

Logarithmic Sobolev inequalities: Regularizing effect of Lévy operators and asymptotic convergence in the Lévy–Fokker–Planck equation

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(Received April 2008; final version received November 2008)

In this paper we study some applications of the Lévy logarithmic Sobolev inequality to the study of the regularity of the solution of the fractal heat equation, i.e. the heat equation where the Laplacian is replaced with the fractional Laplacian. It is also used to the study of the asymptotic behaviour of the Lévy–Ornstein–Uhlenbeck process.

Keywords: Fokker–Planck equation; Ornstein–Uhlenbeck equation; Lévy operator; Φ -entropy inequalities; entropy production method; logarithmic Sobolev inequalities; fractional Laplacian; ultracontractivity

Mathematics Subject Classification: 46N20; 47G20; 35K15

1. Introduction

On one hand, regularity results for the heat equation in \mathbb{R}^d , such as ultracontractivity, can be obtained by using a Euclidean logarithmic Sobolev inequality. On the other hand, the asymptotic behaviour of the Ornstein–Uhlenbeck semigroup, precisely the optimal exponential decay to the equilibrium, is proved by using either Poincaré or logarithmic Sobolev inequalities. See Ref. [2] for a review of the subject.

The heat equation or the Ornstein–Uhlenbeck semigroup are associated with the Laplacian, the infinitesimal generator of the Brownian motion. The Brownian motion makes part of a large class of stochastic processes called Lévy processes. In this note, we would like to describe how the properties we just mentioned (ultracontractivity and exponential decay) are sometimes true if we replace the Laplacian with the infinitesimal operator of a general Lévy process. These generators are integrodifferential and are referred to as Lévy operators.

In Section 2, we give a short introduction to Lévy processes and Lévy operators. Two important inequalities are also given: the Euclidean logarithmic inequality in the case of the α -stable process; and a modified logarithmic Sobolev inequality for infinitely divisible probability measures; the latter inequality generalizes the logarithmic Sobolev inequality given by Gross [10].

In Section 3, we prove that the heat equation associated with a α -stable process satisfies the property of ultracontractivity.

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In Section 4, we consider the Ornstein–Uhlenbeck semigroup or equivalently the Fokker–Planck semi-group associated with a general Lévy operator. We will see that, under proper assumptions on the operator, those semi-groups converge to the unique steady state. Results of Section 4 are presented in full details in Ref. [9] and are extensions of the paper of Biler and Karch [5].

2. Preliminaries

2.1 Lévy operators

Let us recall basic definitions about Lévy operators and introduce notations. See, for example Ref. [3] for further details.

2.1.1 Characteristic exponents and Lévy measures

Let $d \in \mathbb{N}^*$. Because of the definition of a Lévy process in \mathbb{R}^d (a process with stationary and independent increment), the law μ_t of such a process $(X_t)_{t \geq 0}$ at time $t > 0$ is *infinitely divisible*, i.e. it can be written for all $n \geq 1$ under the form

$$\mu_t = \underbrace{\mu_n \star \cdots \star \mu_n}_{n \text{ times}}$$

for some probability μ_n (depending on n). Using this property, it can be shown that the characteristic function $\phi_{X_t}(\xi) := E(\exp(i\xi \cdot X_t))$ (i.e. its Fourier transform) of the law of X_t can be written under the form $\exp(t\psi(\xi))$ for a function ψ called the *characteristic exponent*. The Lévy–Khinchine formula states that ψ can be described with exactly three parameters (σ, b, ν) where σ is a nonnegative symmetric $d \times d$ matrix, $b \in \mathbb{R}^d$ and ν is a nonnegative singular measure on \mathbb{R}^d that satisfies

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int \min(1, |z|^2) \nu(dz) < +\infty. \quad (1)$$

Then ψ can be written under the form

$$\psi(\xi) = -\sigma\xi \cdot \xi + ib \cdot \xi + a(\xi), \quad (2)$$

where a is given by

$$a(\xi) = \int (e^{iz \cdot \xi} - 1 - iz \cdot \xi h(z)) \nu(dz),$$

with $h(z) = 1/(1 + |z|^2)$.

The matrix σ characterizes the diffusion (or Gaussian) part of the operator (with eventually $\sigma = 0$), while b characterizes the drift part and ν is called a Lévy measure; it characterizes the pure jump part. The support of the measure ν represents the possible jumps of the process.

A Lévy operator \mathcal{I} is the infinitesimal generator associated with the Lévy process and the Lévy–Khinchine formula implies that it has the following form

$$\mathcal{I}[u](x) = \operatorname{div}(\sigma \nabla u)(x) + b \cdot \nabla u(x) + \int_{\mathbb{R}^d} (u(x+z) - u(x) - \nabla u(x) \cdot zh(z)) \nu(dz). \quad (3)$$

2.1.2 The pseudo-differential point of view

It is convenient to introduce the operator \mathcal{I}_g associated with the Gaussian part

$$\mathcal{I}_g(u) = \operatorname{div}(\sigma \nabla u) + b \cdot \nabla u$$

and the operator \mathcal{I}_a associated with the pure jump part

$$\mathcal{I}_a(u) = \int_{\mathbb{R}^d} (u(x+z) - u(x) - \nabla u(x) \cdot zh(z)) \nu(dz).$$

The operator \mathcal{I}_a can be seen as a pseudo-differential operator of symbol a

$$\mathcal{I}_a(u) = \mathcal{F}(a \times \mathcal{F}^{-1} u),$$

where \mathcal{F} stands for the Fourier transform (see Theorem 3.3.3, p. 139 of Ref. [3]). Here, we choose the probabilistic convention in defining, for all function $w \in L^1(\mathbb{R}^d)$,

$$\forall \xi \in \mathbb{R}^d, \quad \hat{w}(\xi) = \mathcal{F}(w)(\xi) = \int e^{ix \cdot \xi} w(x) dx. \tag{4}$$

Moreover, using the Fourier interpretation of the Lévy operator one gets the following integration by parts formula: if \mathcal{I} is a Lévy operator with parameters (b, σ, ν) then for any smooth functions u, v one gets

$$\int v \mathcal{I}[u] dx = \int u \check{\mathcal{I}}[v] dx, \tag{5}$$

where $\check{\mathcal{I}}$ is the Lévy operator whose parameters are $(-b, \sigma, \check{\nu})$ with $\check{\nu}(dz) = \nu(-dz)$.

2.1.3 Multi-fractal and α -stable Lévy operators

Lévy operators whose characteristic exponent is positively homogeneous of index $\alpha \in (0, 2]$ are called α -stable. The fractional Laplacian corresponds to a particular α -stable Lévy process with characteristic exponent $\psi(\xi) = |\xi|^\alpha$, where $|\cdot|$ is the Euclidean norm in \mathbb{R}^d . In the case $\alpha \in (0, 2)$, one gets

$$b = 0, \quad \sigma = 0 \quad \text{and} \quad \nu(dz) = \frac{dz}{|z|^{d+\alpha}}.$$

Hence, it is a pure jump process, i.e. it has neither a drift part nor a diffusion one.

Lévy operators whose characteristic exponent can be written as $\psi(\xi) = \sum_{i=1}^n \psi_i(\xi)$ where ψ_i is α_i -homogenous with $\alpha_i \in (0, 2]$, are often referred to as multi-fractal Lévy operators.

The α -stable operators play a central role in this paper. Let $-g_\alpha[\cdot]$ denotes the Lévy operator associated to the α -stable Lévy process whose characteristic exponent is $\psi(\xi) = |\xi|^\alpha, \alpha \in (0, 2]$. In the limit case, when $\alpha = 2$, $g_2[\cdot]$ is the Laplacian operator on \mathbb{R}^d .

2.2 A Euclidean logarithmic Sobolev inequality

The logarithmic Sobolev inequality for the Lebesgue measure is a useful functional inequality in the study of the fractional heat equation: $\partial_t u + g_\alpha[u] = 0$. Such an inequality has been established by A. Cotsiolis and N. K. Tavoularis.

THEOREM 1 ([7]). Let $\alpha \in (0, 2]$ then for any smooth function f on \mathbb{R}^d such that $\int f^2 dx = 1$, the following optimal Euclidean logarithmic Sobolev inequality holds true

$$\text{Ent}_{dx}(f^2) := \int f^2 \log f^2 dx \leq \frac{n}{\alpha} \log \left(\frac{\alpha C^{\alpha/n}}{n \pi^{\alpha/2} e^{1-\alpha}} \int (g_{\alpha/2}[f])^2 dx \right), \tag{6}$$

where $C = (2\Gamma(n/\alpha))/(\alpha\Gamma(n/2))$.

This inequality is a generalization of the classical Euclidean logarithmic Sobolev inequality given by Weissler [11]. We are not convinced about the constant C given in [7]. Nevertheless, even if the constant is not true the paper is correct eventually changing some constants.

2.3 A modified logarithmic Sobolev inequality

In the sequel, we will need another functional inequality proved by Ané and Ledoux [1] in the particular case of the Poisson measure and then generalized by Wu [12] and Chafaï [6] for all infinite measurable laws. In order to state the most general result, we first introduce Φ -entropies.

Let $\Phi : \mathbb{R}^+ \mapsto \mathbb{R}$ a smooth convex function and define the Φ -entropy: for any nonnegative function f ,

$$\text{Ent}_\mu^\Phi(f) := \int \Phi(f) d\mu - \Phi \left(\int f d\mu \right),$$

where μ is a probability measure. When $\Phi(x) = x \log x$, we recover the classical entropy introduced in (6).

For a convex function Φ we note by D_Φ the so-called *Bergman distance* defined by

$$\forall (a, b) \in \mathbb{R}^+, \quad D_\Phi(a, b) := \Phi(a) - \Phi(b) - \Phi'(b)(a - b) \geq 0. \tag{7}$$

THEOREM 2 ([1,6,12]). Assume that Φ satisfies the following properties:

$$\begin{cases} (a, b) \mapsto D_\Phi(a + b, b) \\ (r, y) \mapsto \Phi''(r)y \cdot \sigma y \end{cases} \text{ are convex on } \begin{cases} \{ a + b \geq 0, b \geq 0 \} \\ \mathbb{R}^+ \times \mathbb{R}^{2d} \end{cases}, \tag{8}$$

where D_Φ has been defined by (7).

Consider an infinitely divisible law μ on \mathbb{R}^d . Then for all smooth positive functions v ,

$$\text{Ent}_\mu^\Phi(v) \leq \int \Phi''(v) \nabla v \cdot \sigma \nabla v \mu(dx) + \iint D_\Phi(v(x), v(x+z)) \nu_\mu(dz) \mu(dx), \tag{9}$$

where ν_μ and σ denote, respectively, the Lévy measure and the diffusion matrix associated with μ .

Remark that the drift of the law plays no role in this functional inequality. Inequality (9) is proved in Ref. [12] for $\Phi(x) = x^2$ or $\Phi(x) = x \log x$ and in this general form in Ref. [6].

An important special case is the following one: $\Phi(r) = r^2/2$. A simple computation shows that the Bregman distance D_Φ in this case is $D_\Phi(a, b) = (a - b)^2/2$ so that Ent_μ^Φ reduces to the variance (up to a constant). See also the appendix of Ref. [9] for a proof of this inequality.

3. Regularity of the heat equation driven by a Lévy process

In this section, we study regularity properties of solutions of the fractional heat equation

$$\partial_t u + g_\alpha[u] = 0. \tag{10}$$

In particular, we are interested in the ultracontractivity of this equation.

THEOREM 3. Let $\alpha \in (0, 2)$ and $(P_t)_{t \geq 0}$ denote the semigroup associated with the Equation (10). Consider a smooth initial datum f . Then for all $t > 0$ and $q \geq p \geq 2$

$$\|P_t f\|_q \leq \|f\|_p \left(\frac{\mathcal{A}n(q-p)}{2\alpha t} \right)^{\frac{n(q-p)}{\alpha p q}} p^{n/(q\alpha)} q^{n/(p\alpha)}, \tag{11}$$

where $\|\cdot\|_p$ denotes the $L^p(dx)$ norm and

$$\mathcal{A} = \frac{\alpha((2\Gamma(n/\alpha))/(\alpha\Gamma(n/2)))^{\alpha/n}}{n\pi^{\alpha/2}e^{1-\alpha}}. \tag{12}$$

These results can be found in the case of the Laplacian in Ref. [4]. We would like to mention that in the classical case, inequality (11) for all $q \geq p \geq 2$ and the Euclidean logarithmic Sobolev inequality are equivalent which is not clear in our case.

Letting $q \rightarrow +\infty$ and choosing $p = 2$ in Theorem 3 yields:

COROLLARY 1 (ULTRA CONTRACTIVITY). The semigroup $(P_t)_{t \geq 0}$ is *ultracontractive*, i.e. it satisfies for all smooth function f

$$\|P_t f\|_\infty \leq \|f\|_2 \left(\frac{\mathcal{A}n}{2\alpha t} \right)^{n/(2\alpha)},$$

where \mathcal{A} is given by (12).

We next recall a useful inequality satisfied by Lévy operator. Such an inequality is sometimes called Kato inequality. See for instance the proof given in Ref. [8].

LEMMA 1. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be convex and $u \in C_b^2(\mathbb{R}^N)$. Then, if ϕ is differentiable at $u(x)$, we have:

$$g_\alpha[\phi(u)](x) \leq \phi'(u(x))g_\alpha[u](x).$$

We will also use the simple fact that $\int u g_\alpha[v] dx = \int |\xi|^\alpha \hat{u} \hat{v} dx$. In particular, for all smooth function u on \mathbb{R}^d ,

$$\int u g_\alpha[u] dx = \int (g_{\alpha/2}[u])^2 dx.$$

Proof of Theorem 3. Let $u(t, x)$ denote $P_t f(x)$ and consider an increasing function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\varphi(0) = \alpha$. Define a function $F(t) = \|u(t)\|_{\varphi(t)}$ and let us study its derivative. A computation gives

$$\begin{aligned} \frac{\varphi^2}{\varphi'} F^{\varphi-1} F' &= \text{Ent}_{dx}(|u|^\varphi) + \frac{\varphi^2}{\varphi'} \int |u|^{\varphi-1} \partial_t u dx \\ &= \text{Ent}_{dx}(|u|^\varphi) - \frac{\varphi^2}{\varphi'} \int |u|^{\varphi-1} g_\alpha[u] dx. \end{aligned}$$

Assume that $\varphi \geq 2$. In this case, one can apply Lemma 1 with $\phi(\cdot) = |\cdot|^{\varphi/2}$ and get

$$-\varphi |u|^{\varphi-1} g_\alpha[u] \leq -2 |u|^{\varphi/2} g_\alpha[|u|^{\varphi/2}];$$

integrating over \mathbb{R}^n implies

$$-\varphi \int |u|^{\varphi-1} g_\alpha[u] dx \leq -2 \int |u|^{\varphi/2} g_\alpha[|u|^{\varphi/2}] dx = -2 \int (g_{\alpha/2}[|u|^{\varphi/2}])^2 dx$$

so that

$$\frac{\varphi^2}{\varphi'} F^{\varphi-1} F' \leq \text{Ent}_{dx}((|u|^{\varphi/2})^2) - \frac{2\varphi}{\varphi'} \int (g_{\alpha/2}[|u|^{\varphi/2}])^2 dx.$$

Apply now (6) with $|u|^{\varphi/2} / \sqrt{\int |u|^\varphi dx}$ and get

$$\frac{\varphi^2}{\varphi'} F^{\varphi-1} F' \leq \frac{n}{\alpha} \int |u|^\varphi dx \log \left(\mathcal{A} \frac{\int (g_{\alpha/2}[|u|^{\varphi/2}])^2 dx}{\int |u|^\varphi dx} \right) - \frac{2\varphi}{\varphi'} \int (g_{\alpha/2}[|u|^{\varphi/2}])^2 dx.$$

Use now the concavity of \log ; for any $x \in \mathbb{R}$, we have

$$\frac{\varphi^2}{\varphi'} F^{\varphi-1} F' \leq \left(\frac{n}{\alpha x} - \frac{2\varphi}{\varphi'} \right) \int (g_{\alpha/2}[|u|^{\varphi/2}])^2 dx + \frac{n}{\alpha} \log(\mathcal{A}x/e) \int |u|^\varphi dx.$$

Choose next x such that $2\varphi/\varphi' = n/(\alpha x)$. We now obtain that F satisfies

$$\frac{F'}{F} \leq \frac{n\varphi'}{\alpha\varphi^2} \log \left(\frac{\mathcal{A}n\varphi'}{2e\alpha\varphi} \right),$$

so that for all $t > 0$,

$$\|P_t f\|_{\varphi(t)} = \|u(t)\|_{\varphi(t)} = F(t) \leq F(0) \exp \left\{ \int_0^t \frac{n\varphi'(s)}{\alpha\varphi^2(s)} \log \left(\frac{\mathcal{A}n\varphi'(s)}{2e\alpha\varphi(s)} \right) ds \right\}.$$

We now minimize the right hand side of the previous inequality w.r.t. functions φ such that $\varphi(0) = p$ and $\varphi(t) = q$. Associated Euler's equation reads: $2\varphi'2 = \varphi''\varphi$ so that we choose $\varphi(s) = tpq/((p - q)s + qt)$ and one can check that with such a choice of φ , Inequality (11) is proved. \square

4. Asymptotic behaviour of a Lévy–Fokker–Planck equation

The results presented in this section are coming from Ref. [9]. We are looking for asymptotic behaviour of the solution of a Fokker–Planck equation where the classical Laplacian is replaced with a Lévy operator. Precisely, recalling that \mathcal{I} is defined in (3), we consider the Lévy–Fokker–Planck equation

$$\partial_t u = \mathcal{I}[u] + \operatorname{div}(uF), \quad x \in \mathbb{R}^d, t > 0 \tag{13}$$

submitted to the initial condition

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^d,$$

where u_0 is nonnegative and in $L^1(\mathbb{R}^d)$ and F is a given proper force for which there exists a nonnegative steady state (see below).

4.1 The Φ -entropy and associated Fisher information

In this subsection, we are interested in the (time) derivative of the Φ -entropy associated to the Lévy–Fokker–Planck equation when a steady state is given.

PROPOSITION 1. Assume that there exists u_∞ , a steady state of (13), a positive solution of the equation:

$$\mathcal{I}[u_\infty] + \operatorname{div}(u_\infty F) = 0, \tag{14}$$

such that $\int u_\infty dx = 1$. Assume that the initial condition u_0 is nonnegative and satisfies $\operatorname{Ent}_{u_\infty}^\Phi(u_0/u_\infty) < \infty$.

Then for any convex smooth function $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ and any $t \geq 0$, the solution u of (13) satisfies

$$\begin{aligned} \forall t \geq 0, \quad \frac{d}{dt} \operatorname{Ent}_{u_\infty}^\Phi(v) = & - \int \Phi''(v) \nabla v \cdot \sigma \nabla v u_\infty dx \\ & - \iint D_\Phi(v(t, x), v(t, x - z)) \nu(dz) u_\infty(x) dx, \end{aligned} \tag{15}$$

where $v(t, x) = (u(t, x))/(u_\infty(x))$ and ν is the Lévy measure appearing in the definition of the operator \mathcal{I} and D_Φ is defined in (7).

In order to prove Proposition 1, since the Φ -entropy involves the function $v(t, x) = (u(t, x))/(u_\infty(x))$, its derivative makes appear $\partial_t v$ and it is natural to ask ourselves

which partial differential equation v satisfies. Using (14) one gets by a simple computation

$$\begin{aligned}\partial_t v &= \frac{1}{u_\infty} (\mathcal{I}[u_\infty v] + \operatorname{div}(u_\infty v F)) \\ &= \frac{1}{u_\infty} (\mathcal{I}[u_\infty v] - \mathcal{I}[u_\infty]v) + F \cdot \nabla v =: Lv.\end{aligned}\tag{16}$$

In the case where $\mathcal{I}[u] = \Delta u$ (i.e. σ is the identity matrix, $b = 0$ and $a = 0$), Equation (16) becomes

$$\partial_t v = \Delta v - F \cdot \nabla v$$

and is known as the Ornstein–Uhlenbeck equation. This is the reason why we will refer to Equation (16) as the Lévy–Ornstein–Uhlenbeck equation. We next give a simpler formulation for the Lévy–Ornstein–Uhlenbeck operator.

LEMMA 2 (LÉVY–ORNSTEIN–UHLENBECK EQUATION). If the integrodifferential operator on the right-hand side of (16) is denoted by L , we have for all smooth functions w_1 and w_2

$$\int w_1 L w_2 u_\infty dx = \int (\check{\mathcal{I}}[w_1] - F \cdot \nabla w_1) w_2 u_\infty dx,$$

where $\check{\mathcal{I}}$ is the Lévy operator whose parameters are $(-b, \sigma, \check{\nu})$ with $\check{\nu}(dz) = \nu(-dz)$. This can be expressed by the formula: $L^* = \check{\mathcal{I}} - F \cdot \nabla$ where duality is understood with respect to the measure $u_\infty dx$.

Proof. The main tool is the integration by parts for the operator \mathcal{I} , see Equation (5). For any smooth functions u, v one gets

$$\int v \mathcal{I}[u] dx = \int u \check{\mathcal{I}}[v] dx.$$

If w_1 and w_2 are two smooth functions on \mathbb{R}^d , then

$$\begin{aligned}\int w_1 L w_2 u_\infty dx &= \int w_1 (\mathcal{I}[u_\infty w_2] - \mathcal{I}[u_\infty]w_2 + u_\infty F \cdot \nabla w_2) dx \\ &= \int w_2 \check{\mathcal{I}}[w_1] u_\infty dx - \int \mathcal{I}[u_\infty] w_1 w_2 dx - \int \operatorname{div}(u_\infty w_1 F) w_2 dx \\ &= \int u_\infty (\check{\mathcal{I}}[w_1] - F \cdot \nabla w_1) w_2 dx.\end{aligned}\quad \square$$

Proof of Proposition 1. By using Lemma 2 with $v = u/u_\infty$, we get

$$\begin{aligned}\frac{d}{dt} \operatorname{Ent}_{u_\infty}^\Phi(v) &= \int \Phi'(v) \partial_t v u_\infty dx = \int \Phi'(v) L v u_\infty dx \\ &= \int \check{\mathcal{I}}[\Phi'(v)] v u_\infty dx - \int F \cdot \nabla(\Phi'(v)) v u_\infty dx.\end{aligned}$$

If now one remarks that $r\Phi''(r) = (r\Phi'(r) - \Phi(r))'$, we get

$$\begin{aligned} \frac{d}{dt} \text{Ent}_{u_\infty}^\Phi(v) &= \int v\check{\mathcal{I}}[\Phi'(v)]u_\infty dx - \int F \cdot \nabla(v\Phi'(v) - \Phi(v))u_\infty dx \\ &= \int v\check{\mathcal{I}}[\Phi'(v)]u_\infty dx + \int \text{div}(u_\infty F)(v\Phi'(v) - \Phi(v))dx \\ &= \int v\check{\mathcal{I}}[\Phi'(v)]u_\infty dx - \int \mathcal{I}[u_\infty](v\Phi'(v) - \Phi(v))dx \\ &= \int (v\check{\mathcal{I}}[\Phi'(v)] - \check{\mathcal{I}}[v\Phi'(v)] + \check{\mathcal{I}}[\Phi(v)])u_\infty dx, \\ \frac{d}{dt} \text{Ent}_{u_\infty}^\Phi(v) &= \int (v\check{\mathcal{I}}_g[\Phi'(v)] - \check{\mathcal{I}}_g[v\Phi'(v)] + \check{\mathcal{I}}_g[\Phi(v)])u_\infty dx \\ &\quad + \int (v\check{\mathcal{I}}_a[\Phi'(v)] - \check{\mathcal{I}}_a[v\Phi'(v)] + \check{\mathcal{I}}_a[\Phi(v)])u_\infty dx \\ &= - \int \Phi''(v)\nabla v \cdot \sigma \nabla v u_\infty dx \\ &\quad + \iint (v(x)(\Phi'(v(x+z)) - \Phi'(v(x))) - v(x+z)\Phi'(v(x+z)) \\ &\quad + v(x)\Phi'(v(x)) + \Phi(v(x+z)) - \Phi(v(x)))\check{\nu}(dz)u_\infty(x)dx, \\ \frac{d}{dt} \text{Ent}_{u_\infty}^\Phi(v) &= - \int \Phi''(v)\nabla v \cdot \sigma \nabla v u_\infty dx \\ &\quad - \iint (\Phi(v(x)) - \Phi(v(x+z)) - \Phi'(v(x+z)) \\ &\quad \times (v(x) - v(x+z)))\check{\nu}(dz)u_\infty(x)dx. \end{aligned}$$

Then the definitions of the Bregman distance and of the Lévy measure $\check{\nu}$ give

$$\frac{d}{dt} \text{Ent}_{u_\infty}^\Phi(v) = - \int \Phi''(v)\nabla v \cdot \sigma \nabla v u_\infty dx - \iint D_\Phi(v(x), v(x-z))\nu(dz)u_\infty(x)dx.$$

□

4.2 Exponential decay of the Lévy–Fokker–Planck equation

We give now assumptions such that there exists a steady state of the Lévy–Fokker–planck equation. For this section, we need to assume for that the force is given by $F(x) = x$.

THEOREM 4 (EXPONENTIAL DECAY TO EQUILIBRIUM). Assume that $F(x) = x$ and the operator \mathcal{I} is the infinitesimal generator of a Lévy process whose Lévy measure is denoted by ν . We assume that ν has a density N with respect to the Lebesgue measure and that N satisfies

$$\int_{\mathbb{R}^d \setminus B} \ln|z|N(z)dz < +\infty, \tag{17}$$

where B is the unit ball in \mathbb{R}^d .

- Then there exists a steady state u_∞ , i.e. a nonnegative solution of (14) satisfying $\int u_\infty dx = 1$.
- If moreover N is even and for all $z \in \mathbb{R}^d$,

$$\int_1^{+\infty} N(sz)s^{d-1} ds \leq CN(z) \tag{18}$$

for some constant $C \geq 0$, then for any smooth convex function Φ such that Condition (8) is satisfied, the Φ -entropy of the solution u of (13) goes to 0 exponentially. Precisely, for any nonnegative initial datum u_0 such that $\text{Ent}_{u_\infty}^\Phi(u_0/u_\infty) < \infty$, one gets

$$\forall t \geq 0, \quad \text{Ent}_{u_\infty}^\Phi\left(\frac{u(t)}{u_\infty}\right) \leq e^{-(t/c)} \text{Ent}_{u_\infty}^\Phi\left(\frac{u_0}{u_\infty}\right) \tag{19}$$

with C appearing in (18).

To prove the first part of the Theorem, the existence of the steady state, we need to state the following lemma.

LEMMA 3. Assume that the Lévy measure ν has a density N with respect to the Lebesgue measure and that it satisfies (17). There then exists a steady state u_∞ , i.e. a solution of (14). Moreover, it is an infinitely divisible measure whose characteristic exponent A is defined by

$$A(\xi) = -\xi \cdot \sigma \xi + ib \cdot \xi + \int_0^1 a(s\xi) \frac{ds}{s}. \tag{20}$$

Moreover, parameters of the characteristic exponent A are $(\sigma, b - b_A, N_\infty dx)$ where

$$b_A = \int_0^1 \int z \frac{(1 - \tau^2)|z|^2}{(1 + \tau^2|z|^2)(1 + |z|^2)} d\tau N(z) dz, \tag{21}$$

and

$$N_\infty(z) = \int_1^\infty N(tz)t^{d-1} dt. \tag{22}$$

Note that the Lévy measure ν_∞ associated to the characteristic exponent A has a density N_∞ with respect to the Lebesgue measure.

Remark 1. In the general case, Condition (18) precisely says that $N_\infty \leq CN$ which can be written in terms of measures as follows: $\nu_\infty \leq C\nu$.

Proof of Lemma 3. Let us start as in Ref. [5] in Section 3. At least formally, the Fourier transform \hat{u}_∞ of any steady state u_∞ satisfies

$$\psi(\xi)\hat{u}_\infty + \xi \cdot \nabla \hat{u}_\infty = 0,$$

where ψ is the characteristic exponent of the Lévy operator \mathcal{L} . So that $\hat{u}_\infty = \exp(-A)$ with A such that

$$\nabla A(\xi) \cdot \xi = \psi(\xi).$$

The solution of this equation is precisely given by (20). It is not clear that A is well defined and is the characteristic exponent of an infinitely divisible measure; this is what we prove next. This will imply in particular that $\mathcal{F}^{-1}(\exp(-A))$ is a nonnegative function.

Define the nonnegative N_∞ by Equation (22). This integral of a nonnegative function is finite since for any $R > 0$, if $d\sigma$ denotes the uniform measure on the unit sphere S^{d-1} we get,

$$\int_R^\infty \int_{|D|=1} N(\tau D) \tau^{d-1} d\tau d\sigma(D) = \int_{|y|\geq R} N(y) dy < +\infty.$$

We conclude that for any $r \geq R > 0$ and almost every D on the unit sphere (where the set of null measure depends only on R),

$$r^d N_\infty(rD) = \int_r^\infty N(\tau D) \tau^{d-1} d\tau < +\infty$$

so that $N_\infty(z)$ is well-defined almost everywhere outside B_R . Choose now a sequence $R_n \rightarrow 0$ and conclude.

Let us define $I(r) = \int_{|D|=1} N(rD) d\sigma(D)$ and $I_\infty = \int_{|D|=1} N_\infty(rD) d\sigma(D)$ in an analogous way. The previous equality implies that

$$r^d I_\infty(r) = \int_r^{+\infty} I(\tau) \tau^{d-1} d\tau.$$

We conclude that

$$\begin{aligned} \int_{|z|\leq 1} |z|^2 N_\infty(z) dz &= \int_0^1 I_\infty(r) r^{d+1} dr = \int_0^1 r \int_r^{+\infty} I(\tau) \tau^{d-1} d\tau dr \\ &= \frac{1}{2} \int_{|z|\geq 1} N(z) dz + \frac{1}{2} \int_{|z|\leq 1} |z|^2 N(z) dz < +\infty, \\ \int_{|z|\geq 1} N_\infty(z) dz &= \int_1^{+\infty} I_\infty(r) r^{d-1} dr = \int_1^{+\infty} \frac{1}{r} \int_r^{+\infty} I(\tau) \tau^{d-1} d\tau dr \\ &= \int_{|z|\geq 1} \log|z| N(z) dz. \end{aligned}$$

Hence, we have $\int \min(1, |z|^2)N_\infty(z)dz < +\infty$. We conclude that it is a Lévy measure. Now consider the associated characteristic exponent

$$\tilde{A}(\xi) = ib \cdot \xi - \sigma\xi \cdot \xi + \int (e^{iz \cdot \xi} - 1 - i(z \cdot \xi)h(z))N_\infty(z)dz.$$

Now compute

$$\begin{aligned} \tilde{A}(\xi) - ib \cdot \xi + \sigma\xi \cdot \xi &= \int_1^\infty \int (e^{iz \cdot \xi} - 1 - i(z \cdot \xi)h(z))N(sz)s^{d-1} ds dz \\ &= \int_1^\infty \left\{ \int \left(e^{i\tilde{z} \cdot (\xi/s)} - 1 - i\left(\tilde{z} \cdot \frac{\xi}{s}\right)h\left(\frac{\tilde{z}}{s}\right) \right) N(\tilde{z})d\tilde{z} \right\} \frac{ds}{s} \\ &= \int_1^\infty a\left(\frac{\xi}{s}\right) \frac{ds}{s} - i\xi \cdot \int_1^\infty \left\{ \int \frac{\tilde{z}}{s} \left(h\left(\frac{\tilde{z}}{s}\right) - h(\tilde{z}) \right) N(\tilde{z})d\tilde{z} \right\} \frac{ds}{s} \\ &= A(\xi) - i\xi \cdot b_A, \end{aligned}$$

where b_A is defined by (21). Properties (1) of the Lévy measure ν imply that b_A is well defined. We conclude that A is the characteristic exponent of an infinitely divisible law u_∞ whose drift is $b - b_A$, whose Gaussian part is σ and whose Lévy measure is $N_\infty(z)dz$. \square

Proof of Theorem 4. The proof of the first part is exactly given by Lemma 3.

We now turn to the second part of the theorem. Proposition 1 gives for $t \geq 0$,

$$\begin{aligned} \frac{d}{dt} \text{Ent}_{u_\infty}^\Phi(v) &= - \int \Phi''(v(t, \cdot)) \nabla v \cdot (t, \cdot) \cdot \sigma \nabla v(t, \cdot) u_\infty dx \\ &\quad - \iint D_\Phi(v(t, x), v(t, x - z)) \nu(dz) u_\infty dx \\ &= - \int \Phi''(v(t, \cdot)) \nabla v \cdot (t, \cdot) \cdot \sigma \nabla v(t, \cdot) u_\infty dx \\ &\quad - \iint D_\Phi(v(t, x), v(t, x + z)) \nu(dz) u_\infty dx, \end{aligned}$$

where we used the fact that ν is even. It is now enough to prove the following inequality:

$$\begin{aligned} \text{Ent}_{u_\infty}^\Phi(v) &\leq C \int \Phi''(v(t, \cdot)) \nabla v(t, \cdot) \cdot \sigma \nabla v u_\infty dx \\ &\quad + C \iint D_\Phi(v(x), v(x + z)) \nu(dz) u_\infty(x) dx \end{aligned}$$

for some constant C not depending on v and Gronwall's lemma permits to conclude. But this inequality is a direct consequence of (9) for the infinitely divisible law u_∞ . \square

4.3 Examples

We next discuss the condition we impose in order to get exponential decay, namely Condition (18). We point out that equality in this condition holds true only for α -stable operators and we give a necessary condition on the behaviour of the Lévy measure at infinity if one knows that it decreases faster than $|x|^{-d}$.

PROPOSITION 2.

- Equality $N_\infty = N/\lambda$ holds if and only if ψ is positively homogenous of index $\lambda \in (0, 2]$, i.e.

$$\psi(t\xi) = t^\lambda \psi(\xi) \quad \text{for any } t > 0, \xi \in \mathbb{R}^d.$$

In this case, we get $A = \psi/\lambda$ and $b_A = 0$. Note that in the limit case $\lambda = 2$, then we get $N_\infty = N/2 = 0$.

- If $|x|^d N(x) \rightarrow 0$ as $|x| \rightarrow +\infty$, then the densities N and N_∞ satisfy

$$N = -\text{div}(xN_\infty).$$

- In this case, Condition (18) is equivalent to

$$\begin{cases} N_\infty(tx) \leq N_\infty(x)t^{-d-1/C} & \text{if } t \geq 1, \\ N_\infty(tx) \geq N_\infty(x)t^{-d-1/C} & \text{if } 0 < t \leq 1. \end{cases} \quad (23)$$

Proof. The first item simply follows from the definition of A .

Let us first prove the second item.

$$\begin{aligned} N(x) &= -\left(\lim_{t \rightarrow +\infty} t^d N(tx)\right) + 1^d N(1 \times x) = -\int_1^{+\infty} \frac{d}{dt}(t^d N(tx))dt \\ &= -dN_\infty(x) - x \cdot \nabla N_\infty(x) = -\text{div}(xN_\infty). \end{aligned}$$

To prove the third item, use the first one to rewrite (18) as follows:

$$x \cdot \nabla N_\infty(x) + (d + 1/C)N_\infty(x) \leq 0.$$

Integrate over $[1, t]$ for $t \geq 1$ and $[t, 1]$ for $t \leq 1$ to get the result. □

Example 1. In \mathbb{R} , the Lévy measure $(1/|z|)e^{-|z|}$ does not satisfy Condition (18). Indeed, it is equivalent to

$$\int_1^{+\infty} \frac{e^{-|x|(s-1)}}{s} ds \leq C$$

and the monotone convergence theorem implies that the left hand side of this inequality goes to $+\infty$ as $|x| \rightarrow +\infty$.

Acknowledgement

The first author sincerely thanks H. Ouerdiane for his hospitality during the autumn in Hammamet.

Note

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