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CONVEX ANALYSIS TECHNIQUES FOR HOPF-LAX FORMULAE IN HAMILTON-JACOBI EQUATIONS

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ABSTRACT. The purpose of the present paper is to prove, solely using Convex (and Nonsmooth) analysis techniques, that Hopf-Lax formulae provide explicit solutions for Hamilton-Jacobi equations with merely lower semicontinuous initial data. The substance of these results appears in [1] but the proofs are fundamentally different (we do not use the comparison principle) and a distinct notion of discontinuous solutions is used. Moreover we give a maximum principle for the Lax function. This approach permits us to fully understand the role of the convexity of the data.

INTRODUCTION

The Lax and the Hopf functions are explicit solutions of:

$$\begin{cases} \frac{\partial u}{\partial t} + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0;+\infty), \\ u(.,0) = g(.) & \text{in } \mathbb{R}^n, \end{cases} \tag{1}$$

(where Du stands for the derivative of u with respect to the space variable x) when either H or g is convex. We recall their definition:

$$u_{\mathrm{Lax}}(x,t) = \inf_{y\in\mathbb{R}^n} \sup_{q\in\mathbb{R}^n} \{g(x-y) + \langle y,q\rangle - tH(q)\}, \tag{2}$$

$$u_{\mathrm{Hopf}}(x,t) = \sup_{q\in\mathbb{R}^n} \inf_{y\in\mathbb{R}^n} \{g(x-y) + \langle y,q\rangle - tH(q)\}. \tag{3}$$

These functions have been intensively studied (see for instance [15, 3, 16, 4]) and the latest contribution is [1]. It is proved that for merely lower semicontinuous (lsc for short) and possibly infinite initial data g, the Lax function is a lsc solution of (1) (in the sense of [5]) when the hamiltonian H is convex. It is also proved that the Hopf function is the minimal supersolution of (1) when the initial condition g is convex. In [1], the proofs rely on the famous comparison principle of viscosity sub and supersolutions and on regularization procedures. The aim of the present paper is to use tools from Convex analysis to prove these results, without relying on PDE techniques. Moreover, we show that the Lax function verifies a "maximum principle", that is to say it is the maximal lsc (sub)solution of the Cauchy problem. Note that the definition of lsc solutions we use in this paper is slightly different from [5]. It first appeared in [20]. See also [12] for further results concerning these discontinuous solutions.

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1. Preliminaries

This section is devoted to definitions and results that are used in the present paper.

Discontinuous functions are considered throughout. A solution u of (1) is merely lower semicontinuous (lsc) and it can take the value $+\infty$. It is said to be *extended real-valued.* We refer to the set

$$\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \,:\, u(x,t) < +\infty\}$$

as the *domain* of u and we denote it dom u. If dom u is nonempty, u is said to be *proper.* For such nonsmooth functions, various concepts of subdifferentials were introduced to replace the classical notion of Fréchet derivative. One of them is the *Fréchet subdifferential*; it is defined at any point (x,t) of the domain of u by:

$$\begin{aligned}\partial_F u(x,t) = \{(\zeta,\alpha) \in \mathbb{R}^n \times \mathbb{R}, \alpha(s-t) + \langle \zeta, y-x\rangle \\ \leq u(y,s) - u(x,t) + o(|y-x| + |t-s|)\},\end{aligned}$$

where $o(.)$ is a function such that $\frac{o(x)}{|x|} \to 0$ as $x \to 0$.

1.1. **Lsc solutions.**

Since Crandall and Lions introduced the concept of continuous viscosity solutions of Hamilton-Jacobi equations, these generalized solutions have been intensively studied [9]. For the reader convenience, we recall the definition of a continuous viscosity solution of (1).

Definition 1. *Let $u : \mathbb{R}^n \times [0;+\infty) \to \mathbb{R}$ be a continuous function.*

- *It is a (viscosity) supersolution of* (1) *if for all $(x,t) \in \mathbb{R}^n \times (0,+\infty)$ and for all $(\zeta,\alpha) \in \partial_F u(x,t)$,*

$$\alpha + H(\zeta) \geq 0 \text{ and } u(x,0) \geq g(x).$$

- *It is a (viscosity) subsolution of* (1) *if for all $(x,t) \in \mathbb{R}^n \times (0,+\infty)$ and for all $(\zeta,\alpha) \in -\partial_F(-u)(x,t)$,*

$$\alpha + H(\zeta) \leq 0 \text{ and } u(x,0) \leq g(x).$$

- *It is a continuous viscosity solution of* (1) *if it is a super and a subsolution of* (1).

In 1990, Barron and Jensen [6] introduced (real-valued) lsc solutions for Hamilton-Jacobi equations of evolution type which hamiltonians $H(t,x,u,p)$ are convex in p. It has been shown that for such hamiltonians, a continuous solution of a Hamilton-Jacobi equation can be completely characterized by its subgradients which should satisfy the relation

$$\alpha + H(t,x,u,\zeta) = 0 \;\forall(\zeta,\alpha) \in \partial_F u(x,t) \;\forall(x,t).$$

It has remarkable resemblance with a classical smooth solution concept of Hamilton-Jacobi equations. In [5], Barron extended this definition by authorizing lsc solutions u to be extended real-valued. In [6, 5], the initial condition is not achieved pointwise but in the following way:

$$g(x) = \liminf_{y\to x, s\to 0^+} u(y,s) \text{ for all } x \in \mathbb{R}^n.$$

Analogous results have been obtained by Frankowska [10] for particular hamiltonians. She also provided an equivalent definition of such solutions in terms of directional derivatives and suggested a pointwise interpretation of the initial condition coupled with a one-sided infinitesimal condition on u at $t = 0$.

Soravia [20] introduced a concept of discontinuous viscosity solutions to Dirichlet problems for Hamilton-Jacobi equations with convex hamiltonians. The definition of lsc solutions for Cauchy problems that is given below is (more or less) a special case of it.

Definition 2. *Let $u : \mathbb{R}^n \times [0; +\infty) \to (-\infty; +\infty]$ be a lsc and proper function.*

- *It is a supersolution of* (1) *if for all $(x, t) \in \mathrm{dom}\, u$, $t > 0$, and all $(\zeta, \alpha) \in \partial_F u(x, t)$:*

$$\alpha + H(\zeta) \geq 0 \tag{4}$$

  and for all $x \in \mathbb{R}^n$:

$$u(x, 0) \geq g(x). \tag{5}$$

- *It is a lsc subsolution of* (1) *if for all $(x, t) \in \mathrm{dom}\, u$ and all $(\zeta, \alpha) \in \partial_F u(x, t)$:*

$$\alpha + H(\zeta) \leq 0 \tag{6}$$

  and for all $x \in \mathbb{R}^n$:

$$u(x, 0) \leq g(x). \tag{7}$$

- *It is a lsc solution of* (1) *if it is a super and a subsolution of* (1), *that is for all $(x, t) \in \mathrm{dom}\, u$ and all $(\zeta, \alpha) \in \partial_F u(x, t)$:*

$$\alpha + H(\zeta) = 0 \text{ if } t > 0,$$
$$\alpha + H(\zeta) \leq 0 \text{ if } t = 0,$$

  and for all $x \in \mathbb{R}^n$:

$$u(x, 0) = g(x).$$

In [12], these lsc solutions are characterized in terms of directional derivatives and of approximate decrease properties.

1.2. **Definitions and results from convex analysis.** In this subsection we present basic tools and classical results of Convex analysis. The interested reader is referred to [19, 11] for a complete presentation of them.

We first recall some definitions. The *Legendre-Fenchel conjugate* of a proper function $f : \mathbb{R}^n \to (-\infty; +\infty]$ is defined by the following formula:

$$\text{for all } q \in \mathbb{R}^n, \qquad f^*(q) = \sup_{x \in \mathbb{R}^n} \{\langle x, q \rangle - f(x)\}.$$

The function $(f^*)^*$, that we simply denote by f^{**}, is called the *closed convex hull* of f. If f is lsc and convex, f^{**} coincides with f. The *subdifferential* from Convex analysis of $f : \mathbb{R}^n \to (-\infty; +\infty]$ at $x \in \mathrm{dom}\, f$ is the set

$$\partial f(x) = \{\zeta \in \mathbb{R}^n \ : \ \forall y \in \mathbb{R}^n, \langle y - x, \zeta \rangle \leq f(y) - f(x)\}.$$

When the function f is convex, the two subdifferentials $\partial_F f(x)$ and $\partial f(x)$ coincide at any point x of $\mathbb{R}^n$. The following characterization holds:

$$\zeta \in \partial f(x) \Leftrightarrow f(x) + f^*(\zeta) = \langle x, \zeta \rangle.$$

It is known as *Fenchel's equality*, while *Fenchel's inequality*

$$f^*(\zeta) + f(x) \geq \langle x, \zeta \rangle$$

always holds true. The *indicator function* of a subset $A \subset \mathbb{R}^n$ is denoted by ι_A and is defined by: $\iota_A(z) = 0$ if $z \in A$, $\iota_A(z) = +\infty$ if $z \notin A$. Given two functions $g, h : \mathbb{R}^n \to (-\infty; +\infty]$, the *epi-sum* of g and h is denoted by $g \underset{e}{+} h$ and is defined for all $x \in \mathbb{R}^n$ by:

$$g \underset{e}{+} h(x) = \inf_{y \in \mathbb{R}^n} \{g(x-y) + h(y)\}. \tag{8}$$

The notion of epi-sum is also known as the inf-convolution operation. But it has the following equivalent definition: $g \underset{e}{+} h$ is the only function f such that its strict epigraph (*i.e.* the set of all points $(y, r) \in \mathbb{R}^n \times \mathbb{R}$ such that $f(y) < r$) is the Minkowski sum of the strict epigraph of g and the strict epigraph of h.

A straightforward calculation yields, for all $t > 0$ and $x \in \mathbb{R}^n$:

$$\left\{ \begin{array}{rcl} u_{\mathrm{Lax}}(x,t) & = & g \underset{e}{+} (tH)^*(x) \\ \\ u_{\mathrm{Hopf}}(x,t) & = & (g^* + tH)^*(x) \end{array} \right. \tag{9}$$

(the Legendre-Fenchel conjugates and the epi-sum are calculated with respect to the x variable). Since we want to prove that u_{Lax} is a lsc solution of the Cauchy problem (1), the Fréchet subdifferential of an epi-sum must therefore be studied. Existing results about convex subdifferentials of epi-sums of convex functions (such as stated in [14, 2] for instance) suggested the following lemma.

Lemma 1. *Consider three functions $f, g, h : \mathbb{R}^n \to (-\infty; +\infty]$ and a point $x \in \mathbb{R}^n$ and assume that f is the epi-sum of g and h. If there exists $y \in \mathbb{R}^n$ such that $f(x) = g(x-y) + h(y)$, then:*

$$\partial_F f(x) \subset \partial_F g(x-y) \cap \partial_F h(y).$$

The proof is elementary and we omit it.

We next recall the statement of the so-called multidirectional mean value inequality. We do not give the most general version but we adapt it to our framework. The closed unit ball of $\mathbb{R}^n$ is denoted by B and for any subset $Y \subset \mathbb{R}^n$, $[x, Y]$ refers to the convex hull of $\{x\} \cup Y$.

Theorem 1 ([8, p. 116-117]). *Let Y be a compact convex subset of $\mathbb{R}^n$ and let $x \in \mathrm{dom}\, f$ where $f : \mathbb{R}^n \to (-\infty; +\infty]$ is a lsc proper function. Then for any $r < \inf_{y \in Y}\{f(y) - (x)\}$ and any $\epsilon > 0$, there exists $z \in [x, Y] + \epsilon B$ and $\zeta \in \partial_F f(z)$ such that, for all $y \in Y$,*

$$r < \langle \zeta, y - x \rangle.$$

In [7], the authors studied the subdifferential of the closed convex hull of an extended real-valued function f. They exhibit a *formula linking the subdifferential of f^{**} and the subdifferential of f.* In order to state their main result, we must introduce two other notions.

Definition 3 ([7, Prop 4.5, p. 1669]). *Consider $f : \mathbb{R}^n \to (-\infty; +\infty]$ that is lsc, proper and bounded from below by an affine function. Then we say that f is epi-pointed if the domain of the Legendre-Fenchel conjugate of f has a nonempty interior.*

Definition 4 ([7, Prop 4.4, p. 1668]). *Consider $f : \mathbb{R}^n \to (-\infty; +\infty]$. Under assumptions of Definition 3, the analytical definition of the so-called asymptotic function f_∞ of f is:*

$$f_\infty(d) = \liminf_{t \to 0^+, d' \to d} tf\left(\frac{d'}{t}\right).$$

If f is convex, f_∞ has an alternative analytical definition.

Proposition 1 ([19, p. 66]). *If f is convex, the following equality holds true for all $d \in \mathbb{R}^n$:*

$$f_\infty(d) = \sup_{u \in \operatorname{dom} f} \{f(d+u) - f(u)\}.$$

Observe that in this case the asymptotic function is sublinear and vanishes at 0. We now state the main result of [7].

Theorem 2 ([7, p. 1669]). *Let $f : \mathbb{R}^n \to (-\infty; +\infty]$ be a lsc, proper and epi-pointed function. Then the following holds:*

(i) *For all $x \in \operatorname{dom} f^{**}$, there are points $x_1, \dots, x_p \in \operatorname{dom} f$, positive numbers $\lambda_1, \dots, \lambda_p$ ($p \geq 1$), and possibly points $y_1, \dots, y_q$ in $\operatorname{dom} f_\infty \backslash \{0\}$ such that:*

$$\begin{cases} \displaystyle\sum_{i=1}^{p} \lambda_i = 1, \\ x = \displaystyle\sum_{i=1}^{p} \lambda_i x_i + \sum_{j=1}^{q} y_j, \\ f^{**}(x) = \displaystyle\sum_{i=1}^{p} \lambda_i f(x_i) + \sum_{j=1}^{q} f_\infty(y_j). \end{cases}$$

(ii) *For any decomposition of the type described in* (i), *we have*

$$\partial f^{**}(x) = [\cap_{i=1}^{p} \partial f(x_i)] \cap \left[\cap_{j=1}^{q} \partial f_\infty(y_j)\right].$$

Remark 1. Even if f is not convex, we can define the subdifferential of f in the sense of Convex analysis. In general, it is empty, but by Theorem 2, $\partial f(x_i)$ is nonempty. This implies (see [14, p. 350]) that $f(x_i) = f^{**}(x_i)$.

2. The Lax function

The present section is devoted to the proof of Theorem 3 stated below. We say that the Lax function is *regular* if it is lsc, extended real-valued and if the infimum defining the real number $u_{\text{Lax}}(x,t)$ is attained for any $(x,t) \in \operatorname{dom} u_{\text{Lax}}$.

Theorem 3. *Let $H : \mathbb{R}^n \to \mathbb{R}$ be convex and let g be lsc and proper. Then if the Lax function is regular, it is a lsc solution of* (1) *(in the sense of Definition 2). Moreover, it is the maximal lsc subsolution of* (1).

Remark 2. If the infimum defining $u_{\text{Lax}}(x,t)$ is taken on a bounded set for all (x,t), u_{Lax} is regular. It is the case when g is bounded from below by $-C(1+|x|)$ for some constant $C>0$. This assumption appears in [1].

Remark 3. For the sake of simplicity, we assume that u_{Lax} is regular. But if the lsc closure of u_{Lax} is extended real-valued, it may be proved that it is a lsc solution of our Cauchy problem. Such considerations appear in [13, 17] in an infinite dimensional setting.

Before proving the theorem, we try to explain how we proceed. In order to prove that the Lax function verifies (1), we apply Lemma 1. If it is applied using representation (9), we only get a description of the *partial* Fréchet subdifferential of u_{Lax} with respect to x. Though we try to establish $\alpha + H(\zeta) = 0$ for all (ζ,α) in the subdifferential of u_{Lax}, we loose the interdependence between x and t. This is the reason why we rewrite the Lax function as an epi-sum of two functions with respect to the couple of variables (x,t). This idea is inspired by a theorem from [18]. The author proves that u_{Lax} is a classical solution of our problem under strong assumptions. He uses tools from Convex analysis such as Legendre-Fenchel conjugates and epi-sums. Besides, even if the formula does not appear explicitly, he writes u_{Lax} under the following form:

Lemma 2.

$$u_{\text{Lax}} = \mathcal{G} \underset{e}{+} \mathcal{H}^* \qquad \textit{on } [0;+\infty)\times\mathbb{R}^n,$$

with

$$\begin{cases} \mathcal{G}(y,s) &=& g(y) + \iota_{\{0\}}(s), \\ \mathcal{H}(y,s) &=& \iota_{\mathbb{R}^-}(H(y)+s). \end{cases}$$

Here the epi-sum and the Legendre-Fenchel conjugate are calculated with respect to the couple (y,s).

Proof of Lemma 2. We calculate the Legendre-Fenchel conjugate of $\mathcal{H}$:

$$\begin{aligned} \mathcal{H}^*(y,s) &= \sup_{\alpha,\zeta}\{\alpha s + \langle \zeta, y\rangle - \iota_{\mathbb{R}^-}(\alpha + H(\zeta))\} \\ &= \sup_{\zeta}\ \sup_{\alpha\le -H(\zeta)} \{\alpha s + \langle\zeta,y\rangle\}. \end{aligned}$$

If $s<0$, $\mathcal{H}^*(y,s)=+\infty$. Otherwise: $\mathcal{H}^*(y,s) = \sup_\zeta\{\langle\zeta,y\rangle - sH(\zeta)\} = (sH)^*(y)$.

For $t\ge 0$, this yields:

$$\begin{aligned} \left(\mathcal{G} \underset{e}{+} \mathcal{H}^*\right)(x,t) &= \inf_{s,y}\{g(x-y) + \iota_{\{0\}}(t-s) + \mathcal{H}^*(y,s)\} \\ &= \inf_{y}\{g(x-y) + (tH)^*(y)\} = u_{\text{Lax}}(x,t). \end{aligned}$$

□

Proof of Theorem 3. The initial condition is trivially satisfied. Consider any point $(x,t)\in \operatorname{dom} u_{\text{Lax}}$ and any $(\zeta,\alpha)\in\partial_F u_{\text{Lax}}(x,t)$. Since we assumed that u_{Lax} is regular, there exists (y,s) such that:

$$u_{\text{Lax}}(x,t) = g(x-y) + (tH)^*(y) = \mathcal{G}(x-y,t-t) + \mathcal{H}^*(y,t).$$

We can therefore apply Lemma 1: $(\zeta, \alpha) \in \partial_F \mathcal{H}^*(y,t) \cap \partial_F \mathcal{G}(x-y,0)$. Since $\mathcal{H}^*$ is convex, it follows that $(\zeta,\alpha) \in \partial\mathcal{H}^*(y,t)$. Using the convex duality, we get:

$$(y,t) \in \partial\mathcal{H}(\zeta,\alpha).$$

This implies that (ζ,α) lies in the domain of $\mathcal{H}$. We therefore obtain:

$$\alpha + H(\zeta) \leq 0.$$

Suppose now that $t > 0$. Fenchel's equality yields:

$$\langle \zeta, y\rangle + \alpha t = \mathcal{H}(\zeta,\alpha) + \mathcal{H}^*(y,t) = 0 + (tH)^*(y) = tH^*\left(\frac{y}{t}\right).$$

Use now Fenchel's inequality and get: $\alpha = H^*\left(\frac{y}{t}\right) - \langle \zeta, \frac{y}{t}\rangle \geq -H(\zeta)$.

It remains to prove that the Lax function is the maximal lsc subsolution of (1). Consider any lsc subsolution w. For any $x \in \mathbb{R}^n : w(x,0) \leq g(x) = u_{\text{Lax}}(x,0)$. It therefore remains to prove that for any $(x,t) \in \mathbb{R}^n \times (0;+\infty)$ and any $y \in \operatorname{dom} H^*$:

$$w(x,t) \leq g(x-ty) + tH^*(y).$$

Suppose it is false. There then exists $(x,t) \in \mathbb{R}^n \times (0;+\infty)$, $y \in \operatorname{dom} H^*$ such that:

$$w(x,t) > g(x-ty) + tH^*(y) \geq w(x-ty,0) + tH^*(y).$$

Apply Theorem 1 to the lsc function w between the two points (x,t) and $(x-ty,0)$: for any $\epsilon > 0$, there exists $(z,r) \in [(x,t),(x-ty,0)] + \epsilon B$ and $(x^*,t^*) \in \partial_F w(z,r)$ such that:

$$\begin{aligned} tt^* + \langle ty, x^*\rangle &> tH^*(y) \\ \Rightarrow \quad t^* + \langle y, x^*\rangle - H^*(y) &> 0. \end{aligned}$$

Since w is a lsc subsolution, $t^* + H(x^*) \leq 0$. We conclude that:

$$\langle y, x^*\rangle - H^*(y) - H(x^*) > 0.$$

The last inequality is in contradiction with Fenchel's inequality. □

3. The Hopf function

In this section, we prove Theorem 4 stated below. We did not recall the definition of a continuous viscosity solution but it can be found, as we already mentioned it, in [9].

Theorem 4. *If $H : \mathbb{R}^n \to \mathbb{R}$ is continuous and $g : \mathbb{R}^n \to (-\infty;+\infty]$ is lsc, proper and convex, then the Hopf function is a supersolution and it is a continuous viscosity solution of* (1) *on the interior of* dom u_{Hopf}.

If, moreover, H is bounded from above by a Lipschitz function, then u_{Hopf} is the minimal supersolution of (1).

It is well known that u_{Hopf} is convex with respect to the couple of variables (x,t). But it is a remarkable fact that it can be expressed with the same extended real-valued functions we used to rewrite the Lax function (namely $\mathcal{G}$ and $\mathcal{H}$).

Lemma 3.

$$u_{\text{Hopf}} = (\mathcal{G}^* + \mathcal{H})^* \quad on\ \mathbb{R}^n \times [0;+\infty)$$

where Legendre-Fenchel conjugates are calculated with respect to the couple (y,s).

Proof. First, we calculate $\mathcal{G}^*$:

$$\begin{aligned}\mathcal{G}^*(\zeta,\alpha) &= \sup_{s,y}\left\{\alpha s + \langle \zeta, y\rangle - g(y) - \iota_{\{0\}}(s)\right\}\\ &= \sup_{y}\left\{\langle \zeta, y\rangle - g(y)\right\} = g^*(\zeta).\end{aligned}$$

For $t \geq 0$:

$$\begin{aligned}\left(\mathcal{G}^* + \mathcal{H}\right)^*(x,t) &= \sup_{\alpha,\zeta}\left\{\alpha t + \langle x, \zeta\rangle - \mathcal{G}^*(\zeta,\alpha) - \mathcal{H}(\zeta,\alpha)\right\}\\ &= \sup_{\zeta}\sup_{\alpha\leq -H(\zeta)}\left\{\alpha t + \langle x,\zeta\rangle - g^*(\zeta)\right\}\\ &= \sup_{\zeta}\left\{\langle x,\zeta\rangle - u_0^*(\zeta) - tH(\zeta)\right\} = u_{\text{Hopf}}(x,t).\end{aligned}$$ □

Remark 4. The reader may observe that u_{Hopf} is lsc on $\mathbb{R}^n \times [0;+\infty)$.

Proof of Theorem 4. Let us set $v := \mathcal{G}^* + \mathcal{H}$. Lemma 3 asserts that the Hopf function is the Legendre-Fenchel conjugate of v. The closed convex hull of v, denoted by v^{**}, is used throughout the proof.

We first prove that u_{Hopf} is a supersolution of (1).
Fix $(x,t) \in \operatorname{dom} u_{\text{Hopf}}$, $t > 0$. Then consider $(\zeta,\alpha) \in \partial u_{\text{Hopf}}(x,t) = \partial v^*(x,t)$. This implies that (ζ,α) lies in the domain of v^{**} (the closed convex hull of v), and that $(x,t) \in \partial v^{**}(\zeta,\alpha)$.

- First case: if $v^{**}(\zeta,\alpha) = v(\zeta,\alpha)$.

Then the convex subdifferential $\partial v^{**}(\zeta,\alpha)$ coincides with the convex subdifferential $\partial v(\zeta,\alpha)$ (see [14]). In particular, $(x,t) \in \partial v(\zeta,\alpha)$. Hence for all $\beta \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$:

$$\begin{aligned} t(\beta-\alpha) + \langle x, \xi - \zeta\rangle &\leq v(\xi,\beta) - v(\zeta,\alpha)\\ &\leq g^*(\xi) + \iota_{\mathbb{R}^-}(H(\xi)+\beta) - g^*(\zeta) - \iota_{\mathbb{R}^-}(H(\zeta)+\alpha). \end{aligned} \tag{10}$$

Setting $\xi = \zeta$ and $\beta = -H(\zeta)$, we get: $t(-H(\zeta)-\alpha) \leq -\iota_{\mathbb{R}^-}(H(\zeta)+\alpha)$. Thus, $\iota_{\mathbb{R}^-}(H(\zeta)+\alpha) = 0$ i.e. $H(\zeta)+\alpha \leq 0$ and:

$$\begin{aligned} t(-H(\zeta)-\alpha) &\leq 0\\ \Leftrightarrow H(\zeta)+\alpha &\geq 0. \end{aligned}$$

Finally, we conclude that, in this case, $H(\zeta)+\alpha = 0$.

- Second case: if $v^{**}(\zeta,\alpha) < v(\zeta,\alpha)$.

We remark that $v \geq g^*$, hence $g^* \leq v^{**} \leq v$.
If $\alpha + H(\zeta) \leq 0$, $v(\zeta,\alpha) = g^*(\zeta)$, and using the previous inequality, we obtain:

$$v^{**}(\zeta,\alpha) = v(\zeta,\alpha).$$

We conclude that, in this second case, $\alpha + H(\zeta) > 0$.

Finally, in both cases, $\alpha + H(\zeta) \geq 0$; we thus have proved that u_{Hopf} is a supersolution of (1).

We continue the proof of Theorem 4 by proving that it is a continuous viscosity solution of (1) on the interior of dom u_{Hopf}. We therefore assume that this set is nonempty. Remember that the Hopf function is the Legendre-Fenchel conjugate of v. We conclude that v is epi-pointed (see Definition 3). Consider now any point $(x,t) \in \mathrm{int}(\mathrm{dom}\ u_{\mathrm{Hopf}})$, and any Fréchet supergradient $(\zeta,\alpha) \in \partial^F u_{\mathrm{Hopf}}(x,t)$. Since u_{Hopf} is convex, we know that $\partial u_{\mathrm{Hopf}}(x,t) = \partial_F u_{\mathrm{Hopf}}(x,t)$ is nonempty. We conclude that u_{Hopf} is differentiable at (x,t). This means that there is one and only one $(\zeta,\alpha) \in \partial u_{\mathrm{Hopf}}(x,t)$. There then exists a unique couple (ζ,α) such that:

$$(x,t) \in \partial v^{**}(\zeta,\alpha).$$

We now show that $v^{**}(\zeta,\alpha) = v(\zeta,\alpha)$. Applying Theorem 2 to v, there exists points $(\zeta_1,\alpha_1),\ldots,(\zeta_p,\alpha_p)$ and possibly points $(\xi_1,\beta_1),\ldots,(\xi_q,\beta_q)$ such that $\sum_{i=1}^{p} \lambda_i = 1$ and:

$$(x,t) \in \partial v^{**}(\zeta_i,\alpha_i).$$

This implies that $p = 1$, $\alpha_1 = \alpha$ and $\zeta_1 = \zeta$ and $\sum_{j=1}^{q}(\xi_j,\beta_j) = 0$. Hence, the following equality holds true:

$$\begin{aligned}
v^{**}(\zeta,\alpha) &= v(\zeta,\alpha) + \sum_{j=1}^{q} v_\infty(\xi_j,\beta_j) \\
&\geq v(\zeta,\alpha) + v_\infty\left(\sum_{j=1}^{q}(\xi_j,\beta_j)\right) \\
&= v(\zeta,\alpha) \geq v^{**}(\zeta,\alpha).
\end{aligned}$$

We used the fact that v_∞ is sublinear and equals 0 at 0. Since $v^{**}(\zeta,\alpha) = v(\zeta,\alpha)$, we proved above that $\alpha + H(\zeta) = 0 \leq 0$. We conclude that u_{Hopf} is a continuous viscosity solution of (1) on $\mathrm{int}(\mathrm{dom}\ u_{\mathrm{Hopf}})$.

To achieve the proof of Theorem 4, we must prove that the Hopf function is the minimal supersolution of (1). Consider a supersolution w of (1), and let us prove that $w \geq u_{\mathrm{Hopf}}$. By assumption, H is bounded from above by a Lipschitz continuous function. There then exists a Lipschitz continuous function H_1 such that w is a supersolution of (1) with $H = H_1$. We can therefore assume that H is Lipschitz continuous. Remember that $u_{\mathrm{Hopf}}(x,t) = \sup_\zeta\{\langle\zeta,x\rangle - g^*(\zeta) - tH(\zeta)\}$. Let us consider some $\zeta_0 \in \mathrm{dom}\ g^*$, and define a new function w_1 as follows:

$$w_1(x,t) = w(x,t) - \langle\zeta_0,x\rangle + g^*(\zeta_0) + tH(\zeta_0).$$

We have to prove that $w_1 \geq 0$. We first remark that w_1 is a supersolution of the following Cauchy problem:

$$\frac{\partial w}{\partial t} + G(Dw) = 0 \quad \text{in } \mathbb{R}^n \times (0;+\infty), \tag{11}$$

$$w(.,0) = 0 \quad \text{in } \mathbb{R}^n, \tag{12}$$

where G denotes the new Hamiltonian defined for all ζ by $G(\zeta) = H(\zeta+\zeta_0) - H(\zeta_0)$. Indeed,

$$w_1(0,x) = w(0,x) - \langle \zeta_0, x\rangle + g^*(\zeta_0) \geq g(x) - \langle \zeta_0, x\rangle + g^*(\zeta_0) \geq 0.$$

Moreover, for all $(x,t) \in \operatorname{dom} w_1$, for all $(\zeta,\alpha) \in \partial_F w_1(x,t)$:

$$(\zeta,\alpha) = (\zeta_1,\alpha_1) + (-\zeta_0, H(\zeta_0)),$$

with $\alpha_1 + H(\zeta_1) \geq 0$. Hence

$$\alpha + G(\zeta) = \alpha + H(\zeta+\zeta_0) - H(\zeta_0) = \alpha_1 + H(\zeta_1) \geq 0.$$

The reader may remark that $G(0) = 0$ and that G is a Lipschitz continuous function. We denote by K a Lipschitz constant of G.

Suppose that there exists some $(\bar{x},\bar{t})$ such that $w_1(\bar{x},\bar{t}) \leq -\Delta < 0$. Let us fix $R > 0$ and let $B(\bar{x},R)$ denote the closed ball centered at $\bar{x}$ of radius R. The lower semicontinuity of w_1 implies that there exists $\underline{t} \in]0,\bar{t}[$, such that for all $x \in B(\bar{x},R)$:

$$0 \leq w_1(x,\underline{t}) + \frac{\Delta}{2}. \tag{13}$$

Combining (13) with $w_1(\bar{x},\bar{t}) \leq -\Delta$, we obtain:

$$\frac{\Delta}{2} \leq w_1(x,\underline{t}) - w_1(\bar{x},\bar{t}),$$

for all $x \in B(\bar{x},R)$. We next apply the mean value Theorem 1 to the lsc function w_1 with $Y = B(\bar{x},R) \times \{\underline{t}\}$ as the closed convex set on which w_1 is bounded from below.

$$\begin{aligned} &\forall \epsilon > 0, \exists (z,\tau) \in [(\bar{x},\bar{t}), Y] + \epsilon B(0,1), \exists (\zeta,\alpha) \in \partial_F w_1(z,\tau) \ : \\ &\qquad \forall x \in B(\bar{x},R),\ \frac{\Delta}{3} \leq \langle (\zeta,\alpha), (x,\underline{t}) - (\bar{x},\bar{t})\rangle . \end{aligned} \tag{14}$$

Observe that $\tau \in [\underline{t} - \epsilon, \bar{t} + \epsilon]$. We therefore choose $\epsilon < \underline{t}$ in order to ensure $\tau > 0$. Since w_1 is a supersolution of (11)-(12): $\alpha + G(\zeta) \geq 0$. Now (14) yields

$$\frac{\Delta}{3} \leq \alpha(\underline{t} - \bar{t}) - R|\zeta| \leq G(\zeta)(\bar{t} - \underline{t}) - R|\zeta|.$$

Since G is Lipschitz and $G(0) = 0$, we conclude that:

$$\frac{\Delta}{3} \leq (K(\bar{t} - \underline{t}) - R)|\zeta| \leq (K\bar{t} - R)|\zeta|.$$

This yields a contradiction for all R large enough. The proof of Theorem 4 is therefore complete. $\square$

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