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The Lévy–Fokker–Planck equation: Φ-entropies and convergence to equilibrium

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Abstract. In this paper, we study a Fokker–Planck equation of the form $u_t = \mathcal{I}[u] + \mathrm{div}(xu)$, where the operator $\mathcal{I}$, which is usually the Laplacian, is replaced here with a general Lévy operator. We prove by the entropy production method the exponential decay in time of the solution to the steady state of the associated stationary equation.

Keywords: Fokker–Planck equation, Lévy operator, Φ-entropy inequalities, entropy production method, logarithmic Sobolev inequalities, fractional Laplacian

1. Introduction and main results

1.1. The equation at stake

In this paper, we are interested in the long time behaviour of the following generalized Fokker–Planck equation:

$$\partial_t u = \mathcal{I}[u] + \mathrm{div}(u\nabla V), \quad x \in \mathbb{R}^d, t > 0, \tag{1}$$

submitted to the initial condition:

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^d, \tag{2}$$

where u_0 is nonnegative and in $L^1(\mathbb{R}^d)$ and V is a given proper potential for which there exists a nonnegative steady state (see below). The operator $\mathcal{I}$ is a Lévy operator

$$\mathcal{I}[u](x) = \mathrm{div}(\sigma\nabla u)(x) - b\cdot\nabla u(x) + \int_{\mathbb{R}^d}\big(u(x+z) - u(x) - \nabla u(x)\cdot zh(z)\big)\nu(\mathrm{d}z) \tag{3}$$

with parameters (b,σ,ν), where $b = (b_i) \in \mathbb{R}^d$, σ is a symmetric semi-definite $d\times d$ matrix $\sigma = (\sigma_{i,j})$ and ν denotes a nonnegative singular measure on $\mathbb{R}^d$ that satisfies

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int \min\big(1,|z|^2\big)\nu(\mathrm{d}z) < +\infty; \tag{4}$$

h is a truncature function and we fix it on this article: for any $z \in \mathbb{R}^d$, $h(z) = 1/(1+|z|^2)$.

Remark that this equation is conservative: for any $t > 0$, $\int u(t,x)\,\mathrm{d}x = \int u_0(x)\,\mathrm{d}x$ and we will therefore assume without loss of generality that we have $\int u_0(x)\,\mathrm{d}x = 1$. The diffusion equation (1) is a natural generalization of the well-known Fokker–Planck equation given by

$$\partial_t u = \Delta u + \mathrm{div}(u\nabla V), \quad x \in \mathbb{R}^d, t > 0, \tag{5}$$

and since $\mathcal{I}$ defined by (3) is the infinitesimal generator of a Lévy process, we refer to (1) as the Lévy–Fokker–Planck equation.

1.2. Motivations and goals

As pointed out by Biler and Karch in [7], Eq. (1) is studied at least for two important reasons. First, it is deeply connected with stochastic differential equations driven by Lévy stable noise, see e.g. [11]. Secondly, in the case of a quadratic potential $V(x) = \frac{1}{2}|x|^2$ and the fractional Laplacian $\mathcal{I} = -(-\Delta)^{\alpha/2}$ (even for general α-stable Lévy process – see Section 2 for a definition), it permits to describe the long time behaviour of the solution of

$$\partial_t U + (-\Delta)^{\alpha/2}[U] = 0 \tag{6}$$

with a nonnegative initial condition U_0 in $L^1(\mathbb{R}^d)$. Indeed, the reader can check that the function $u(t,x) = R^d(t)U(\tau(t), xR(t))$ satisfies (1) as soon as one chooses $R(t) = \mathrm{e}^t$ and $\tau(t) = (\mathrm{e}^{\alpha t} - 1)/\alpha$ and $U_0(x) = u_0(x)$. Consequently, if $K(t,x)$ denotes the Green function associated with the fractional Laplacian,

$$u(t,x) = \mathrm{e}^{dt} \int u_0\left(\frac{\mathrm{e}^{\alpha t} - 1}{\alpha} x - y\right) K\left(\mathrm{e}^t, y\right) \mathrm{d}y.$$

On the one hand, U vanishes as $t \to +\infty$ and on the other hand, u is expected to tend towards the steady state of (1), that is, towards the nonnegative stationary solution u_∞ of (1) with the quadratic potential $V(x) = \frac{1}{2}|x|^2$:

$$\mathcal{I}[u_\infty] + \mathrm{div}(u_\infty x) = 0 \tag{7}$$

such that $\int u_\infty\,\mathrm{d}x = \int u_0\,\mathrm{d}x = 1$. We see that dealing with the quadratic potential is of particular interest and we will be able to say more in this case.

In full generality, we expect that the solution u of (1) converges towards the unique nonnegative solution of:

$$\mathcal{I}[u_\infty] + \mathrm{div}(u_\infty \nabla V) = 0 \tag{8}$$

such that $\int u_\infty = \int u_0 = 1$. The function u_∞ is called a steady state and conditions on V must ensure that such u_∞ exists and it is nonnegative.

1.3. Known results

Consider a smooth convex function $\Phi : \mathbb{R}^+ \mapsto \mathbb{R}$ and u_∞ positive such that $\int u_\infty \, \mathrm{d}x = 1$ and define the Φ-entropy: for any nonnegative function f,

$$\mathrm{Ent}^{\Phi}_{u_\infty}(f) := \int \Phi(f) u_\infty \, \mathrm{d}x - \Phi\biggl(\int f u_\infty \, \mathrm{d}x\biggr).$$

Jensen's inequality gives that $\mathrm{Ent}^{\Phi}_{u_\infty}(f) \geqslant 0$. Let u_0 be a nonnegative initial condition of (1) or (5) then the maximum principle ensures that

$$E_\Phi(u_0)(t) = \mathrm{Ent}^{\Phi}_{u_\infty}\left(\frac{u(t)}{u_\infty}\right)$$

is well defined for $t > 0$.

In the case of the classical Fokker–Planck equation (5), by using functional inequalities as Poincaré, logarithmic Sobolev or Φ-entropy inequalities, and under proper assumptions on the potential V such as the Bakry–Emery criterion (Γ_2), one obtains exponential decays to zero of $E_\Phi(u_0)$. Then the solution u of (5) converges towards the steady state u_∞ in the sense of Φ-entropy. Methods to prove such results are usually based on entropy/entropy-production tools. See [1,4,5,8] for different methods and applications.

For our Eq. (1), Biler and Karch prove in [7] that $E_\Phi(u_0)$ decreases in time for general confinement potentials V; they also prove that for $\Phi(r) = r^2/2$ and for the quadratic potential $V(x) = \frac{1}{2}|x|^2$, there exists $C = C(u_0, \mathcal{I})$ and ε such that:

$$E_{|\cdot|^2/2}(u_0)(t) \leqslant C\mathrm{e}^{-\varepsilon t}. \tag{9}$$

To prove the latter result, they assume that the symbol ψ (see Section 2.1) satisfies for some real number $\alpha \in (0, 2]$:

$$\begin{cases} 0 < \liminf\limits_{\xi \to 0} \dfrac{\psi(\xi)}{|\xi|^\alpha} \leqslant \limsup\limits_{\xi \to 0} \dfrac{\psi(\xi)}{|\xi|^\alpha} < +\infty, \\ 0 < \inf \dfrac{\psi(\xi)}{|\xi|^2}. \end{cases} \tag{10}$$

In particular, the second assumption says more or less that there is a nontrivial Gaussian part ($\sigma \neq 0$).

1.4. Main results

The results of Biler and Karch we just described are the starting point of this paper. We tried to understand if one can generalize them and obtain an exponential decay for any convex function Φ under a sharper form. Our two main contributions are:

(1) to exhibit the so-called Fisher information associated with the Φ-entropies E_Φ,
(2) and to take advantage of this result to prove an exponential decay of the Φ-entropies for a class of convex functions Φ and for a larger class of operators.

Let us describe our results more precisely. Let us consider the Bregman distance associated with Φ:

$$\forall (a,b) \in \mathbb{R}^+, \quad D_\Phi(a,b) := \Phi(a) - \Phi(b) - \Phi'(b)(a-b) \geqslant 0. \tag{11}$$

Our first result states that for all initial nonnegative datum u_0 s.t. $\int u_0 \,\mathrm{d}x = 1$:

$$\begin{aligned} \forall t \geqslant 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} E_\Phi(u_0)(t) = &- \int \Phi''(v)\nabla v \cdot \sigma \nabla v u_\infty \,\mathrm{d}x \\ &- \iint D_\Phi\big(v(t,x), v(t,x-z)\big)\nu(\mathrm{d}z) u_\infty(x)\,\mathrm{d}x, \end{aligned} \tag{12}$$

where $v(t,x) = \frac{u(t,x)}{u_\infty(x)}$ and ν is the Lévy measure appearing in the definition of the operator $\mathcal{I}$, see (4). The first result of Biler and Karch is a straightforward consequence of this formula and they were very close to it (see their proof of the entropy decreasing property). In the particular case of $\Phi(x) = x^2$, the right-hand side of (12) can be seen as the opposite of a Dirichlet form (see Chapter 3 of [3]).

The main result of this paper is the following theorem.

Theorem 1 (Exponential decay to equilibrium). *Assume that $V(x) = \frac{1}{2}|x|^2$ and the operator $\mathcal{I}$ is the infinitesimal generator of a Lévy process whose Lévy measure is denoted ν. We assume that ν has a density N with respect to the Lebesgue measure and that N satisfies*

$$\int_{\mathbb{R}^d \setminus B} \ln|z| N(z)\,\mathrm{d}z < +\infty, \tag{13}$$

where B is the unit ball in $\mathbb{R}^d$. Then there exists a steady state u_∞, i.e. a nonnegative solution of (7) *satisfying $\int u_\infty \,\mathrm{d}x = 1$.*

If moreover N is even and for all $z \in \mathbb{R}^d$,

$$\int_1^{+\infty} N(sz)s^{d-1}\,\mathrm{d}s \leqslant CN(z) \tag{14}$$

for some constant $C \geqslant 0$, then for any smooth convex function Φ such that

$$\begin{cases} (a,b) \mapsto D_\Phi(a+b,b), \\ (r,y) \mapsto \Phi''(r)y \cdot \sigma y, \end{cases} \quad \text{are convex on} \quad \begin{cases} \{a+b \geqslant 0, b \geqslant 0\}, \\ \mathbb{R}^+ \times \mathbb{R}^{2d}, \end{cases} \tag{15}$$

the Φ-entropy of the solution u of (1) *and* (2) *goes to* 0 *exponentially. Precisely, for any nonnegative initial datum u_0 such that $\mathrm{Ent}^\Phi_{u_\infty}(\frac{u_0}{u_\infty}) < \infty$, one gets:*

$$\forall t \geqslant 0, \quad \mathrm{Ent}^\Phi_{u_\infty}\left(\frac{u(t)}{u_\infty}\right) \leqslant \mathrm{e}^{-t/C}\,\mathrm{Ent}^\Phi_{u_\infty}\left(\frac{u_0}{u_\infty}\right) \tag{16}$$

with C appearing in (14). *In addition, u_∞ is the unique nonnegative solution of* (7).

Condition (14) is fulfilled by any α-stable Lévy process and in this case, $C = \alpha^{-1}$. It is also satisfied by multi-fractal operators. See the next section for definitions. Furthermore, it should be compared with the first part of (10). Indeed, freely speaking, it says that the symbol lies between two α-stable ones for small ξ.

Conditions (15) appear in [8] (see ($\mathcal{H}1$) and ($\mathcal{H}2$) in this paper). Basic but important examples of such convex functions are given by $\Phi(x) = x^p - 1 - p(x-1)$ with $p \in (1,2]$, $\Phi(x) = x \log x - x + 1$. When $p = 2$, $D_\Phi(a,b) = (a-b)^2$ and in the latter case, $D_\Phi(a,b) = a \ln \frac{a}{b} + (b-a)$. In Theorem 4.4 of [9] it is proved that condition (15) is equivalent to either Φ is affine or Φ is C^2, $\Phi'' > 0$ on $\mathbb{R}^+$ and $1/\Phi''$ is concave on $\mathbb{R}^+$.

The paper is organized as follows. In Section 2, we recall basic facts about Lévy operators and associated Lévy measures. We also recall a modified logarithmic Sobolev inequality due to Wu and Chafaï (Theorem 2). Section 3 is devoted to derivating Φ-entropies with respect to time for general potentials V. The last section is devoted to the proof of Theorem 1, a discussion about condition (14) and some examples.

Notation. $x \cdot y$ denotes the usual scalar product in $\mathbb{R}^d$.

2. Preliminaries

2.1. Lévy operators

Let us next recall basic definitions about Lévy operators and introduce notations. See [3,6] for further details.

2.1.1. Characteristic exponents and Lévy measures

Because of the definition of a Lévy process (see Chapter 1 of [3]), the law μ_t of such a process $(X_t)_{t \geqslant 0}$ at time $t > 0$ is *infinitely divisible*, i.e. it can be written for all $n \geqslant 1$ under the form

$$\mu_t = \underbrace{\mu_n \star \cdots \star \mu_n}_{n \text{ times}}$$

for an arbitrary probability μ_n. Using this property, it can be shown that the characteristic function $\phi_{X_t}(\xi)$ (i.e. its Fourier transform) of the law of X_t can be written under the form $\exp(t\psi(\xi))$ for a function ψ called the *characteristic exponent*. The Lévy–Khinchine formula states that ψ can be written in the form

$$\psi(\xi) = -\sigma\xi \cdot \xi + \mathrm{i}b \cdot \xi + a(\xi), \tag{17}$$

where a is associated with the pure jump part of the Lévy process

$$a(\xi) = \int \left(\mathrm{e}^{\mathrm{i}z\cdot\xi} - 1 - \mathrm{i}(z \cdot \xi)h(z)\right)\nu(\mathrm{d}z).$$

Recall that $h(z) = 1/(1 + |z|^2)$ and ν is the Lévy measure which satisfies (4). The matrix σ characterizes the diffusion (or Gaussian) part of the operator (with eventually $\sigma = 0$), while b and ν characterize the drift part and the pure jump part, respectively. The support of the measure ν represents the possible jumps of the process. See [3] for further details. A Lévy operator $\mathcal{I}$ is the infinitesimal generator associated with the Lévy process and the Lévy–Khinchine formula implies that it has the form (3).

2.1.2. The pseudo-differential point of view

It is convenient to introduce the operator $\mathcal{I}_g$ associated with the Gaussian part,

$$\mathcal{I}_g(u) = \mathrm{div}(\sigma \nabla u) - b \cdot \nabla u$$

and the operator $\mathcal{I}_a$ associated with the pure jump part

$$\mathcal{I}_a(u) = \int_{\mathbb{R}^d} \big(u(x+z) - u(x) - \nabla u(x) \cdot z h(z)\big) \nu(\mathrm{d}z).$$

The operator $\mathcal{I}_a$ can be seen as a pseudo-differential operator of symbol a:

$$\mathcal{I}_a(u) = \mathcal{F}\big(a \times \mathcal{F}^{-1} u\big)$$

where $\mathcal{F}$ stands for the Fourier transform. We choose the probabilistic convention in defining, for all function $w \in L^1(\mathbb{R}^d)$:

$$\forall \xi \in \mathbb{R}^d, \quad \hat{w}(\xi) = \mathcal{F}(w)(\xi) = \int \mathrm{e}^{\mathrm{i}x \cdot \xi} w(x)\, \mathrm{d}x. \tag{18}$$

2.1.3. Multi-fractal and α-stable Lévy operators

Lévy operators whose characteristic exponent is positively homogeneous of index $\alpha \in (0, 2]$ are called *α-stable*. The fractional Laplacian corresponds to a particular α-stable Lévy process whose characteristic exponent is $\psi(\xi) = |\xi|^\alpha$. In the case $\alpha \in (0, 2)$, one gets

$$b = 0, \qquad \sigma = 0 \quad \text{and} \quad \nu(\mathrm{d}z) = \frac{\mathrm{d}z}{|z|^{d+\alpha}}.$$

Hence, it is a pure jump process, i.e. it has neither a drift part nor a diffusion one. Note that if $\alpha = 2$ the Lévy operator has no jump part, it is classical Laplacian. The α-stable operators play a central role in this paper since they are the ones for which we have equality in condition (14). See the discussion in the last section.

Lévy operators whose characteristic exponent can be written as $\psi(\xi) = \sum_{i=1}^n \psi_i(\xi)$ where ψ_i is α_i-homogeneous with $\alpha_i \in (0, 2]$, are often referred to as multi-fractal Lévy operators. These operators are typically those which satisfy condition (14).

2.2. A modified logarithmic Sobolev inequality

The functional inequality we will need in the sequel is given in the following theorem. It was proved by Wu and Chafaï.

Theorem 2 ([8,12]). *Assume that a smooth function Φ satisfies* (15) *and consider an infinitely divisible law μ. Then for all smooth positive functions v,*

$$\mathrm{Ent}_\mu^\Phi(v) \leqslant \int \Phi''(v) \nabla v \cdot \sigma \nabla v \mu(\mathrm{d}x) + \iint D_\Phi\big(v(x), v(x+z)\big) \nu_\mu(\mathrm{d}z) \mu(\mathrm{d}x), \tag{19}$$

where ν_μ and σ denote respectively the Lévy measure and the diffusion matrix associated with μ.

Remark that the drift of the law plays no role in this functional inequality. Equation (19) is proved in [12] for $\Phi(x) = x^2$ or $\Phi(x) = x \log x$ and in this general form in [8] but only for a pure jump Lévy process. The two articles are generalization of the logarithmic Sobolev inequality for Poisson measure given in [2]. For the completeness of this article, we give a sketch of the proof of this theorem in the Appendix.

2.2.1. The important special case $\Phi(r) = r^2/2$

A simple computation shows that the Bregman distance in this case is $D_\Phi(a, b) = (a - b)^2$. In this case, the Φ-entropy $\mathrm{Ent}^\Phi_{u_\infty}$ reduces to the variance. One can next consider the "carré du champ" operator $\Gamma^{\mathcal{I}}$ associated with the operator $\mathcal{I}$ defined for smooth functions u and v by:

$$\begin{aligned}\Gamma^{\mathcal{I}}[u, v] &= \mathcal{I}[uv] - u\mathcal{I}[v] - v\mathcal{I}[u] \\ &= 2\nabla u \cdot \sigma \nabla v + \int \big(u(x + z) - u(x)\big)\big(v(x + z) - v(x)\big)\nu(\mathrm{d}z).\end{aligned}$$

We shed some light on the fact that one can obtain in that case inequality (19) by using Γ_2 associated with $\mathcal{I} = \mathcal{I}_\mu$ defined as follows:

$$\Gamma^{\mathcal{I}}_2(u, v) = \mathcal{I}\big[\Gamma^{\mathcal{I}}[u, v]\big] - \Gamma^{\mathcal{I}}\big[u, \mathcal{I}[v]\big] - \Gamma^{\mathcal{I}}\big[v, \mathcal{I}[u]\big].$$

Inequality (19) can be obtained by remarking that (assume $\sigma = 0$ for simplicity):

$$\Gamma^{\mathcal{I}}_2(u, u) = \iint \big(u(x + z + z') - u(x + z) - u(x + z') + u(x)\big)^2 \nu(\mathrm{d}z)\nu(\mathrm{d}z') \geqslant 0,$$

with ν the Lévy measure associated to $\mathcal{I}$. This computation was already done by Chafaï and Malrieu in a nice but unpublished note [10].

3. Φ-entropy and associated Fisher information

In this section, we are interested in the (time) derivative of $E_\Phi(u_0)(t)$ in the case where a steady state is given. Precisely, the following theorem is proved.

Proposition 1. *Assume that the initial condition u_0 is nonnegative and satisfies $\mathrm{Ent}^\Phi_{u_\infty}(\frac{u_0}{u_\infty}) < \infty$ with u_∞, a solution of* (8)*. Then for any convex smooth function $\Phi : \mathbb{R}^+ \to \mathbb{R}$ and any $t \geqslant 0$, the solution u of* (1) *and* (2) *satisfies* (12) *where $v(t, x) = \frac{u(t,x)}{u_\infty(x)}$ and ν is the Lévy measure associated to $\mathcal{I}$.*

Remark 1. In the special case $\Phi(x) = x^2/2$, Proposition 1 can be reformulated as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\Phi(u_0)(t) = -\int \Gamma^{\mathcal{I}}[v, v] u_\infty \,\mathrm{d}x \leqslant 0.$$

As in the case of the classical Fokker–Planck equation, the energy does not depend on the potential V.

In order to prove Proposition 1, since the Φ-entropy involves the function $v(t,x) = \frac{u(t,x)}{u_\infty(x)}$, its derivative makes appear $\partial_t v$ and it is natural to ask ourselves which partial differential equation v satisfies. A simple computation gives:

$$\begin{aligned}\partial_t v &= \frac{1}{u_\infty}\big(\mathcal{I}[u_\infty v] + \operatorname{div}(u_\infty v \nabla V)\big)\\ &= \frac{1}{u_\infty}\big(\mathcal{I}[u_\infty v] - \mathcal{I}[u_\infty]v\big) + \nabla V\cdot\nabla v =: Lv. \end{aligned} \tag{20}$$

In the case where $\mathcal{I}[u] = \Delta u$ (i.e. σ is the identity matrix, $b = 0$ and $a = 0$), Eq. (20) writes:

$$\partial_t v = \Delta v - \nabla V\cdot\nabla v$$

and is known as the Ornstein–Uhlenbeck equation. This is the reason why we will refer to Eq. (20) as the Lévy–Ornstein–Uhlenbeck equation. We next give a simpler formulation for the Lévy–Ornstein–Uhlenbeck operator.

Lemma 1 (Lévy–Ornstein–Uhlenbeck equation). *If the integrodifferential operator on the right-hand side of* (20) *is denoted L, we have for all smooth functions w_1 and w_2:*

$$\int w_1(Lw_2)u_\infty\,\mathrm{d}x = \int\big(\check{\mathcal{I}}[w_1] - \nabla V\cdot\nabla w_1\big)w_2 u_\infty\,\mathrm{d}x,$$

where $\check{\mathcal{I}}$ is the Lévy operator whose parameters are $(-b,\sigma,\check{\nu})$ with $\check{\nu}(\mathrm{d}z) = \nu(-\mathrm{d}z)$. This can be expressed by the formula: $L^* = \check{\mathcal{I}} - \nabla V\cdot\nabla$ where duality is understood with respect to the measure $u_\infty\,\mathrm{d}x$.

Proof. The main tool is the integration by parts for the operator $\mathcal{I}$, for any smooth functions u, v one gets

$$\int v\mathcal{I}[u]\,\mathrm{d}x = \int u\check{\mathcal{I}}[v]\,\mathrm{d}x.$$

If w_1 and w_2 are two smooth functions on $\mathbb{R}^d$, then:

$$\begin{aligned}\int w_1(Lw_2)u_\infty\,\mathrm{d}x &= \int w_1\big(\mathcal{I}[u_\infty w_2] - \mathcal{I}[u_\infty]w_2 + u_\infty\nabla V\cdot\nabla w_2\big)\,\mathrm{d}x\\ &= \int w_2\check{\mathcal{I}}[w_1]u_\infty\,\mathrm{d}x - \int\mathcal{I}[u_\infty]w_1w_2\,\mathrm{d}x - \int\operatorname{div}(u_\infty w_1\nabla V)w_2\,\mathrm{d}x\\ &= \int u_\infty\big(\check{\mathcal{I}}[w_1] - \nabla V\cdot\nabla w_1\big)w_2\,\mathrm{d}x. \qquad\square\end{aligned}$$

Proof of Proposition 1. By using Lemma 1 with $v = u/u_\infty$, we get:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\Phi(u_0)(t) &= \int\Phi'(v)\partial_t v u_\infty\,\mathrm{d}x = \int\Phi'(v)(Lv)u_\infty\,\mathrm{d}x\\ &= \int\check{\mathcal{I}}\big[\Phi'(v)\big]vu_\infty\,\mathrm{d}x - \int\nabla V\cdot\nabla\big(\Phi'(v)\big)vu_\infty\,\mathrm{d}x.\end{aligned}$$

If now one remarks that $r\Phi''(r) = (r\Phi'(r) - \Phi(r))'$, we get:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_\Phi(u_0)(t) &= \int v\check{\mathcal{I}}\big[\Phi'(v)\big]u_\infty\,\mathrm{d}x - \int \nabla V\cdot\nabla\big(v\Phi'(v)-\Phi(v)\big)u_\infty\,\mathrm{d}x\\
&= \int v\check{\mathcal{I}}\big[\Phi'(v)\big]u_\infty\,\mathrm{d}x + \int \operatorname{div}(u_\infty\nabla V)\big(v\Phi'(v)-\Phi(v)\big)\,\mathrm{d}x\\
&= \int v\check{\mathcal{I}}\big[\Phi'(v)\big]u_\infty\,\mathrm{d}x - \int \mathcal{I}[u_\infty]\big(v\Phi'(v)-\Phi(v)\big)\,\mathrm{d}x\\
&= \int \big(v\check{\mathcal{I}}\big[\Phi'(v)\big] - \check{\mathcal{I}}\big[v\Phi'(v)\big] + \check{\mathcal{I}}\big[\Phi(v)\big]\big)u_\infty\,\mathrm{d}x,\\
\frac{\mathrm{d}}{\mathrm{d}t}E_\Phi(u_0)(t) &= \int \big(v\check{\mathcal{I}}_g\big[\Phi'(v)\big] - \check{\mathcal{I}}_g\big[v\Phi'(v)\big] + \check{\mathcal{I}}_g\big[\Phi(v)\big]\big)u_\infty\,\mathrm{d}x\\
&\quad + \int \big(v\check{\mathcal{I}}_a\big[\Phi'(v)\big] - \check{\mathcal{I}}_a\big[v\Phi'(v)\big] + \check{\mathcal{I}}_a\big[\Phi(v)\big]\big)u_\infty\,\mathrm{d}x\\
&= -\int \Phi''(v)\nabla v\cdot\sigma\nabla v u_\infty\,\mathrm{d}x\\
&\quad + \iint \big(v(x)\big(\Phi'\big(v(x+z)\big)-\Phi'\big(v(x)\big)\big) - v(x+z)\Phi'\big(v(x+z)\big)\\
&\qquad + v(x)\Phi'\big(v(x)\big) + \Phi\big(v(x+z)\big) - \Phi\big(v(x)\big)\big)\check{\nu}(\mathrm{d}z)u_\infty(x)\,\mathrm{d}x,\\
\frac{\mathrm{d}}{\mathrm{d}t}E_\Phi(u_0)(t) &= -\int \Phi''(v)\nabla v\cdot\sigma\nabla v u_\infty\,\mathrm{d}x\\
&\quad - \iint \big(\Phi\big(v(x)\big) - \Phi\big(v(x+z)\big) - \Phi'\big(v(x+z)\big)\\
&\qquad \times \big(v(x)-v(x+z)\big)\big)\check{\nu}(\mathrm{d}z)u_\infty(x)\,\mathrm{d}x.
\end{aligned}
$$

Then the definitions of the Bregman distance and of the Lévy measure $\check{\nu}$ give

$$
\frac{\mathrm{d}}{\mathrm{d}t}E_\Phi(u_0)(t) = -\int \Phi''(v)\nabla v\cdot\sigma\nabla v u_\infty\,\mathrm{d}x - \iint D_\Phi\big(v(x),v(x-z)\big)\nu(\mathrm{d}z)u_\infty(x)\,\mathrm{d}x. \qquad \square
$$

4. The proof of Theorem 1, discussion and examples

To prove the first part of the Theorem, the existence of the steady state, we need to state the following lemma.

Lemma 2. *Assume that the Lévy measure ν has a density N with respect to the Lebesgue measure and that it satisfies* (13). *There then exists a steady state u_∞, i.e. a solution of* (7). *Moreover, it is an infinitely divisible measure whose characteristic exponent A is defined by:*

$$
A(\xi) = -\xi\cdot\sigma\xi + \mathrm{i}b\cdot\xi + \int_0^1 a(s\xi)\frac{\mathrm{d}s}{s}. \tag{21}
$$

Moreover, parameters of the characteristic exponent A are $(\sigma, b - b_A, N_\infty \,\mathrm{d}x)$ where

$$b_A = \iint_0^1 z \frac{(1-\tau^2)|z|^2}{(1+\tau^2|z|^2)(1+|z|^2)} \,\mathrm{d}\tau\, N(z)\,\mathrm{d}z, \tag{22}$$

and

$$N_\infty(z) = \int_1^\infty N(tz)t^{d-1}\,\mathrm{d}t. \tag{23}$$

Note that the Lévy measure ν_∞ associated to the characteristic exponent A has a density N_∞ with respect to the Lebesgue measure.

Remark 2. In the general case, condition (14) precisely says that $N_\infty \leqslant CN$ which can be written in terms of measures as follows: $\nu_\infty \leqslant C\nu$.

Proof of Lemma 2. Let us start as in [7]. At least formally, the Fourier transform $\hat{u}_\infty$ of any steady state u_∞ satisfies

$$\psi(\xi)\hat{u}_\infty + \xi \cdot \nabla \hat{u}_\infty = 0$$

so that $\hat{u}_\infty = \exp(-A)$ with A such that:

$$\nabla A(\xi) \cdot \xi = \psi(\xi).$$

The solution of this equation is precisely given by (21). It is not clear that A is well defined and is the characteristic exponent of an infinitely divisible measure; this is what we prove next. This will imply in particular that $\mathcal{F}^{-1}(\exp(-A))$ is a nonnegative function (see [6]).

Define the nonnegative N_∞ by Eq. (23). This integral of a nonnegative function is finite since for any $R > 0$, if $\mathrm{d}\sigma$ denotes the uniform measure on the unit sphere S^{d-1} we get,

$$\int_R^\infty \int_{|D|=1} N(\tau D)\tau^{d-1}\,\mathrm{d}\tau\,\mathrm{d}\sigma(D) = \int_{|y|\geqslant R} N(y)\,\mathrm{d}y < +\infty.$$

We conclude that for any $r \geqslant R > 0$ and almost every D on the unit sphere (where the set of null measure depends only on R),

$$r^d N_\infty(rD) = \int_r^\infty N(\tau D)\tau^{d-1}\,\mathrm{d}\tau < +\infty$$

so that $N_\infty(z)$ is well defined almost everywhere outside B_R. Choose now a sequence $R_n \to 0$ and conclude.

Let us define $I(r) = \int_{|D|=1} N(rD)\,\mathrm{d}\sigma(D)$ and I_∞ in an analogous way. The previous equality implies that: $r^d I_\infty(r) = \int_r^{+\infty} I(\tau)\tau^{d-1}\,\mathrm{d}\tau$. We conclude that:

$$\begin{aligned}
\int_{|z|\leqslant 1} |z|^2 N_\infty(z)\,\mathrm{d}z &= \int_0^1 I_\infty(r) r^{d+1}\,\mathrm{d}r = \int_0^1 r \int_r^{+\infty} I(\tau)\tau^{d-1}\,\mathrm{d}\tau\,\mathrm{d}r \\
&= \frac{1}{2}\int_{|z|\geqslant 1} N(z)\,\mathrm{d}z + \frac{1}{2}\int_{|z|\leqslant 1} |z|^2 N(z)\,\mathrm{d}z < +\infty, \\
\int_{|z|\geqslant 1} N_\infty(z)\,\mathrm{d}z &= \int_1^{+\infty} I_\infty(r) r^{d-1}\,\mathrm{d}r = \int_1^{+\infty} \frac{1}{r}\int_r^{+\infty} I(\tau)\tau^{d-1}\,\mathrm{d}\tau\,\mathrm{d}r \\
&= \int_{|z|\geqslant 1} \ln|z| N(z)\,\mathrm{d}z.
\end{aligned}$$

Hence we have $\int \min(1,|z|^2) N_\infty(z)\,\mathrm{d}z < +\infty$. We conclude that it is a Lévy measure. Now consider the associated characteristic exponent:

$$\tilde{A}(\xi) = \mathrm{i}b\cdot\xi - \sigma\xi\cdot\xi + \int \big(\mathrm{e}^{\mathrm{i}z\cdot\xi} - 1 - \mathrm{i}(z\cdot\xi)h(z)\big) N_\infty(z)\,\mathrm{d}z.$$

Now compute:

$$\begin{aligned}
\tilde{A}(\xi) - \mathrm{i}b\cdot\xi + \sigma\xi\cdot\xi &= \iint_1^\infty \big(\mathrm{e}^{\mathrm{i}z\cdot\xi} - 1 - i(z\cdot\xi)h(z)\big) N(sz) s^{d-1}\,\mathrm{d}s\,\mathrm{d}z \\
&= \int_1^\infty \left\{\int \left(\mathrm{e}^{\mathrm{i}\tilde{z}\cdot\xi/s} - 1 - \mathrm{i}\left(\tilde{z}\cdot\frac{\xi}{s}\right) h\left(\frac{\tilde{z}}{s}\right)\right) N(\tilde{z})\,\mathrm{d}\tilde{z}\right\}\frac{\mathrm{d}s}{s} \\
&= \int_1^\infty a\left(\frac{\xi}{s}\right)\frac{\mathrm{d}s}{s} - \mathrm{i}\xi\cdot\int_1^\infty \left\{\int \frac{\tilde{z}}{s}\left(h\left(\frac{\tilde{z}}{s}\right) - h(\tilde{z})\right) N(\tilde{z})\,\mathrm{d}\tilde{z}\right\}\frac{\mathrm{d}s}{s} \\
&= A(\xi) - \mathrm{i}\xi\cdot b_A,
\end{aligned}$$

where b_A is defined by (22). Properties (4) of the Lévy measure ν imply that b_A is well defined. We conclude that A is the characteristic exponent of an infinitely divisible law u_∞ whose drift is $b - b_A$, whose Gaussian part is σ and whose Lévy measure is $N_\infty(z)\,\mathrm{d}z$. □

Proof of Theorem 1. The proof of the first part is exactly given by Lemma 2.

We now turn to the second part of the theorem. Proposition 1 gives for $t \geqslant 0$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Ent}^{\Phi}_{u_\infty}\big(v(t,\cdot)\big) &= -\int \Phi''\big(v(t,\cdot)\big)\nabla v(t,\cdot)\cdot\sigma\nabla v(t,\cdot) u_\infty\,\mathrm{d}x \\
&\quad - \iint D_\Phi\big(v(t,x), v(t,x-z)\big)\nu(\mathrm{d}z) u_\infty\,\mathrm{d}x \\
&= -\int \Phi''\big(v(t,\cdot)\big)\nabla v(t,\cdot)\cdot\sigma\nabla v(t,\cdot) u_\infty\,\mathrm{d}x \\
&\quad - \iint D_\Phi\big(v(t,x), v(t,x+z)\big)\nu(\mathrm{d}z) u_\infty\,\mathrm{d}x,
\end{aligned}$$

where we used the fact that ν is even. It is now enough to prove the following inequality

$$\mathrm{Ent}_{u_\infty}^{\Phi}(v) \leqslant C\int \Phi''\big(v(t,\cdot)\big)\nabla v(t,\cdot)\cdot\sigma\nabla v u_\infty\,\mathrm{d}x + C\iint D_\Phi\big(v(x),v(x+z)\big)\nu(\mathrm{d}z)u_\infty(x)\,\mathrm{d}x$$

for some constant C not depending on v and Gronwall's lemma permits to conclude. But this inequality is a direct consequence of (19) for the infinitely divisible law u_∞. Remark also that if one considers another nonnegative steady state $\tilde{u}_\infty$ such that $\mathrm{Ent}_{u_\infty}^{\Phi}(\tilde{u}_\infty/u_\infty) < +\infty$ for some Φ strictly convex and satisfying the assumptions of the theorem, then $\mathrm{Ent}_{u_\infty}^{\Phi}(\tilde{u}_\infty/u_\infty) = 0$ and we conclude that $\tilde{u}_\infty = u_\infty$. □

4.1. Discussion of condition (14) *and examples*

Let us discuss the condition we impose in order to get exponential decay. We point out that equality in this condition holds true only for α-stable operators and we give a necessary condition on the behaviour of the Lévy measure at infinity if one knows that it decreases faster than $|x|^{-d}$.

Proposition 2.

- *Equality $N_\infty = N/\lambda$ holds if and only if ψ is positively homogeneous of index $\lambda \in (0,2]$, i.e.*

  $$\psi(t\xi) = t^\lambda\psi(\xi) \quad \text{for any } t > 0, \xi \in \mathbb{R}^d.$$

  In this case, we get $A = \psi/\lambda$ and $b_A = 0$. Note that in the limit case $\lambda = 2$, then we get $N_\infty = N/2 = 0$.
- *If $|x|^d N(x) \to 0$ as $|x| \to +\infty$, then the densities N and N_∞ satisfy:*

  $$N = -\operatorname{div}(xN_\infty).$$

- *In this case, condition* (14) *is equivalent to:*

  $$\begin{cases} N_\infty(tx) \leqslant N_\infty(x)t^{-d-1/C} & \text{if } t \geqslant 1, \\ N_\infty(tx) \geqslant N_\infty(x)t^{-d-1/C} & \text{if } 0 < t \leqslant 1. \end{cases} \tag{24}$$

Proof. The first item simply follows from the definition of A.

Let us first prove the second item.

$$\begin{aligned} N(x) &= -\Big(\lim_{t\to+\infty} t^d N(tx)\Big) + 1^d N(1\times x) = -\int_1^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\big(t^d N(tx)\big)\,\mathrm{d}t \\ &= -\mathrm{d}N_\infty(x) - x\cdot\nabla N_\infty(x) = -\operatorname{div}(xN_\infty). \end{aligned}$$

To prove the third item, use the first one to rewrite (14) as follows:

$$x\cdot\nabla N_\infty(x) + \left(d + \frac{1}{C}\right)N_\infty(x) \leqslant 0.$$

Integrate over $[1,t]$ for $t \geqslant 1$ and $[t,1]$ for $t \leqslant 1$ to get the result. □

Example 1. In $\mathbb{R}$, the Lévy measure $\frac{1}{|z|}\mathrm{e}^{-|z|}$ does not satisfy condition (14). Indeed, it is equivalent to:

$$\int_1^{+\infty} \frac{\mathrm{e}^{-|x|(s-1)}}{s}\,\mathrm{d}s \leqslant C$$

and the monotone convergence theorem implies that the left-hand side of this inequality goes to $+\infty$ as $|x| \to +\infty$.

Appendix

This is a sketch of the proof given by Chafaï. The difference is that we treat here the case of a Lévy operator with eventually a diffusion part ($\sigma \neq 0$ in (17)).

Proof of Theorem 2. Let μ be the infinitely divisible law. There then exists ψ a characteristic exponent such that the characteristic function of μ is $\exp\psi$. Now consider $K(t,\cdot)$ the law associated with $\exp(t\psi)$. If $\mathcal{I}$ defined by (3), we know that in this case the solution of

$$\begin{aligned}&\partial_t u = \mathcal{I}[u],\\ &u(0,x) = u_0(x)\end{aligned}$$

is $K(t,\cdot)\star u_0$. Let v be a smooth and positive function on $\mathbb{R}$. Define next for $s \in [0,t]$:

$$\Psi(s) = K(s,\cdot)\star \Phi\big(K(t-s,\cdot)\star v\big).$$

It is convenient to write g for $K(t-s,\cdot)\star v$.

One gets, for $s \in [0,t]$,

$$\begin{aligned}\Psi'(s) &= K(s,\cdot)\star\big(\mathcal{I}\big(\Phi(g)\big) - \Phi'(g)\big(\mathcal{I}(g)\big)\big)\\ &= K(s,\cdot)\star\big(\Phi''(g)\nabla g\cdot\sigma\nabla g\big) + K(s,\cdot)\star\bigg(\int D_\Phi\big(g(\cdot+z),g(\cdot)\big)\nu_\mu(\mathrm{d}z)\bigg).\end{aligned}$$

Using the fact that $(a,b)\mapsto D_\Phi(a+b,a)$ and $(x,y)\mapsto \Phi''(x)y\cdot\sigma y$ are convex, one gets by Jensen's inequality,

$$\Psi'(s) \leqslant K(t,\cdot)\star\big(\Phi''(v)\nabla v\cdot\sigma\nabla v\big) + K(t,\cdot)\star\bigg(\int D_\Phi\big(v(\cdot+z),v(\cdot)\big)\nu_\mu(\mathrm{d}z)\bigg).$$

Integrate over $s\in[0,t]$, we obtain for $t \geqslant 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}&K(t,\cdot)\star\big(\Phi(v)\big)(x) - \Phi\big(K(t,\cdot)\star v\big)(x)\\ &\quad\leqslant tK(t,\cdot)\star\big(\Phi''(v)\nabla v\cdot\sigma\nabla v\big)(x) + tK(t,\cdot)\star\bigg(\int D_\Phi\big(v(\cdot+z),v(\cdot)\big)\nu_\mu(\mathrm{d}z)\bigg)(x).\end{aligned}\tag{25}$$

Then choose $t = 1$ and $x = 0$ in inequality (25) and get:

$$\mathrm{Ent}_{K_\psi}^{\Phi}(v) \leqslant \int \Phi''(v)\nabla v \cdot \sigma \nabla v K_\psi(\mathrm{d}x) + \iint D_\Phi\big(v(x), v(x+z)\big)\nu_\mu(\mathrm{d}z)K_\psi(\mathrm{d}x). \qquad \square$$

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