The equation permits to see that it is in fact $\mu$-integrable.

3. It is not hard to prove that this definition is equivalent to the one given in [22].

4. The definition still makes sense for sublinear functions but we will not use this notion of unbounded solution in the following.

Throughout the paper and unless otherwise stated, subsolution (resp. supersolution and solution) refers to viscosity subsolution (resp. viscosity supersolution and viscosity solution).

1.2. The kernel associated with the non-local operator

The semi-group generated by $g$ is formally given by the convolution with the kernel (defined for $t > 0$ and $x \in \mathbb{R}^N$),

$$K(t, x) = \mathcal{F}(e^{-t|\cdot|^2})(x).$$

Let us recall the main properties of $K$ (see [14]).

$$K \in C^\infty((0, +\infty) \times \mathbb{R}^N) \quad \text{and} \quad K \geq 0,$$

$$\forall (t, x) \in (0, +\infty) \times \mathbb{R}^N, K(t, x) = t^{-N/\lambda}K(1, t^{-1/\lambda}x)$$

for all $m \geq 0$ and all multi-index $\alpha, |\alpha| = m$, there exists $B_m$ such that

$$\forall (t, x) \in (0, +\infty) \times \mathbb{R}^N, |\partial^\alpha_x K(t, x)| \leq t^{-(N+m)/\lambda} \frac{B_m}{(1 + t^{-1/(N+1)}|x|^{N+1})},$$

$$\|K(t)\|_{L^1(\mathbb{R}^N)} = 1 \quad \text{and} \quad \|\nabla K(t)\|_{L^1(\mathbb{R}^N)} = K_1 t^{-1/\lambda}.$$ 

An easy consequence of the main result of [14] is the fact that $K$ is the kernel of the semi-group generated by $g$ for bounded continuous (even $L^\infty$) data.

**Proposition 1** (Droniou et al. [14]). Consider $u_0 \in C_b(\mathbb{R}^N)$. Then $K(t, \cdot) * u_0(\cdot)$ is a $C^\infty$ (in $(t, x)$) solution of $\partial_t u + g[u] = 0$ submitted to the initial condition $u(t, \cdot) = u_0(\cdot)$.

2. Stability, uniqueness and existence of continuous solutions

This section is devoted to stability, uniqueness and existence results. In the stationary case, similar results were established in [22]. Nevertheless, our stability results are more general and the proof of uniqueness is simpler. The techniques used here are classical.

For the sake of completeness, we state and prove a general result of discontinuous stability for subsolutions of (1). Recall that the upper semi-limit of $(u_n)_n \geq 1$, a uniformly
bounded from above sequence of usc functions is defined by:
\[
\limsup^* u_n(t, x) = \limsup_{(s, y) \to (t, x), n \to +\infty} u_n(s, y).
\]
This function is usc. If the sequence is constant \( u_n = u \) for any \( n \), \( \limsup^* u_n \) (resp. \( \liminf^* u_n \)) is the upper usc (resp. lower lsc) envelope of \( u \) and is denoted by \( u^* \) (resp. \( u_* \)). See [4,5,12] for more details about semi-limits, semi-continuous envelopes and their use in the viscosity solution theory.

**Theorem 1** (Stability). Suppose that \( H \) is continuous and \( (u_n)_{n \geq 1} \) is a sequence of viscosity subsolutions of (1) that is locally uniformly bounded from above. Then \( \limsup^* u_n \) is a viscosity subsolution of (1).

**Remark 3.** An analogous result for supersolution can easily be stated and proved. Hence one can pass to the limit in (1) w.r.t. the local uniform convergence.

**Proof.** Let \( u \) denotes \( \limsup^* u_n \) and \( (\alpha, p, 2\sigma I) \in \partial^P u(t, x) \) and it is well-known (see for instance [12]) that there then exists \( (t_n, x_n) \to (t, x) \) and \( (k_n)_{n \geq 1} \) such that \( u(t, x) = \lim_{n} u_{k_n}(t_n, x_n) \) and \( (\alpha_n, p_n, \sigma_n) \to (\alpha, p, \sigma) \) such that \( (\alpha_n, p_n, 2\sigma_n I) \in \partial^P u_{k_n}(t_n, x_n) \). In particular \( (\alpha_n, p_n) \in \partial^P u_{k_n}(t_n, x_n) \) and since \( u_{k_n} \) is a subsolution of (1), we get,
\[
\alpha_n + H(t_n, x_n, u_{k_n}(t_n, x_n), p_n) - \int_{\mathbb{R}^N \setminus \{0\}} (u_{k_n}(x_n + z) - u_{k_n}(x_n) - p_n \cdot z) d\mu(z) \leq 0.
\]
We therefore must pass to the upper limit in the integral to conclude. This is an easy consequence of Fatou’s lemma. \( \Box \)

We next state stability of subsolutions w.r.t. the “sup” operation; this property is used when constructing a solution by Perron’s method. The proof is analogous to the proof of Theorem 1 and is classical; we omit it.

**Proposition 2.** Consider \( (u_\alpha)_{\alpha \in A} \), a family of viscosity subsolutions of (1) that is locally uniformly bounded from above. Then \( u = (\sup\{u_\alpha : \alpha \in A\})^* \) is a viscosity subsolution of (1).

We now turn to strong uniqueness results. It permits to compare sub- and supersolutions of (1).

**Theorem 2** (Comparison principle). Assume (A1)–(A3). Let \( T > 0 \) and \( u_0 \) be a bounded uniformly continuous function. Suppose that \( u \) is a bounded subsolution of (1) on \([0, T) \times \mathbb{R}^N\) and \( v \) is a bounded supersolution of (1) on \([0, T) \times \mathbb{R}^N\). If \( u(0, x) \leq u_0(x) \) and \( v(0, x) \geq u_0(x) \) then \( u \leq v \) on \([0; T) \times \mathbb{R}^N\).
Remark 4. A comparison result for unbounded sub- and supersolution can be proven in the class of sublinear functions. Since we will not use such an extension, we do not prove it.

Proof. First, we make a classical change of variables so that the Hamiltonian is nondecreasing w.r.t. $u$. Set $\lambda_1 = (\gamma R_0) - 1$ where $\gamma R_0$ is given by (A1) and $R_0 = \|u\|_{\infty} + \|v\|_{\infty}$. The functions $U(t, x) = e^{-\lambda_1 t} u(t, x)$ and $V(t, x) = e^{-\lambda_1 t} v(t, x)$ are, respectively, sub- and supersolution of:

\[
\tilde{c}_t W + \lambda_1 W + e^{-\lambda_1 t} H(t, x, e^{\lambda_1 t} W, e^{\lambda_1 t} \nabla W) + g[W] = 0.
\]  

(11)

It suffices to prove a comparison result for this equation.

Let $M = \sup_{[0, T) \times \mathbb{R}^N} (U - V)$. We must prove that $M \leq 0$. Suppose that $M > 0$ and let us exhibit a contradiction. Consider a function $\phi \in C^2(\mathbb{R}^N)$ such that:

\[
|\nabla \phi| + |D^2 \phi| \leq C \quad \text{and} \quad \lim_{|x| \to +\infty} \phi(x) = +\infty
\]

and four parameters $\varepsilon, \upsilon, \alpha, \gamma > 0$. Define

\[
M_{\varepsilon, \alpha, \gamma} := \sup_{[0, T) \times \mathbb{R}^N} \left\{ U(t, x) - V(s, y) - \frac{|x - y|^2}{2\varepsilon} - \frac{(s - t)^2}{2\upsilon} - \alpha \phi(x) - \frac{\gamma}{T - t} \right\}.
\]

There exists $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{\upsilon}) \in [0, T) \times [0, T) \times \mathbb{R}^N \times \mathbb{R}^N$ where the supremum is attained. We remark that

\[
M_{\varepsilon, \alpha, \gamma} \geq \sup_{[0, T) \times \mathbb{R}^N} \left\{ U(t, x) - V(t, x) - \alpha \phi(x) - \frac{\gamma}{T - t} \right\} > 0
\]

for $\alpha$ and $\gamma$ small enough (we use here that $M > 0$). In the following $\alpha, \varepsilon \leq 1$.

First case: Suppose that there exists $\varepsilon_n \to 0$, $\alpha_p \to 0$ et $\upsilon_q \to 0$ such that $M_{\varepsilon_n, \alpha_p, \upsilon_q}$ is attained at $\tilde{t} = 0$ or $\tilde{\upsilon} = 0$. Then we claim that $M_{\varepsilon_n, \alpha_p, \upsilon_q} - \gamma / T = \lim_{q \to +\infty} M_{\varepsilon_n, \alpha_p, \upsilon_q} > 0$ where

\[
M_{\varepsilon, \alpha} := \sup_{x, y \in \mathbb{R}^N} \left\{ U(0, x) - V(0, y) - \frac{|x - y|^2}{2\varepsilon} - \alpha \phi(x) \right\} \leq \sup_{x, y \in \mathbb{R}^N} \left\{ u_0(x) - u_0(y) - \frac{|x - y|^2}{2\varepsilon} \right\}.
\]
Since \( u_0 \) is bounded and uniformly continuous, the right-hand side tends to 0 as \( \varepsilon \to 0 \). Hence \(-\gamma / T \geq 0\) is a contradiction.

Second case: Suppose that for any \( \varepsilon, \alpha, \nu > 0 \) small enough, the supremum is attained at \( \tilde{t} > 0 \) and \( \tilde{s} > 0 \). We first get that,

\[
\lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \lim_{\nu \to 0} U(\tilde{t}, \tilde{x}) - V(\tilde{s}, \tilde{y}) \geq \frac{\gamma}{T} > 0, \tag{12}
\]

\[
\lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \lim_{\nu \to 0} \frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} + \frac{(\tilde{x} - \tilde{t})^2}{2\alpha} + \alpha \varphi(\tilde{x}) = 0, \tag{13}
\]

\[
\frac{|\tilde{x} - \tilde{y}|^2}{\varepsilon} \leq 2(\|u\|_\infty + \|v\|_\infty) = 2R_0. \tag{14}
\]

Then,

\[
\left( \frac{\gamma}{(T - \tilde{t})^2} + \frac{\tilde{t} - \tilde{s}}{\nu}, \bar{p} + \alpha \nabla \varphi(\tilde{x}) \right) \in \partial^P U(\tilde{t}, \tilde{x}) \quad \text{and} \quad \left( \frac{\tilde{t} - \tilde{s}}{\nu}, \bar{p} \right) \in \partial^P V(\tilde{s}, \tilde{y}),
\]

where \( \bar{p} = \frac{\tilde{x} - \tilde{y}}{\varepsilon} \). Since \( U \) is a subsolution and \( V \) is a supersolution of (11), we get,

\[
\frac{\gamma}{(T - \tilde{t})^2} + \frac{\tilde{t} - \tilde{s}}{\nu} + \lambda_1 U(\tilde{t}, \tilde{x}) + e^{-\lambda_1 \tilde{t}} H(\tilde{t}, \tilde{x}, e^{\lambda_1 \tilde{t}} U(\tilde{t}, \tilde{x}), \bar{p} + \alpha \nabla \varphi(\tilde{x}))
+ g[U](\tilde{t}, \tilde{x}, \bar{p} + \nabla \varphi(\tilde{x})) \leq 0,
\]

\[
\frac{\tilde{t} - \tilde{s}}{\varepsilon} + \lambda_1 V(\tilde{s}, \tilde{y}) + e^{-\lambda_1 \tilde{s}} H(\tilde{s}, \tilde{y}, e^{\lambda_1 \tilde{s}} V(\tilde{s}, \tilde{y}), \bar{p} + \nabla \varphi(\tilde{y})) + g[V](\tilde{s}, \tilde{y}, \bar{p}) \geq 0.
\]

Substracting the two inequalities, using (12), (A1) and the definition of \( \lambda_1 \), it comes,

\[
\frac{\gamma}{T^2} \leq e^{-\lambda_1 \tilde{t}} H(\tilde{t}, \tilde{x}, e^{\lambda_1 \tilde{t}} V(\tilde{s}, \tilde{y}), \bar{p} + \alpha \nabla \varphi(\tilde{x})) - e^{-\lambda_1 \tilde{s}} H(\tilde{s}, \tilde{y}, e^{\lambda_1 \tilde{s}} V(\tilde{s}, \tilde{y}), \bar{p})
+ g[V](\tilde{s}, \tilde{y}, \bar{p}) - g[U](\tilde{t}, \tilde{x}, \bar{p} + \nabla \varphi(\tilde{x})). \tag{15}
\]

Using the fact that \( U(\tilde{t}, \tilde{x} + z) - V(\tilde{t}, \tilde{y} + z) - \alpha \varphi(\tilde{x} + z) \leq U(\tilde{t}, \tilde{x}) - V(\tilde{t}, \tilde{y}) - \alpha \varphi(\tilde{x}), \) we get,

\[
g[V](\tilde{s}, \tilde{y}, \bar{p}) - g[U](\tilde{t}, \tilde{x}, \bar{p} + \nabla \varphi(\tilde{x})) \leq - \alpha g[\phi](\tilde{x}). \tag{16}
\]

Combining (15) and (16), we obtain

\[
0 < \frac{\gamma}{T^2} \leq e^{-\lambda_1 \tilde{t}} H(\tilde{t}, \tilde{x}, e^{\lambda_1 \tilde{t}} V(\tilde{s}, \tilde{y}), \bar{p} + \alpha \nabla \varphi(\tilde{x})) - e^{-\lambda_1 \tilde{s}} H(\tilde{s}, \tilde{y}, e^{\lambda_1 \tilde{s}} V(\tilde{s}, \tilde{y}), \bar{p})
- \alpha g[\phi](\tilde{x}).
\]
Now let \( \nu \to 0 \) and use (A2) with \( R_0 \) and (A3) with \( R_e = \sqrt{\frac{2R_0}{\varepsilon}} + C \) (use (14)):

\[
\frac{\dot{y}}{2T} \leq C_{R_0}(1 + |\overline{p}| + C\alpha)|\overline{x} - \overline{y}| + C_{R_e}C\alpha - 2g[\phi](\overline{x}) = C_{R_0}|\overline{x} - \overline{y}| + C_{R_0} \frac{|\overline{x} - \overline{y}|^2}{\varepsilon} + C_{R_e}C\alpha.
\]

Using (13), we see that the right-hand side tends to 0 as \( \alpha \to 0 \) and \( \varepsilon \to 0 \) successively; we therefore get the desired contradiction. \( \square \)

In order to prove the existence of a solution of (1) in \([0, +\infty) \times \mathbb{R}^N\), we must strengthen assumption (A1). We suppose that either \( \gamma_R \) is positive (that is \( H \) is nondecreasing w.r.t. \( u \)) or that it does not depend on \( R \) (that is \( H \) is Lipschitz continuous w.r.t. \( u \) uniformly in \((x, p))\). With classical change of variables, the second case reduces to first one:

(A1\'') \( H \) is nondecreasing w.r.t. \( u \).

We use Perron’s method to prove the following result.

**Theorem 3 (Existence).** Assume (A1\'')–(A4). For any \( u_0 : \mathbb{R}^N \to \mathbb{R} \) bounded and uniformly continuous, there exists a (unique) viscosity solution of (1) in \([0, +\infty) \times \mathbb{R}^N\) such that \( u(0, x) = u_0(x) \).

**Proof.** Suppose we already constructed solutions for initial conditions that are \( C^2_b \). Then if \( u_0 \) is bounded and uniformly continuous, there exists \( (u^0_n)_{n \geq 1} \) that converges uniformly to \( u_0 \). Let \( u_q \) be the associated solution of (1). One can easily see that \( v^+_q = u_q \pm \varepsilon^it\|u^0_0 - u_q^0\|_{\infty} \) are, respectively, a super- and a subsolution of (1) and \( v^+_q(0, x) \geq u^0_0(x) \geq v^-_q(0, x) \). Using the comparison principle, we then conclude that \( \|u^p_0 - u_q^0\|_{\infty} \leq \varepsilon^it\|u^0_0 - u_q^0\|_{\infty} \) so that the sequence \( (u^p_n)_{n \geq 1} \) satisfies Cauchy criterion and thus it converges uniformly to a bounded continuous function \( u \). Using the stability of solutions, we conclude that \( u \) is a solution of (1).

Let us construct a solution for a \( C^2_b \) initial condition. Define \( u^\pm(t, x) = u_0(x) \pm Ct \) with \( C \) such that:

\[
C \geq C_0 + C_{R_0}R_0 + 2\|D^2u_0\|_\infty \int_{B\setminus\{0\}} |z|^2 d\mu(z) + 2R_0 \int_{B^c} |z| d\mu(z) \\
\geq |H(x, u_0(x), \nabla u_0)| + |g[u_0]|,
\]

where \( C_0 \) is given by (A4), \( R_0 = \|u_0\|_{W^{1, \infty}(\mathbb{R}^N)} \) and \( C_{R_0} \) is given by (A3). The functions \( u^+ \) and \( u^- \) are, respectively, a super- and a subsolution of (1). Moreover, both \( u^+ \) and \( u^- \) satisfy the initial condition in a strong sense:

\[
(u^-)_*(0, x) = u^-(0, x) = (u^+)^*(0, x) = u^+(0, x) = u_0(x).
\]
Consider now the set
\[ S = \{ w : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}, \text{ subsolution of (1), } w \leq u^+ \} \]
and define \( u = (\sup\{ w : w \in S \})^* \). By Proposition 2, \( u \) is a subsolution of (1). Using the barriers \( u^- \) and \( u^+ \), we also get that \( u \) satisfies the initial condition. Consider now \( u_* \). We remark that \( u_*(0, x) \leq (u^-)_*(0, x) = u_0(x) \). Thus if we prove that \( u_* \) is a supersolution of (1), the comparison principle yields \( u_* \geq u \) and we conclude that \( u \) is continuous, that it is a solution of (1) and that it satisfies the initial condition.

It remains to prove that \( u_* \) is a supersolution of (1). Suppose that it is false and let us construct a subsolution \( U \in S \) such that \( U > u \) at least at one point. This will contradict the definition of \( u \). Thus, suppose that there exists \((t, x) \in (0, +\infty) \times \mathbb{R}^N \) and \((\varepsilon, p) \in \partial \rho u_*(t, x)\) such that,
\[
\alpha + H(t, x, u_*(t, x), p) - \int_{\mathbb{R}^N \setminus \{0\}} (u_*(t, x + z) - u_*(t, x) - p \cdot z) \, d\mu(z) \leq -\theta < 0 \tag{17}
\]
and for all \( z \in B_{r_0} \),
\[
u_*(t + \tau, x + z) - u_*(t, x) - p \cdot z \geq \alpha \tau - \sigma |z|^2 + o(|\tau|).
\]
Note that in (17), the integral can be \(+\infty\). Define on \((t - \varepsilon, t + \varepsilon) \times B_r(x)\):
\[
Q(s, y) = u_*(t, x) + \alpha(s - t) + p \cdot (y - x) - \sigma |y - x|^2 + \delta - \gamma(|y - x|^2 + |s - t|),
\]
where \( \varepsilon, \delta, \gamma \) are constants to be fixed later and \( r \leq r_0 \). Thus,
\[
u(s, y) \geq u_*(s, y) \geq u_*(t, x) + \alpha(s - t) + p \cdot (y - x) - \sigma |y - x|^2 + o(|s - t|)
\]
\[
\geq Q(s, y) - \delta + \gamma |y - x|^2 + (\gamma |s - t| + o(|s - t|)).
\]
We can choose \( \varepsilon \) small enough such that for all \((s, y) \in (t - \varepsilon, t + \varepsilon) \times B_r(x)\):
\[
u(s, y) \geq Q(s, y) - \delta/2 + \gamma |y - x|^2.
\]
Choose next \( \delta = \gamma r^2/4 \) so that for \((s, y) \in (t - \varepsilon, t + \varepsilon) \times (B_r(x) \setminus B_{r/2}(x))\),
\[
u(s, y) \geq Q(s, y) - \gamma r^2/8 + \gamma r^2/4 = Q(s, y) + \gamma r^2/8 > Q(s, y).
\]
Now define a function $U$ by

$$U = \begin{cases} \max(u, Q) & \text{in } (t - \varepsilon, t + \varepsilon) \times B_r(x), \\ u & \text{elsewhere}. \end{cases}$$

Let us prove that $U$ is a subsolution of (1). Consider $(s, y) \in (0, +\infty) \times \mathbb{R}^N$ and $(\beta, q) \in \partial^P U(s, y)$.

**First case:** Suppose that $U(s, y) = u(s, y)$. Then $(\beta, q) \in \partial^P u(s, y)$. Since $u$ is a subsolution of (1), we get,

$$\beta + H(s, y, U(s, y), q) - \int_{\mathbb{R}^N \setminus \{0\}} (U(s, y + z) - U(s, y) - q \cdot z) \, d\mu(z)$$

$$\leq \beta + H(s, y, u(s, y), q) - \int_{\mathbb{R}^N \setminus \{0\}} (u(s, y + z) - u(s, y) - q \cdot z) \, d\mu(z) \leq 0.$$

**Second case:** Suppose that $U(s, y) = Q(s, y) > u(s, y)$. Then $(s, y) \in (t - \varepsilon, t + \varepsilon) \times B_r(x)$ and $(\beta, q) \in \partial^P Q(s, y)$; in particular, $\beta = \alpha - \gamma e$ with $|e| \leq 1$, $q = p - 2(\sigma + \gamma)(y - x)$. We claim that if $\varepsilon = r^2$, then

$$\lim \inf_{r \to 0} \int_{\mathbb{R}^N \setminus \{0\}} (U(s, y + z) - U(s, y) - q \cdot z) \, d\mu(z)$$

$$\geq \int_{\mathbb{R}^N \setminus \{0\}} (u_*(t, x + z) - u_*(t, x) - p \cdot z) \, d\mu(z).$$

To see this, write $\int_{\mathbb{R}^N \setminus \{0\}} = \int_{B_r \setminus \{0\}} + \int_{B_r^c}$ and study each term:

$$\int_{B_r \setminus \{0\}} (U(s, y + z) - U(s, y) - q \cdot z) \, d\mu(z)$$

$$= \int_{B_r \setminus \{0\}} \left( \frac{1}{2} D^2 Qz \cdot z \right) d\mu(z) = (\sigma + \gamma) \int_{B_r \setminus \{0\}} |z|^2 d\mu(z) \to 0$$

as $r \to 0$.

$$\int_{B_r^c} (U(s, y + z) - U(s, y) - q \cdot z) \, d\mu(z)$$

$$\geq \int_{B_r^c} (u_*(s, y + z) - Q(s, y) - q \cdot z) \, d\mu(z)$$

$$\geq \int_{B_r^c} (u_*(s, y + z) - u_*(t, x) - \alpha(s - t) - p \cdot (y - x) - \delta - q \cdot z) \, d\mu(z).$$
The integrand of the right-hand side converges to \([u_*(t, x + z) - u_*(t, x) - p \cdot z] \mathbb{1}_{\mathbb{R}^N \setminus \{0\}}(z)\). Hence, it suffices to exhibit a lower bound independent of \(r\) and integrable to conclude by using Fatou’s lemma. On \(B^c_{r_0}\), we choose \(C(1 + |z|)\) for \(C\) large enough. On \(B_{r_0} \setminus B_r\), we have

\[
\begin{align*}
  u_*(s, y + z) - u_*(t, x) - x(s - t) - p \cdot (y - x) - \delta - q \cdot z \\
  \geq -\sigma |z + y - x|^2 - Cr^2 - C|z| \geq - C(r^2 + |z|^2) \geq - C|z|^2
\end{align*}
\]

for \(C\) large enough and we are done.

Suppose first that \(\int_{\mathbb{R}^N \setminus \{0\}}(u_*(t, x + z) - u_*(t, x) - p \cdot z) d\mu(z) = +\infty\). Then for \(r\) small enough, we have:

\[
\beta + H(s, y, U(s, y), q) - \int_{\mathbb{R}^N \setminus \{0\}}(U(s, y + z) - U(s, y) - q \cdot z) d\mu(z) \leq 0.
\]

If now \(\int_{\mathbb{R}^N \setminus \{0\}}(u_*(t, x + z) - u_*(t, x) - p \cdot z) d\mu(z) < +\infty\), then,

\[
\begin{align*}
  \beta + H(s, y, U(s, y), q) - \int_{\mathbb{R}^N \setminus \{0\}}(U(s, y + z) - U(s, y) - q \cdot z) d\mu(z) \\
  \leq - \theta + \gamma - H(t, x, u_*(t, x), p) + H(s, y, U(s, y), q) \\
  + \int_{\mathbb{R}^N \setminus \{0\}}(u_*(t, x + z) - u_*(t, x) - p \cdot z) d\mu(z) \\
  - \int_{\mathbb{R}^N \setminus \{0\}}(U(s, y + z) - U(s, y) - q \cdot z) d\mu(z).
\end{align*}
\]

Choosing \(\gamma = \theta/2\) and \(r\) small enough permits to conclude that \(U\) is a subsolution of (1).

By the comparison principle, since \(U(0, x) = u(0, x)\), we have \(U \leq u^+\). Thus \(U \in \mathcal{S}\). Moreover, if \((t_n, x_n)\) is a sequence such that \(u_*(t, x) = \lim_n u(t_n, x_n)\), we get,

\[
\limsup_{n \to \infty} U(t_n, x_n) \geq \lim_{n \to \infty} Q(t_n, x_n) - u_*(t, x) = \delta > 0.
\]

There then exists \((s, y)\) such that \(U(s, y) > u(s, y)\) which is a contradiction. The proof is now complete. \(\square\)

### 3. Regularizing effect

In this section, if the natural assumptions that ensure the existence and the uniqueness of a continuous (viscosity) solution of (1) are slightly strengthened the solution is
in fact $C^2$ in $x$ and $C^1$ in $t$. We also show that $C^\infty$ regularity is obtained if assumptions are further strengthened (Theorem 5). We use techniques and ideas introduced in (A3′) For any $R > 0$, there exists $C_R > 0$ s.t. $\hat{\partial}_u H, \nabla_p H, \nabla^2_{\partial_p} H, \nabla_p \hat{\partial}_u H$ and $\nabla^2_{\partial_p} H$ are bounded by $C_R$ on $[0, T) \times \mathbb{R}^N \times [-R, R] \times B_R$. [14].

**Theorem 4** ($C^{1,2}$ regularity). Assume that $H$ satisfies (A1′)–(A2)–(A3′)–(A4) and consider an initial condition $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. Then the (unique) viscosity solution of (1) is $C^2$ in the space variable and $C^1$ in the time variable in $]0; +\infty[ \times \mathbb{R}^N$.

**Proof.** We first remark that the viscosity solution $u$ of (1) remains (globally) Lipschitz continuous at any time $t > 0$. This fact is well-known for local equations (see [13,21,18,3]) and the classical proof can be adapted to our situation; this is the reason why we omit details.

**Lemma 2.** For any $t \in [0, T)$, $\|u(t, \cdot)\|_{W^{1,\infty}(\mathbb{R}^N)} \leq M_T$ with $M_T$ that only depends on $\|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$, $C_0$ and $T$.

**Proof (Sketch).** The comparison principle gives immediately: $\|u\|_{\infty} \leq \|u_0\|_{\infty} + C_0 T$ where $C_0$ denotes the constant in (A4). Next, define: $u^\varepsilon(t, x) = \sup_{y \in \mathbb{R}^N} \{u(t, y) - e^{Kt} |x-y|^2 \varepsilon\}$ with $K = 4C\|u\|_{\infty}$ from (A2) and verify that it is a viscosity subsolution of:

$$\hat{\partial}_t u^\varepsilon + H(t, x, u^\varepsilon, \nabla u^\varepsilon) + g[u^\varepsilon](t, x) \leq \frac{K\varepsilon}{16}.$$ 

The non-local term makes no trouble since

$$u^\varepsilon(t, x + z) - u^\varepsilon(t, x) - p \cdot z \geq u(t, x_\varepsilon + z) - u(t, x_\varepsilon) - p \cdot z,$$

where $x_\varepsilon$ denotes a point such that $u^\varepsilon(t, x) = u(t, x_\varepsilon) = e^{Kt} |x-x_\varepsilon|^2 \varepsilon$. The comparison principle yields:

$$u^\varepsilon(t, x) \leq u(t, x) + \frac{K\varepsilon}{16} t + \sup_{x \in \mathbb{R}^N} \{u_0(x) - u_0(x)\}.$$ 

Using the definition of $u^\varepsilon$ and the fact that $u_0$ is Lipchitz continuous, we get

$$u(t, y) \leq u(t, x) + (Kt/16 + \|\nabla u_0\|_{\infty}^2/2)\varepsilon + e^{Kt} \frac{|y-x|^2}{2\varepsilon}.$$ 

Optimizing w.r.t. $\varepsilon$, we finally obtain

$$u(t, y) \leq u(t, x) + e^{Kt/2}(K/8 + \|\nabla u_0\|_{\infty}^2)^{1/2} |y-x|. \quad \square$$
We next construct a solution using Duhamel’s integral representation of (1). More precisely, we look for functions satisfying:

\[
v(t, x) = K(t, \cdot) * v_0(\cdot)(x) - \int_0^t K(t - s, \cdot) * H(s, x, v(s, \cdot), \nabla v(s, \cdot))(x) \, ds.
\]  

(19)

Lemma 3. Let \( v_0 \in W^{1, \infty}(\mathbb{R}^N) \). There exists \( T_1 > 0 \), that depends only on \( \lambda, N \) and \( \|v_0\|_{W^{1, \infty}(\mathbb{R}^N)} \), and \( v \in C_b(]0, T_1[ \times \mathbb{R}^N) \) such that \( \nabla v \in C_b(]0, T_1[ \times \mathbb{R}^N) \) and (19) holds true.

Remark 5. If \( C_R \) in (A2) and (A3) does not depend on \( R \) (\( C_R = \overline{C} \)), then \( T_1 \) in Lemma 3 only depends on \( \lambda, K_1 \) and \( \overline{C} \). Hence we can construct classical solutions of (1) in \([0, +\infty) \times \mathbb{R}^N\) without using viscosity solutions (time regularity is studied below).

Proof of Lemma 3. We use a contracting fixed point theorem. Consider the space

\[
E_1 = \{ v \in C_b(]0, T_1[ \times \mathbb{R}^N), \nabla v \in C_b(]0, T_1[ \times \mathbb{R}^N) \}
\]

endowed with its natural norm \( \|v\|_{E_1} = \|v\|_{C_b(]0, T_1[ \times \mathbb{R}^N)} + \|\nabla v\|_{C_b(]0, T_1[ \times \mathbb{R}^N)} \). We define

\[
\psi_1(v)(t, x) = K(t, \cdot) * u_0(\cdot)(x) - \int_0^t K(t - s, \cdot) * H(s, x, v(s, \cdot), \nabla v(s, \cdot))(x) \, ds.
\]  

(20)

Let us first show that \( \psi_1 \) maps \( E_1 \) into \( E_1 \). Consider \( v \in E_1 \) such that \( \|v\|_{E_1} \leq R_1 \). By Proposition 1, \( K(t, \cdot) * u_0(\cdot) \) is \( C^1 \) in space and \( K(t, \cdot) * u_0(\cdot) \) and its gradient are continuous in \((t, x)\). Let \( \Phi(v)(t, x) = \int_0^t (K(t - s, \cdot) * H(x, v(s, \cdot), \nabla v(s, \cdot))(x) \, ds \). Then defining

\[
\mathcal{H}(s, x) = H(s, x, v(s, x), \nabla v(s, x))1_{]0, T_1[}(s),
\]

\[
\mathcal{K}(s, x) = K(s, x)1_{]0, T_1[}(s),
\]

we have: \( \Phi(v) = \mathcal{H} * \mathcal{K} \) where the convolution is computed w.r.t. \((t, x)\). The function \( \mathcal{K} \) is continuous in \((t, x)\) in \( ]0, T_1[ \times \mathbb{R}^N \) and, using (A3)–(A4),

\[
|\mathcal{H}(s, x) \mathcal{K}(t - s, x - y)| \leq (C_0 + C_R, R_1) \mathcal{K}(t - s, x - y)
\]

and the right-hand side is integrable since \( \int_{\mathbb{R}^N} \mathcal{K}(t, x) \, dx = T_1 \) (see estimate (10)). The theorem of continuity under the integral sign ensures that \( \Phi(v) \) is continuous in
We also have the following upper bound:

\[ |\psi_1(v)(t, x)| \leq \|u_0\|_\infty + (C_0 + C_{R_1} R_1) T_1. \]

Since \( K(t, x) \) is continuously differentiable and

\[ |H(s, x, v(s, x), \nabla v(s, x)) \nabla K(t - s, x - y)| \leq (C_0 + C_{R_1} R_1) |\nabla K(t - s, x - y)| \]

and \( |\nabla K(t - s, x - y)| \) is integrable with \( \|\nabla K(t - s, x - y)\|_{L^1(0, T; \mathbb{R}^N)} = \frac{\lambda K_1}{\lambda - 1} T_1^{(\lambda - 1)/\lambda} \) (see estimate (10)), we see that \( \psi_1(v) \) is continuously differentiable in \( x \) and

\[
\nabla \psi_1(v)(t, x) = K(t) * \nabla v_0(x) - \int_0^t ((\nabla K)(t - s) * H(s, x, v(s, \cdot), \nabla v(s, \cdot))(x) ds,
\]

\[ |\nabla \psi_1(v)(t, x)| \leq \|\nabla u_0\|_\infty + (C_0 + C_{R_1} R_1) K_1 \frac{\lambda}{\lambda - 1} T_1^{(\lambda - 1)/\lambda}. \]

We conclude that \( \psi_1(v) \in E_1 \) and

\[ \|\psi_1(v)\|_{E_1} \leq R_0 + (C_0 + C_{R_1} R_1) \left( T_1 + K_1 \frac{\lambda}{\lambda - 1} T_1^{(\lambda - 1)/\lambda} \right) \]

if \( \|v_0\|_{W^{1,\infty}(\mathbb{R}^N)} \leq R_0 \). Choose \( R_1 = 2R_0 \) and \( T_1 \) such that

\[ (C_0 + C_{R_1} R_1) \left( T_1 + K_1 \frac{\lambda}{\lambda - 1} T_1^{(\lambda - 1)/\lambda} \right) \leq R_0. \]

This implies that \( \psi_1 \) maps \( B_{R_1} \), the closed ball of \( E_1 \) of radius \( R_1 \), into itself. Moreover, this condition ensures that \( \psi_1 \) is a contraction:

\[ \|\psi_1(v) - \psi_1(w)\|_{E_1} \leq C_{R_1} \left( T_1 + K_1 \frac{\lambda}{\lambda - 1} T_1^{(\lambda - 1)/\lambda} \right) \|u - v\|_{E_1} \]

\[ \leq \frac{R_0}{R_1} \|u - v\|_{E_1} = \frac{1}{2} \|u - v\|_{E_1}. \]

By the Banach fixed point theorem, there then exists a unique fixed point \( v \in B_{R_1} \).

Let us turn to second order regularity in \( x \).

**Lemma 4.** The function \( v \) constructed in Lemma 3 is continuously twice differentiable in \( x \) in \( [0, T_2] \times \mathbb{R}^N \), with \( T_2 \leq T_1 \) that only depends on \( \lambda, N \) and \( \|v_0\|_{W^{1,\infty}(\mathbb{R}^N)} \). Moreover \( t^{1/\lambda} D^2 v \) is bounded in \( [0, T_2] \times \mathbb{R}^N \).
Proof. Remark that $\overline{w} = \nabla v$ verifies:

$$\overline{w} = K(t, \cdot) * w_0(\cdot) - \int_0^t \nabla K(t - s, \cdot) * H(s, \cdot, v(s, \cdot), \overline{w}(s, \cdot))(x) \, ds$$

and

$$\|\overline{w}\|_{C_b([0,T] \times \mathbb{R}^N)} \leq R_1$$

(21)

with $w_0 = \nabla v_0$. Consider the space

$$E_2 = \{ w \in C_b([0,T_2] \times \mathbb{R}^N, \mathbb{R}^N), t^{1/2} Dw \in C_b([0,T_2] \times \mathbb{R}^N) \}$$

defined by its natural norm $\|w\|_{E_2} = \|w\|_{C_b([0,T_2] \times \mathbb{R}^N)} + \|t^{1/2} Dw\|_{C_b([0,T_2] \times \mathbb{R}^N)}$.

We consider the map $\psi_2$ defined by

$$\psi_2(w)(t,x) = K(t, \cdot) * w_0(\cdot)(x) - \int_0^t \nabla K(t - s, \cdot) * H(s, \cdot, v(s, \cdot), w(s, \cdot))(x) \, ds$$

with $w_0 = \nabla v_0$. Choose $w$ such that $\|w\|_{E_2} \leq R_2$ with $R_2 \geq R_1$. Remark first that

$$|H(s, x, v(s, x), w(s, x))| \leq C_0 + 2 CR_2 R_2.$$

Moreover, $x \mapsto H(s, x, v(s, x), w(s, x))$ is differentiable on $]0, T_2[ \times \mathbb{R}^N$ and:

$$\nabla (H(s, x, v(s, x), w(s, x))) = \nabla_x H(s, x, v(s, x), w(s, x))$$

$$+ \partial_u H(s, x, v(s, x), w(s, x)) \nabla v(s, x)$$

$$+ Dw(s, x) \nabla_p H(s, x, v(s, x), w(s, x))$$

$$|\nabla (H(s, x, v(s, x), w(s, x)))| \leq CR_2 (1 + 2 R_2) + CR_2 R_2 s^{-1/2}$$

if $\|w\|_{E_2} \leq R_2$ (we used $R_2 \geq R_1 \geq \|\nabla v\|_\infty$). Using the theorem of continuity and differentiability under the integral sign, we conclude that $\psi_2$ maps $E_2$ into $E_2$ and

$$D\psi_2(w)(t,x) = w_0(\cdot) * \nabla K(t, \cdot)(x)$$

$$- \int_0^t \nabla K(t - s, \cdot) * \nabla (H(s, \cdot, v(s, \cdot), w(s, \cdot)))(x) \, ds,$$

where $* \otimes$ is defined as follows: if $F, G : \mathbb{R}^N \to \mathbb{R}^N$, $F \otimes G(x) = \int F(y) \otimes G(x - y) \, dy$. Recall that $\otimes$ denote the tensor product.

We also have the following estimates:

$$|\psi_2(w)(t,x)| \leq R_0 + (C_0 + 2 CR_2 R_2) K_1 \frac{\lambda}{\lambda - 1} T_2^{(\lambda - 1)/\lambda}$$

$$|t^{1/2} D\psi_2(w)(t,x)| \leq K_1 R_0 + C R_2 \left( (1 + 2 R_2) \frac{\lambda}{\lambda - 1} K_1 T_2 + R_2 \gamma K_1 T_2^{(\lambda - 1)/\lambda} \right),$$
We now choose max(2(1\(K_1\))R_0, 1) = 2(1 + \(K_1\))R_0 \geq R_1 and \(T_2 \leq T_1\) such that
\[
(C_0 + 2C_2 R_2)K_1 \frac{\hat{\lambda}}{\hat{\lambda} - 1} T_2^{(\hat{\lambda} - 1)/\hat{\lambda}} + C R_2 \left( (1 + 2R_2) \frac{\hat{\lambda}}{\hat{\lambda} - 1} K_1 T_2 + R_2 \hat{\gamma}_\lambda K_1 T_2^{(\hat{\lambda} - 1)/\hat{\lambda}} \right)
\geq \min \left( \frac{1}{2}, (1 + K_1)R_0 \right).
\] (22)

This condition thus ensures that \(\psi_2\) maps \(B_{R_2}\), the closed ball of \(E_2\) of radius \(R_2\), into itself and that it is a contraction for the norm \(E_2\). Hence, there is a unique fixed point \(w\). Moreover if \(w_1, w_2\) lie in \(D_{R_2}\), the closed ball of \(C_b([0, T[\times \mathbb{R}^N]) of radius \(R_2 \geq R_1\), (22) implies that
\[
\|\psi_2(w_1) - \psi_2(w_2)\|_{C_b([0,T_2[\times \mathbb{R}^N)} \leq \frac{1}{4} \|w_1 - w_2\|_{C_b([0,T_2[\times \mathbb{R}^N)}
\]
and \(\psi_2\) is also a contraction in \(D_{R_2} \subset C_b([0, T[\times \mathbb{R}^N]). Using (21), we conclude that the fixed point we just constructed coincide with \(\bar{w}\). The proof is now complete. □

We next prove that the function \(v\) constructed in Lemma 3 is \(C^1\) in the time variable \(t\) and that it satisfies (1). This lemma is adapted from [14, p. 512].

**Lemma 5.** Suppose that \(w \in C_b([0, T_2[\times \mathbb{R}^N]) is \(C^2\) in \(x\) such that \(\nabla w, D^2 w \in C_b([0, T_2[\times \mathbb{R}^N))\). Then \(\Phi(w)(t, x) = \int_0^t K(t - s, \cdot) * w(s, \cdot)(x) ds\) is \(C^1\) w.r.t. \(t \in [0, T_2[ and \(\partial_t \Phi(w)(t, x) = w(t, x) - g[\Phi(w)](t, x)\).

**Proof.** It is enough to prove the result for \(t \in ]\delta_0, T_2 - \delta_0[\) for any \(\delta_0 \in [0, T_2/2[\). Fix such a \(\delta_0\), consider \(\delta \in [0, \delta_0[\) and define \(\Phi_\delta(w)(t, x) = \int_0^t K(t - s, \cdot) * w(s, \cdot)(x) ds\) in \(]\delta_0, T_2 - \delta_0[\times \mathbb{R}^N\). It is easy to see that \(\Phi_\delta(w)\) converges uniformly to \(\Phi(w)\) in \(]\delta_0, T_2 - \delta_0[\times \mathbb{R}^N\). We next prove that \(\Phi_\delta(w)\) is continuously differentiable in \(]\delta_0, T_2 - \delta_0[\times \mathbb{R}^N\) and we compute its time derivative. To do so, consider \(\phi : [(t, s, x) : t \in ]\delta, T_2 - \delta[\times \mathbb{R}^N : s \leq t - \delta_2[ \rightarrow \mathbb{R}\) defined by \(\phi(t, s, x) = K(t - s, \cdot) * w(s, \cdot)(x)\). It is enough to prove that \(\phi\) and \(\partial_t \phi\) are bounded and continuous to get that \(t \mapsto \int_0^t \phi(t, s, x) ds\) is continuously differentiable and its time derivative equals
\[
\phi(t, t - \delta, x) + \int_0^{t - \delta} \partial_t \phi(t, s, x) ds.
\]
The function $\phi$ satisfies $\|\phi\|_\infty \leq \|w\|_\infty$ and its continuity is a consequence of the theorem of continuity under the integral sign. Using Proposition 1, we can assert that $\phi$ is differentiable in time and $\partial_t \phi(t,s,x) = -g[\phi(t,s,\cdot)](x)$. The space derivatives $\nabla \phi, D^2 \phi$ are bounded since $\nabla w, D^2 w$ are bounded. It follows that $g[\phi]$ is bounded. We conclude that $\Phi_\delta(w)$ is differentiable in time and, using Fubini’s theorem:

$$\partial_t \Phi_\delta(w)(t,x) = \partial_t \Phi_\delta(w)(t,x) = K(\delta, \cdot) \ast w(t - \delta, \cdot)(x) - g[\Phi_\delta(w)](t,x).$$

It is now easy to see that $\partial_t \Phi_\delta(w)$ converges to the continuous function $w(t,x) - g[\Phi(w)](t,x)$ as $\delta \to 0$. Since $\Phi_\delta(w)$ converges uniformly to $\Phi(w)$ on $]\delta_0, T_2 - \delta_0[ \times \mathbb{R}^N$ and remains bounded, it also converges in the distribution sense. We conclude that $\partial_t \Phi(w) = w(t,x) - g[\Phi(w)](t,x)$ and the proof is complete. □

Apply Lemma 5 to the continuous and bounded function $w = H(x, v(t,x), \nabla v(t,x))$:

$$\partial_t v(t,x) = -g[K(t, \cdot) \ast v_0(\cdot)](x) - H(x, v(t,x), \nabla v(t,x))$$

$$+ g\left[\int_0^t K(t - s, \cdot) \ast H(x, v(s,\cdot), \nabla v(s,\cdot))\right]$$

$$= -H(t, x, v(t,x), \nabla v(t,x)) - g[v(t,\cdot)](x).$$

Hence $v$ is the viscosity solution of (1) in $]0, T_2[ \times \mathbb{R}^N$ and its Fréchet derivatives $\partial_t v, \nabla v, D^2 v$ exist.

Consider now the viscosity solution $u$ of (1) in $(0, +\infty) \times \mathbb{R}^N$ and fix $T > 0$. Lemma 2 implies that for any $t \in [0, T]$, $\|u(t,\cdot)\|_{W^{1,\infty}(\mathbb{R}^N)} \leq M_T$. For any $T_0 \in [0, T]$, $v(t,x) = u(T_0 + t, x)$ is a viscosity solution of (1) in $[0, +\infty[ \times \mathbb{R}^N$ with initial data $v_0(t,x) = u(T_0, x) \in W^{1,\infty}(\mathbb{R}^N)$. By Lemmas 3–5, there exists $T_2 > 0$ that depends only on $\lambda$, $N$ and $M_T$ such that $v$ is $C^2$ in $x$ and $C^1$ in $t$ in $]0, T_2[ \times \mathbb{R}^N$; this implies that $u$ has the same regularity in $]T_0, T_0 + T_2[ \times \mathbb{R}^N$. Since $T_0$ and $T$ are arbitrary, the proof is complete. □

We conclude this section with the following regularity result which asserts the existence of a solution of (1) that is infinitely differentiable in time and space.

**Theorem 5 (C^\infty regularity).** Let $H \in C^\infty(\mathbb{R}^N)$. The unique viscosity solution of

$$\partial_t u + H(\nabla u) + g[u] = 0$$

with initial data $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ is $C^\infty$ in both time and space variables in $]0; +\infty[ \times \mathbb{R}^N$.

**Remarks 6.**

1. If $N = 1$, the result is an immediate consequence of the integral representation of $\partial_x u$, (21), and of the main result of [14].

2. An analogous result with an Hamiltonian $H$ depending on $t, x$ and $u$ can be stated and proved under suitable assumptions. The ideas are exactly the same as the ones
presented here. We choose to restrict ourselves to $H(\nabla u)$ so that technical difficulties do not hide the key points of the proof.

**Proof.** We first prove that $u$ is $C^\infty$ with respect to $x$.

*Space regularity.* We already proved that the (unique) viscosity solution $u$ of (1) is $C^2$ in $x$ and $C^1$ in $t$ and that $t^{1/2} D^2 u$ is bounded in $[0, T_2] \times \mathbb{R}^N$. Then construct an “integral” solution on $[T_2/2, 3T_2/2]$ with Lemmas 3 and 4. It coincides with $u$ in $[T_2/2, 3T_2/2] \times \mathbb{R}^N$ and $\nabla u, D^2 u$ are bounded in $[T_2, 3T_2/2]$ by a constant $C$ only depending on $\lambda, N$ and $\|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$. Iterating this process, we conclude that $D^2 u$ is bounded in $[t_0, +\infty] \times \mathbb{R}^N$ by a constant only depending on $\lambda, N$, $\|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$ and $t_0$.

We now prove by induction that $u$ is $C^k$ in the space variable in $(0, +\infty) \times \mathbb{R}^N$ and that $D^k u$ is bounded on $[t_0, +\infty] \times \mathbb{R}^N$ by a constant only depending on $\lambda, N$, $\|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$ and $t_0$. We proved this assertion at rank $k = 2$. Suppose it is true at any rank $i$ for $2 \leq i \leq k + 1$ and let us prove it at rank $k + 2$. Let us fix $t_0 > 0$. Then $\bar{W}(t, x) = \nabla u(t_0 + t, x)$ satisfies for any $t > 0$:

$$
\bar{W}(t, x) = K(t, \cdot) \ast W_0(x) - \int_0^t \nabla K(t - s, \cdot) \ast H(\bar{W}(s, \cdot))(x) \, ds,
$$

where $W_0(\cdot) = \bar{W}(0, \cdot) = \nabla u(t_0, \cdot))$. By assumption, we know that $\bar{W}$ is $C^k$ in space and its $k$ first derivatives are bounded in $(0, +\infty) \times \mathbb{R}^N$.

Remark that if $v$ is sufficiently regular and $j \in \{1, \ldots, k + 1\}$:

$$
D^j(\nabla K(t - s) \ast H(v(s))) = \nabla K(t - s) \ast (D^j v \circ \nabla H(v))(s) + \nabla K(t - s) \ast G_j(v, Dv, \ldots, D^{j-1}v)(s),
$$

where $G_j$ is $C^\infty$ and $\circ$ denotes the contraction product of tensors. Consider the space

$$
E_{k+1} = \{v \in C_b([0, T_{k+1}] \times \mathbb{R}^N), \nabla v, \ldots, D^k v \in C_b([0, T_{k+1}] \times \mathbb{R}^N),
$$

$$
t^{1/2} D^{k+1} v \in C_b([0, T_{k+1}] \times \mathbb{R}^N)\}
$$

endowed with its natural norm $\|v\|_{E_{k+1}} = \|v\|_0 + \|v\|_k + \|t^{1/2} D^{k+1} v\|_0$ where

$$
\|v\|_0 = \|v\|_{C_b([0, T_{k+1}] \times \mathbb{R}^N)} \quad \text{and} \quad \|v\|_k = \|v\|_0 + \sum_{i=1}^k \|D^i v\|_0.
$$

We consider $\psi_2$ defined in the proof of Lemma 4:

$$
\psi_2(W)(t, x) = K(t, \cdot) \ast W_0(\cdot)(x) - \int_0^t \nabla K(t - s, \cdot) \ast H(W(s, \cdot))(x) \, ds
$$
\( D^i \psi_2(W)(t,x) = K(t,\cdot) * D^i W_0(\cdot)(x) \)
- \( \int_0^t \nabla K(t-s,\cdot) * \otimes (D^i W(s,\cdot) \odot \nabla H(W(s,\cdot))) (x) \, ds \)
- \( \int_0^t \nabla K(t-s,\cdot) * \otimes G_t(W(s,\cdot), DW(s,\cdot), \ldots, D^{i-1} W(s,\cdot))(x) \, ds \)

\( D^{k+1} \psi_2(W)(t,x) = D^k W_0 * \otimes \nabla K(t)(x) \)
- \( \int_0^t \nabla K(t-s,\cdot) * \otimes (D^{k+1} W(s,\cdot) \odot \nabla H(W(s,\cdot))) (x) \, ds \)
- \( \int_0^t \nabla K(t-s,\cdot) * \otimes G_{k+1}(W(s,\cdot), DW(s,\cdot), \ldots, D^k W(s,\cdot))(x) \, ds, \)

where \( i \in \{1,\ldots,k\} \). Now estimate each term:

\[
|\psi_2(W)(t,x)| \leq \|W\|_0 + \frac{\lambda K_1}{\lambda - 1} T_{k+1}^{(\lambda - 1)/\lambda} (C_0 + C_{\|W\|_0} \|W\|_0) 
\]

\[
|D^i \psi_2(W)(t,x)| \leq \|D^i W\|_0 + \frac{\lambda K_1}{\lambda - 1} T_{k+1}^{(\lambda - 1)/\lambda} (C_{\|W\|_0} \|D^i W\|_0 + D_{\|W\|_k} \|W\|_k) 
\]

\[
|t^{1/\lambda} D^{k+1} \psi_2(W)(t,x)| \leq K_1 \|D^k W\|_0 + \frac{\lambda K_1}{\lambda - 1} T_{k+1} D_{\|W\|_k} \|W\|_k 
+ K_1^{\gamma_\lambda} T_{k+1}^{(\lambda - 1)/\lambda} C_{\|W\|_0} \|t^{1/\lambda} D^k W\|_0, 
\]

where \( D_{\|W\|_k} \) only depends on \( \|W\|_k \). If \( W \) is such that \( \|W\|_{E_{k+1}} \leq R_{k+1} \), then:

\[
\|\psi_2(W)\|_{E_{k+1}} \leq (1 + K_1) \|W\|_k + \frac{\lambda K_1}{\lambda - 1} T_{k+1}^{(\lambda - 1)/\lambda} (C_0 + C_{R_{k+1}} R_{k+1}) 
+ \frac{\lambda K_1}{\lambda - 1} T_{k+1} D_{R_{k+1}} R_{k+1} + K_1^{\gamma_\lambda} T_{k+1}^{(\lambda - 1)/\lambda} C_{R_{k+1}} R_{k+1}. 
\]

If now one chooses \( R_{k+1} = 2(1 + K_1) \|W\|_k \) and \( T_{k+1} \) such that:

\[
\frac{\lambda K_1}{\lambda - 1} T_{k+1}^{(\lambda - 1)/\lambda} (C_0 + C_{R_{k+1}} R_{k+1}) + \frac{\lambda K_1}{\lambda - 1} T_{k+1} D_{R_{k+1}} R_{k+1} 
+ K_1^{\gamma_\lambda} T_{k+1}^{(\lambda - 1)/\lambda} C_{R_{k+1}} R_{k+1} \leq (1 + K_1) \|W\|_k, 
\]

we ensure that \( \psi_2 \) maps \( B_{R_{k+1}} \) into itself (in the space \( E_{k+1} \)). There then exists a fixed point \( W_{k+1} \in F_{k+1} \). Moreover one can check that it is a contraction map in the subspace \( F_k \subset E_{k+1} \) defined by

\[
F_k = \{ v \in C_b([0, T_{k+1} \times \mathbb{R}^N], \nabla v, \ldots, D^k v) \subset C_b([0, T_{k+1} \times \mathbb{R}^N]) \} 
\]
endowed with its natural norm \( \|v\|_k \); since \( \|W_{k+1}\|_{F_k} \leq \|W_{k+1}\|_{E_{k+1}} = R_{k+1} \) and \( \|W\|_{F_k} \leq R_{k+1} \), we conclude that \( W_{k+1} = W \). Finally, since \( T_{k+1} \) only depends on \( \lambda, N, t_0 \) and \( \|u_0\|_{W^{1,\infty}(\mathbb{R}^N)} \), by arguing as at the beginning of this proof, we conclude that \( u \) is \( C^{k+2} \) on \([t_0, +\infty[ \times \mathbb{R}^N \) and that \( D^{k+2}u \) is bounded in \([2t_0, +\infty[ \times \mathbb{R}^N \) by a constant that only depends on \( \lambda, N, t_0 \) and \( \|u_0\|_{W^{1,\infty}(\mathbb{R}^N)} \). Since \( t_0 \) is arbitrary, this achieves the proof of space regularity. We now turn to time regularity.

**Time regularity.** We first prove that \( \nabla u \) is \( C^1 \) in time. In order to do so, we represent \( \nabla u \) in the following way:

\[
\nabla u(t, x) = K(t, \cdot) \ast \nabla u_0(\cdot)(x) - \int_0^t K(t - s, \cdot) \ast D^2 u(s, \cdot) \nabla H(\nabla u(s, \cdot))(x) \, ds
\]

and we apply Lemma 5 to prove that the second term of the right-hand side is \( C^1 \) in \( t \) (we already know that the first one is \( C^1 \) in time). Next, since \( \partial_t u(t, x) = -g[u](t, x) - H(\nabla u(t, x)) \), we see that \( \partial_t u \) is bounded in \([t_0, +\infty[ \times \mathbb{R}^N \); the theorem of differentiability under the integral sign ensures that \( \partial_t u \) has second-order spatial derivatives that they are bounded in \([t_0, +\infty[ \times \mathbb{R}^N \). Hence, \( \partial_t u \) is differentiable w.r.t. \( t \) and

\[

\partial_t^2 u(t, x) = -g[\partial_t u](t, x) - \nabla H(\nabla u(t, x)) \cdot \partial_t(\nabla u)(t, x).
\]

This process can be iterated to conclude. \( \square \)

4. **An error estimate**

In this section, we compare the solution of the Hamilton–Jacobi equation with a vanishing Lévy operator (4) with the solution of the pure Hamilton–Jacobi equation (5) (we impose the same initial condition (2) to both equations).

**Theorem 6.** Assume (A0)–(A4) and consider \( u_0 \in W^{1,\infty}(\mathbb{R}^N) \). There then exists a constant \( C > 0 \) only depending on \( H \) and \( u_0 \) and \( T \) such that, if \( u^\varepsilon \) and \( u \), respectively, denote the solutions of (4) and (5) such that \( u^\varepsilon(0, \cdot) = u(0, \cdot) = u_0(\cdot) \), then for all \( t \in [0, T] \):

\[
\|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq C\varepsilon^{1/\lambda} \sqrt{t}.
\]

**Remarks 7.** 1. Using the fact that \( u^\varepsilon \) is \( C^2 \) in \( x \) and \( C^1 \) in \( t \) and the bound on \( t^{1/\lambda} D^2 u^\varepsilon \), we get an error estimate of the form \( C\varepsilon^{1/\lambda} t^{1-1/\lambda} \), which is less precise than the one of Theorem 6.

2. About the optimality of the estimate, the power in \( \varepsilon \) cannot be improved: choosing \( H = 0, u_0(z) = \min(|z|, 1) \) and \( x = 0 \), we get \( u^\varepsilon(t, 0) - u(t, 0) = C\varepsilon^{1/\lambda}(t^{1/\lambda} + o_\varepsilon(1)) \). We do not know if one can do better about the power in \( t \).
Proof. Let us define

\[
M = \sup_{t \in [0, T), x, y \in \mathbb{R}^N} \left\{ u(t, x) - u^\varepsilon(t, y) - \frac{|x - y|^2}{2\varepsilon} - \frac{\beta}{2} |x|^2 - \eta t - \frac{\gamma}{T - t} \right\}.
\]

Since \( u \) and \( u^\varepsilon \) are bounded, this supremum is attained. We now prove that if one chooses \( \eta, \gamma \) and \( \beta \) properly, the supremum cannot be achieved at \( t = 0 \).

Consider:

\[
M_v = \sup_{t, s \in [0, T), x, y \in \mathbb{R}^N} \left\{ u(t, x) - u^\varepsilon(s, y) - \frac{|x - y|^2}{2\varepsilon} - \frac{(s - t)^2}{2\varepsilon} - \frac{\beta}{2} |x|^2 - \eta t - \frac{\gamma}{T - t} \right\}.
\]

It is classical to prove that \( M_v \) tends to \( M \) as \( v \to 0 \). Let \((t_v, s_v, x_v, y_v)\) denote a point where the supremum is attained. We have:

\[
q_v = \frac{x_v - y_v}{\varepsilon} \quad \text{and} \quad \left( \eta + \frac{\gamma}{(T - t_v)^2} + \frac{t_v - s_v}{v}, q_v + \beta x_v \right) \in \partial^p u(t_v, x_v)
\]

\[\left( \frac{t_v - s_v}{v}, q_v \right) \in \partial^p u^\varepsilon(s_v, y_v).\]

Since \( u^\varepsilon \) is regular, its subgradient is the set \((\partial_t u^\varepsilon, \nabla u^\varepsilon)\); using Remark 2.2, we can take \( r = 0 \) in the viscosity formulation of (1). Since \( u \) is a viscosity solution of (5) and \( u^\varepsilon \) is viscosity (classical) solution of (4), we get:

\[
\eta + \frac{\gamma}{(T - t_v)^2} + \frac{t_v - s_v}{v} + H(t_v, x_v, u(t_v, x_v), q_v + \beta x_v) \leq 0
\]

\[
\frac{t_v - s_v}{v} + H(s_v, y_v, u^\varepsilon(s_v, y_v), q_v) - \varepsilon \int_{\mathbb{R}^N \setminus \{0\}} \left[ u^\varepsilon(s_v, y_v + z) - u^\varepsilon(s_v, y_v) - q_v \cdot z \right] d\mu(z)
\]

\[
\geq 0.
\]

Subtracting these two inequalities yields:

\[
\eta + \frac{\gamma}{(T - t_v)^2} + H(t_v, x_v, u(t_v, x_v), q_v + \beta x_v) - H(s_v, y_v, u^\varepsilon(s_v, y_v), q_v)
\]

\[+\varepsilon \int_{\mathbb{R}^N \setminus \{0\}} \left[ u^\varepsilon(s_v, y_v + z) - u^\varepsilon(s_v, y_v) - q_v \cdot z \right] d\mu(z) \leq 0.
\]

Now let \( v \to 0 \). We can ensure that \((t_v, s_v, x_v, y_v) \to (\bar{t}, \bar{x}, \bar{y})\) such that \( M \) is achieved at \((\bar{t}, \bar{x}, \bar{y})\). We can pass to the limit in the integral thanks to Fatou’s lemma. We obtain:

\[
\eta + \frac{\gamma}{(T - \bar{t})^2} + H(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), \bar{q} + \beta \bar{x}) - H(\bar{t}, \bar{y}, u^\varepsilon(\bar{t}, \bar{y}), \bar{q})
\]

\[+\varepsilon \int_{\mathbb{R}^N \setminus \{0\}} \left[ u^\varepsilon(\bar{t}, \bar{y} + z) - u^\varepsilon(\bar{t}, \bar{y}) - \bar{q} \cdot z \right] d\mu(z) \leq 0.
\]
Notice that \( u(\bar{t}, \bar{x}) - u^\varepsilon(\bar{x}, \bar{y}) - \frac{\gamma}{T - \bar{t}} \geq - \frac{\gamma}{T - \bar{t}} \) which implies that \( u(\bar{t}, \bar{x}) \geq u^\varepsilon(\bar{x}, \bar{y}) \). Since \( u \) is Lipschitz continuous, we now that \( |\nabla u|_\infty \leq C \) and \( |\bar{x} - \bar{y}| \leq C \bar{x} \). We easily get \( \beta|\bar{x}|^2 \leq C \) and thus \( \beta \bar{x} \leq C \sqrt{\beta} \). Using (A1'), (A2) and (A3) we therefore get for \( \varepsilon \leq 1 \):

\[
\eta + \frac{\gamma}{(T - \bar{t})^2} - C \varepsilon - C \sqrt{\beta}
\]

\[+ \varepsilon \int_{\mathbb{R}^N \setminus \{0\}} [u^\varepsilon(\bar{t}, \bar{y} + z) - u^\varepsilon(\bar{t}, \bar{y}) - \bar{q} \cdot z] d\mu(z) \leq 0. \tag{23}
\]

We now make a change of variables \( r = \varepsilon^{-1/\lambda} z \) in the remaining integral:

\[
\varepsilon \int_{\mathbb{R}^N \setminus \{0\}} [u^\varepsilon(\bar{t}, \bar{y} + z) - u^\varepsilon(\bar{t}, \bar{y}) - \bar{q} \cdot z] d\mu(z)
\]

\[= \mu_0 \varepsilon \int_{\mathbb{R}^N \setminus \{0\}} [u^\varepsilon(\bar{t}, \bar{y} + \varepsilon^{1/\lambda} r) - u^\varepsilon(\bar{t}, \bar{y}) - \varepsilon^{1/\lambda} \bar{q} \cdot r] |\varepsilon^{1/\lambda} r|^{-N-\lambda} \varepsilon^{N/\lambda} d\mu(r)
\]

\[= \int_{\mathbb{R}^N \setminus \{0\}} [u^\varepsilon(\bar{t}, \bar{y} + \varepsilon^{1/\lambda} r) - u^\varepsilon(\bar{t}, \bar{y}) - \varepsilon^{1/\lambda} \bar{q} \cdot r] d\mu(r) = \int_{B_{\varepsilon} \setminus \{0\}} \ldots + \int_{B_{\varepsilon}^c} \ldots,
\]

where \( B \) denotes the unit ball. Using the fact that \( u^\varepsilon \) is Lipschitz and its Lipschitz constant is bounded independently of \( \varepsilon \) can be proven as we did when \( \varepsilon = 1 \). Rewriting (23) yields,

\[
\eta + \frac{\gamma}{(T - \bar{t})^2} - C \varepsilon - C \sqrt{\beta} - C \varepsilon^{1/\lambda} - \frac{C \varepsilon^{2/\lambda}}{2\varepsilon} \leq 0.
\]

Now choosing \( \eta = C(\varepsilon^{1/\lambda} + \varepsilon^{2/\lambda}/\varepsilon) \) and \( \gamma = CT^2 \sqrt{\beta} \) yields,

\[
\frac{\gamma}{(T - \bar{t})^2} - \frac{\gamma}{T^2} \leq 0
\]

which contradicts the fact that \( \bar{t} > 0 \).

We conclude that

\[
u(t, x) - u^\varepsilon(t, y) - \frac{|x - y|^2}{2\varepsilon} - \frac{\beta}{2} |x|^2 - \eta t - \frac{CT^2 \sqrt{\beta}}{T - t} \leq \sup_{x, y \in \mathbb{R}^N} |u_0(x) - u_0(y) - \frac{|x - y|^2}{2\varepsilon}| \leq \alpha \|\nabla u_0\|_2^2/2
\]
and letting $\beta \to 0$,
\[ u(t, x) \leq u^\varepsilon(t, y) + \frac{|x - y|^2}{2\varepsilon} + C(\varepsilon^{1/\lambda} + \varepsilon^{2/\lambda}/\varepsilon)t + \varepsilon \frac{\|\nabla u_0\|^2}{2}. \]

Choosing $x = y$ and $\varepsilon = \varepsilon^{1/\lambda}/\sqrt{t}$, we finally get,
\[ u(t, x) \leq u^\varepsilon(t, x) + C\varepsilon^{1/\lambda}/\sqrt{t}. \]

We can argue similarly to get the other inequality. The proof is now complete. □

Appendix A. Proof of Lemma 1

To prove Lemma 1, we use Lemma 5.1 from [15, p.17].

Lemma A.1 (Droniou [15]). There exists $\mu_0 \in \mathbb{R}$ such that
\[ g[u](x) = -\mu_0 \cdot |-(N + \lambda) + 2| \Delta u, \]
where $\ast$ denotes the convolution.

It is not proven that $\mu_0$ is positive. To see this, let us fix $u \in \mathcal{S}(\mathbb{R}^N)$ and write $\mu_0(\lambda)$ to enhance the fact that it is a function of $\lambda$. Since it never vanishes and it is continuous w.r.t. $\lambda$ (use the theorem of continuity under the integral sign), it suffices to prove that $\lim_{\lambda \to 2} \mu_0(\lambda) > 0$ to conclude. We know that $g_\lambda[u] \to -\frac{1}{4\pi} \Delta u$ as $\lambda \to 2$ and $g_\lambda[u] = \mu_0(\lambda) D(\lambda)(D(\lambda)^{-1}| |^{-N+2-\lambda}) \ast \Delta u$ where $D(\lambda) = \| |^{-N+2-\lambda} \|_{L^1(B)}$. Since the limit of $D(\lambda)^{-1}| |^{-N+2-\lambda}$ as $\lambda \to 2$, in the distribution sense, is the Dirac mass at the origin, we conclude that $\mu_0$ is positive.

Let $\varepsilon$ denote $-(N + \lambda) + 2$. We first remark that if $x$ is fixed and if one defines $\tilde{u}(y) = u(y) - u(x) - \nabla u(x) \cdot y$, then $\Delta \tilde{u}(y) = \Delta u(y)$. Combining this fact with Lemma A.1 yields:

\[ \frac{1}{\mu_0} g[u](x) = \lim_{\varepsilon \to 0+} \int_{\varepsilon \leq |z| \leq 1/\varepsilon} |z|^2 \Delta \tilde{u}(x + z) + \lim_{\varepsilon \to 0+} \int_{\varepsilon \leq |z| \leq 1/\varepsilon} \Delta(|z|^2) \tilde{u}(x + z) \]
\[ + \int_{|z| = \varepsilon} \text{or } |z| = 1/\varepsilon \left( |z|^2 \frac{\partial \tilde{u}}{\partial n}(x + z) - \tilde{u}(x + z) \frac{\partial |z|^2}{\partial n} \right). \]

Easy computation gives $\Delta(|z|^2) = \varepsilon(N + \varepsilon - 2)|z|^{\varepsilon - 2} = (N + \lambda - 2)\lambda |z|^{-(N + \lambda)}$. Let us set $\nu_0 = \mu_0(N + \lambda - 2)\lambda > 0$. Thus, it remains to prove that the second term of the right-hand side goes to 0 as $\varepsilon \to 0$. We use the fact that $\tilde{u}$ is sublinear and $\frac{\partial \tilde{u}}{\partial n}$ is bounded.
A.1. Details dans la preuve du Lemme 2

Notons \( x_\varepsilon \) un point tel que \( u^\varepsilon(t, x) = u(t, x_\varepsilon) - e^{Kt} \frac{|x-x_\varepsilon|^2}{2\varepsilon} \). On prouve alors le

**Lemma A.2.** Si \((x, p) \in D^{1,+}u^\varepsilon(t, x)\), alors \((z + Ke^{Kt} \frac{|x-x_\varepsilon|^2}{2\varepsilon}) \in D^{1,+}u(t, x_\varepsilon)\) et \( p = e^{Kt} \frac{x-x_\varepsilon}{\varepsilon} \).

**Proof.** On commence par écrire la définition du sur-différentiel. Pour \((s, y)\) proche de \((t, x)\).

\[
\begin{align*}
\dot{z}(s-t) + p \cdot (y-x) & \geq u(s, y) - u^\varepsilon(t, x) + o(|s-t| - \sigma|y-x|^2) \\
& \geq u(s, z) - e^{Ks} \frac{|y-z|^2}{2\varepsilon} - u(t, x_\varepsilon) + e^{Kt} \frac{|x-x_\varepsilon|^2}{2\varepsilon} \\
& \quad + o(|s-t| - \sigma|y-x|^2).
\end{align*}
\]

En prenant alors \( z = x_\varepsilon \) et \( y = x + d \), on obtient que \( p = e^{Kt} \frac{x-x_\varepsilon}{\varepsilon} \). Puis en choisissant y tel que \( y-x = z-x_\varepsilon \), on obtient:

\[
\begin{align*}
\dot{z}(s-t) + p \cdot (y-x) & \geq u(s, z) - u(t, x_\varepsilon) - e^{Ks} \frac{|x-x_\varepsilon|^2}{2\varepsilon} + e^{Kt} \frac{|x-x_\varepsilon|^2}{2\varepsilon} \\
& \quad + o(|s-t| - \sigma|y-x|^2).
\end{align*}
\]

Il suffit alors de voir cela comme une fonction-test en temps pour conclure. \( \square \)
Ensuite, on estime:

\[
H(t, x, u^\varepsilon(t, x), p) \leq H(t, x_\varepsilon, u^\varepsilon(t, x), p) + C_{\|u\|_\infty} \left(1 + |p|\right) |x_\varepsilon - x|
\]

\[
\leq H\left(t, x_\varepsilon, u(t, x), p\right) - e^{Kt} \frac{|x - x_\varepsilon|^2}{2\varepsilon} + (1 + |p|) |x_\varepsilon - x|
\]

\[
\leq H(t, x_\varepsilon, u(t, x), p) + C_{\|u\|_\infty} \left(1 + e^{Kt} \frac{|x - x_\varepsilon|}{\varepsilon}\right) |x - x_\varepsilon|.
\]

Comme \(u\) est une solution de (1), on a:

\[
\alpha + Ke^{Kt} \frac{|x - x_\varepsilon|^2}{2\varepsilon} + H(t, x_\varepsilon, u(t, x), p) + g^+[u](t, x_\varepsilon, p) \leq 0
\]

et donc:

\[
\alpha + H(t, x, u^\varepsilon(t, x), p) + g^+[u^\varepsilon](t, x)
\]

\[
\leq -Ke^{Kt} \frac{|x - x_\varepsilon|^2}{2\varepsilon} + C_{\|u\|_\infty} \left(1 + e^{Kt} \frac{|x - x_\varepsilon|}{\varepsilon}\right) |x - x_\varepsilon|.
\]

On choisit alors \(K = 4C_{\|u\|_\infty}\) et on obtient:

\[
\alpha + H(t, x, u^\varepsilon(t, x), p) + g^+[u^\varepsilon](t, x) \leq C_{\|u\|_\infty} \sup_r \left(r - e^{Kt} r^2 / \varepsilon\right) = \frac{C_{\|u\|_\infty} \varepsilon}{4} e^{-Kt}
\]

\[
\leq (K/16)e.
\]

A.2. Remarque supplémentaire après le théorème 5

Notice that it is an alternative way to prove that the derivative of the viscosity solution of (5) is the entropy solution of the associated scalar conservation law (results of [14] are needed).

A.3. Dans la preuve de l’estimation d’erreur

On a pour tout \(s, t, x, y\),

\[
u(t, x) - u^\varepsilon(s, y) - \frac{|x - y|^2}{2\alpha} - \frac{(s - t)^2}{2\alpha} - \frac{\beta|\eta|^2}{2\alpha} - \frac{\gamma}{T - t} \geq u(t, x) - u^\varepsilon(s, y) - \frac{|x - y|^2}{2\alpha} - \frac{(s - t)^2}{2\alpha} - \frac{\beta|\eta|^2}{2\alpha} - \frac{\gamma}{T - t}
\]
donc en particulier pour tout $y$

$$u^\varepsilon(s, y) \geq u^\varepsilon(s, y) + \frac{|x_v - y_v|^2}{2\varepsilon} - \frac{|x_v - y|^2}{2\varepsilon}$$

$$= u^\varepsilon(s, y) + \frac{|x_v - y_v|^2}{\lambda}, y - y_v - \frac{1}{2\lambda} |y - y_v|^2.$$

(on peut même prendre n’importe quel $r$).

A.4. Optimalité de l’estimation

Si $u$ est solution de $\partial_t u + \varepsilon g[u] = 0$, alors $v(t, x) = u(t, \varepsilon^{1/\lambda} x)$ est solution de $\partial_t v + g[v] = 0$. Ainsi, $v = K(t) \ast v_0$ et donc $u(t, x) = v(t, \varepsilon^{-1/\lambda} x) = \int K(t, y) v_0(\varepsilon^{-1/\lambda} x - y) = \int K(t, y) u_0(x - \varepsilon^{1/\lambda} y)$. Donc pour $u_0(z) = \min(|z|, 1)$ et $x = 0$, on trouve: $v(t, 0) = \int_B K(t, y) \varepsilon^{1/\lambda}|y|dy + \int_{B'\lambda} \cdots = \varepsilon^{1/\lambda}(t^{1/\lambda} \int_{B_{\lambda - 1/\lambda}} K(1, y)|y|dy + \int_{B'_{\lambda - 1/\lambda}} K(1, y)dy).$

Acknowledgments

The author would like to thank J. Droniou for the fruitful discussions they had together.

References