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On vectorial Hamilton–Jacobi equations

by

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Abstract: We consider the generalized Hopf and Lax functions associated with a vector-valued hamiltonian and we prove that they still provide lower semicontinuous solutions for the corresponding vectorial Hamilton–Jacobi equation in a very general context. Uniqueness of these generalized solutions is also investigated.
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1. Introduction

We deal with Hamilton-Jacobi equations

$$\begin{cases} D_t u(x,t) + H(D_x u(x,t)) = 0 & \text{in } X \times T_+ \\ u(x,0) = g(x) & \text{in } X \end{cases} \tag{1.1}$$

in which the variable t is not real but vectorial and the hamiltonian H is a vector-valued mapping. For instance, multitime Hamilton–Jacobi equations introduced by Lions and Rochet (1986) enter into this framework.

Let X, T be two real Banach spaces. Their respective topological duals are denoted by X^*, T^*. Consider a closed convex cone $T_+ \subset T$ and define a vectorial preorder on T in the following way: for any $s, t \in T$,

$$s \preccurlyeq t \quad \Leftrightarrow \quad t - s \in T_+. \tag{1.2}$$

The bilinear couplings between X and X^* and between T and T^* are both denoted by $\langle \cdot, \cdot \rangle$. The set of all continuous linear forms defined on X which are nonnegative on T_+ is denoted by T_+^* :

$$T_+^* = \{t^* \in T^* : \forall t \in T_+, \langle t, t^* \rangle \geq 0\}.$$

The w^*−closed convex cone T_+^* coincides with the set of continuous linear forms defined on T that are nondecreasing with respect to (1.2). Moreover, T_+^* induces a vectorial preorder on T^* : for any $s^*, t^* \in T^*$,

$$s^* \preccurlyeq_* t^* \quad \Leftrightarrow \quad t^* - s^* \in T_+^*. \tag{1.3}$$

Let us introduce a mapping H defined on a nonempty subset $\operatorname{dom} H$ of X^* with values in T^* :

$$H : \operatorname{dom} H \subset X^* \longrightarrow T^* \tag{1.4}$$

and a lower semicontinuous (lsc for short) proper function

$$g : X \longrightarrow \mathbb{R} \cup \{+\infty\}.$$

We say that H is *T_+^*-convex* if $\operatorname{dom} H$ is convex and if for any $x^*, y^* \in \operatorname{dom} H$, $\lambda \in [0,1]$, one has

$$H(\lambda x^* + (1-\lambda)y^*) \preccurlyeq_* \lambda H(x^*) + (1-\lambda)H(y^*).$$

If we define the epigraph of H by

$$\operatorname{epi} H = \{(x^*, t^*) \in X^* \times T^* : t^* \succcurlyeq_* H(x^*)\},$$

then the T_+^*-convexity of H is equivalent to the convexity of its epigraph.

Before making it more precise in what sense (1.1) is solved, we need to recall what a subgradient is. For a given function $u : X \times T_+ \to \mathbb{R} \cup \{+\infty\}$, a couple of vectors $(x^*, t^*) \in X^* \times T^*$ is a so-called *Fréchet subgradient* of u at a point $(x,t) \in X \times T_+$ if for any $(y,s) \in X \times T_+$,

$$\langle y - x, x^* \rangle + \langle s - t, t^* \rangle \le u(y,s) - u(x,t) + o(|(y,s) - (x,t)|) \tag{1.5}$$

where $o(\cdot)$ is a function such that $o(x)/|x| \to 0$ as $x \to 0$. The couple $(x^*, t^*) \in X^* \times T^*$ is said to be a *subgradient in the sense of convex analysis* if (1.5) is true with $o(\cdot) \equiv 0$. The set of all Fréchet subgradients (resp. subgradients in the sense of convex analysis) is referred to as the *Fréchet subdifferential* (resp. *subdifferential in the sense of convex analysis*) of u at (x,t) and is denoted by $\partial_F u(x,t)$ (resp. $\partial u(x,t)$).

The generalized solutions of (1.1) are defined by adapting the Crandall-Lions' notion of viscosity solution, Crandall and Lions (1983), or some extensions of it, Barron and R. Jensen (1990), Frankowska (1993). A lsc proper function $u : X \times T_+ \to \mathbb{R} \cup \{+\infty\}$ is a *subsolution* of (1.1) if for any $(x,t) \in X \times T_+$, any $(x^*, t^*) \in \partial_F u(x,t)$, one has $t^* + H(x^*) \preccurlyeq_* 0$ and if $u(x,0) \le g(x)$ for any $x \in X$. The function u is a *supersolution* of (1.1) if for any $(x,t) \in X \times \operatorname{int} T_+$, any $(x^*, t^*) \in \partial_F u(x,t)$, one has $t^* + H(x^*) \succcurlyeq_* 0$ and if $u(x,0) \ge g(x)$ for any $x \in X$. The function u is a *solution* of (1.1) if it is both a subsolution and a supersolution. Note that the concept of supersolution is inoperative if the topological interior of the ordering convex cone T_+ is empty.

The reader may observe that if $\operatorname{int} T_+$ is nonempty, then the dual cone T_+^* is pointed so that T^* is partially ordered instead of partially preordered. Therefore any solution u of (1.1) satisfies for any $(x^*, t^*) \in \partial_F u(x, t)$ with $t \in \operatorname{int} T_+$: $t^* = -H(x^*)$. A referee kindly pointed out to us that the previous observation remains valid if one extends the concept of supersolution by replacing the topological interior of T_+ with

$$T_+^i = \{t \in T : \forall t^* \in T_+^* \setminus \{0\}, \langle t, t^* \rangle > 0\}.$$

Nevertheless, although many results of the paper can be stated with this concept of supersolution, our proof of Proposition 7 requires the topological interior of T_+ to be nonempty.

The paper is organized as follows. In Section 2, we introduce the generalized Hopf function and describe its subdifferential (Proposition 3). When g and $\operatorname{epi} H$ are convex, we prove that u_{Hopf} is a solution of (1.1) (Theorem 1). The generalized Lax function is considered in Section 3. Under a regularity assumption we prove that u_{Lax} is a solution of (1.1) (Theorem 2). Next we prove that the lsc convex hull of u_{Lax} coincides with u_{Hopf} (Theorem 3). The uniqueness of the solution of (1.1) is studied in Section 4; we prove that u_{Lax} and u_{Hopf} are respectively the greatest lsc subsolution and the lowest weakly lsc supersolution (Theorems 4, 5, 6). Several examples are presented in Section 5.

The remaining part of the present section is devoted to definitions and notations that are used throughout.

Let Z denote an arbitrary Banach space and consider a function $f : Z \to \mathbb{R} \cup \{+\infty\}$. The Legendre-Fenchel conjugate of f is denoted by f^* and is defined on Z^* by the following formula:

$$f^*(z^*) = \sup_{z \in Z} \{\langle z^*, z \rangle - f(z)\}.$$

The function $f^{**} = (f^*)^*$ defined on Z instead of Z^{**} turns out to be the greatest lsc and convex function bounding f from below. It is known as the *lsc convex hull* of f while the *lsc hull* of f is denoted by $\underline{f}$ and is defined by:

$$\underline{f}(z) = \liminf_{y \to z} f(y).$$

As usual $\Gamma_0(Z)$ denotes the set of lsc proper convex functions defined on Z and $\Gamma_0(Z^*)$ denotes the set of weakly-$\star$ lsc proper convex functions defined on Z^*. Subgradients (in the sense of convex analysis) $z^* \in \partial f(z)$ are characterized by the so-called *Fenchel's equality*:

$$\langle z^*, z \rangle = f(z) + f^*(z^*),$$

while *Fenchel's inequality* holds true for any z^*, z :

$$\langle z^*, z \rangle \geq f(z) + f^*(z^*).$$

Consider two arbitrary sets $A, B \subset Z$. Then, $[A, B]$ denotes the convex hull of $A \cup B$. To finish with, the indicator function of A is denoted by ι_A and is defined by setting $\iota_A(z) = 0$ if $z \in A$ and $\iota_A(z) = +\infty$ if $z \notin A$.

2. The generalized Hopf function

In this section, we assume that g is lsc and proper and we consider a mapping H as in (1.4). For any $t \in T_+$, let us define the composite function

$$(t \circ H)(x^*) = \begin{cases} \langle t, H(x^*)\rangle & \text{if } x^* \in \operatorname{dom} H, \\ +\infty & \text{if not.} \end{cases}$$

Observe that $t \circ H$ is convex if H is T_+^*-convex. The *generalized Hopf function* is defined as a certain Legendre-Fenchel conjugate with respect to the x variable: for any $(x,t) \in X \times T_+$,

$$u_{\mathrm{Hopf}}(x,t) = (g^* + t \circ H)^*(x) \tag{2.1}$$

that is to say

$$u_{\mathrm{Hopf}}(x,t) = \sup_{x^* \in \operatorname{dom} g^* \cap \operatorname{dom} H} \langle x, x^*\rangle - g^*(x^*) - \langle t, H(x^*)\rangle. \tag{2.2}$$

In order to ensure that u_{Hopf} does not equal $-\infty$, we assume that

$$\operatorname{dom} g^* \cap \operatorname{dom} H \neq \emptyset. \tag{2.3}$$

Throughout, some functions u are only defined on $X \times T_+$ (as u_{Hopf}). It is convenient to set $u(x,t) = +\infty$ for $(x,t) \notin X \times T_+$ so that u is defined on the whole space $X \times T$.

PROPOSITION 1 *The Hopf function belongs to* $\Gamma_0(X \times T)$ *and one has*

$$u_{\mathrm{Hopf}}(\cdot, 0) \leq g. \tag{2.4}$$

Equality holds true in (2.4) if g is convex and if $\operatorname{dom} g^* \subset \operatorname{dom} H$.

Proof. From (2.2), we get that u_{Hopf} is the supremum of a family of continuous linear functions on $X \times T$. Moreover, one has

$$u_{\mathrm{Hopf}}(x,0) = (g^* + \iota_{\operatorname{dom} H})^*(x) \leq g(x).$$

This implies that u_{Hopf} is proper and that $u_{\mathrm{Hopf}}(\cdot,0) = g^{**} = g$ whenever $\operatorname{dom} g^* \subset \operatorname{dom} H$ and g is convex. ■

We now explain how to rewrite u_{Hopf} as a Legendre-Fenchel conjugate with respect to the (x,t) variable (see Imbert, 2001, for the scalar case). Let us define a function $\Phi \in \Gamma_0(X^* \times T^*)$ by

$$\Phi(x^*, t^*) = g^*(x^*), \tag{2.5}$$

and let us introduce the symmetrical of the epigraph of H with respect to the X^*-axis:

$$\widehat{\operatorname{epi}}\, H = \{(x^*, t^*) \in X^* \times T^* : H(x^*) \preccurlyeq_* -t^*\},$$

We claim that

PROPOSITION 2 $u_{\text{Hopf}} = (\Phi + \iota_{\widehat{\text{epi}\, H}})^*$.

The following corollary provides upper and lower estimates of the Legendre-Fenchel conjugate of u_{Hopf}. For an arbitrary set A, $\overline{\text{co}}A$ denotes the w^*-closed convex hull of A.

COROLLARY 1 $\Phi + \iota_{\overline{\text{co}}\,\widehat{\text{epi}\, H}} \leq u^*_{\text{Hopf}} \leq \Phi + \iota_{\widehat{\text{epi}\, H}}$.

Let us study the Fréchet subdifferential of the Hopf function. Consider an arbitrary point $(x,t) \in X \times T_+$ and an arbitrary subgradient $(x^*,t^*) \in \partial_F u_{\text{Hopf}}(x,t) = \partial u_{\text{Hopf}}(x,t)$. By Corollary 1 and (2.5), we know that $x^* \in \text{dom}\, g^*$ and $(x^*,t^*) \in \overline{\text{co}}\,\widehat{\text{epi}}\, H$. Using Fenchel's equality, we get

$$\langle x, x^* \rangle + \langle t, t^* \rangle \geq u_{\text{Hopf}}(x,t) + g^*(x^*). \tag{2.6}$$

Besides, we notice that $x^* \in \partial u_{\text{Hopf}}(.,t)(x)$. Hence Fenchel's equality and (2.1) yield

$$\begin{aligned}\langle x, x^* \rangle &= u_{\text{Hopf}}(x,t) + (g^* + t \circ H)^{**}(x^*) \\ &\leq u_{\text{Hopf}}(x,t) + g^*(x^*) + (t \circ H)(x^*).\end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we finally obtain that

$$0 \leq \langle t, t^* \rangle + (t \circ H)(x^*).$$

Let us gather what we just proved in the following proposition.

PROPOSITION 3 *For any $(x,t) \in X \times T_+$ and any $(x^*,t^*) \in \partial_F u_{\text{Hopf}}(x,t)$, one has*

$$(x^*,t^*) \in \overline{\text{co}}\,\widehat{\text{epi}}\, H \tag{2.8}$$

$$x^* \in \text{dom}\, g^* \tag{2.9}$$

$$\langle t, t^* \rangle + (t \circ H)(x^*) \geq 0 \tag{2.10}$$

REMARK 1 *In the scalar case ($T = \mathbb{R}$), when $\text{dom}\, g^* \subset \text{dom}\, H$, (2.9), (2.10) and Proposition 3 entail that u_{Hopf} is a supersolution of (1.1) (see Alvarez, Barron and Ishii, 1999).*

In view of (2.8), it seems interesting to investigate what happens when the epigraph of H is w^*-closed and convex.

THEOREM 1 *Assume that g is lsc and proper, that $\text{epi}\, H$ is w^*-closed and convex, and that (2.3) holds. Then, for any $(x^*,t^*) \in \partial_F u_{\text{Hopf}}(x,t)$, $t \in T_+$ (resp. $t \in \text{int}\, T_+$), we have $t^* + H(x^*) \preccurlyeq_* 0$ (resp. $t^* + H(x^*) = 0$). In particular, u_{Hopf} is a subsolution of (1.1). Moreover, if $\text{dom}\, g^* \subset \text{dom}\, H$ and g is convex, then u_{Hopf} is a solution of (1.1).*

Proof. Let $(x^*, t^*) \in \partial_F u_{\text{Hopf}}(x,t) = \partial u_{\text{Hopf}}(x,t)$. As $\operatorname{epi} H$ is w^*-closed and convex, (2.8) reads $H(x^*) \preccurlyeq_* -t^*$ and, since $u_{\text{Hopf}}(.,0) \leq g$, u_{Hopf} is a subsolution. Moreover, by (2.10), one has $\langle t, H(x^*)+t^* \rangle \geq 0$. Now since $H(x^*)+t^* \preccurlyeq_* 0$, the linear form $s \mapsto \langle s, H(x^*)+t^* \rangle$ is nonpositive on T_+. Therefore, if t belongs to $\operatorname{int} T_+$, then t is a local maximum of the linear form, so that $H(x^*)+t^* = 0$. Consequently, when $\operatorname{dom} g^* \subset \operatorname{dom} H$, u_{Hopf} is a solution of (1.1). ■

The study of the Hopf function when $g = \iota_{\{0\}}$ will be useful in the following. In this case, $g^* = 0$, $\Phi = 0$ and we have (see Proposition 2)

$$u_{\text{Hopf}}(x,t) = \iota^*_{\widehat{\operatorname{epi} H}}(x,t) = \begin{cases} (t \circ H)^*(x) & \text{if } (x,t) \in X \times T_+ \\ +\infty & \text{if not.} \end{cases}$$

It therefore follows from Theorem 1 that

Corollary 2 *Assume that* $\operatorname{epi} H$ *is w^*-closed and convex and consider a point* $(x,t) \in X \times T_+$. *Then for any* $(x^*,t^*) \in \partial \iota^*_{\widehat{\operatorname{epi} H}}(x,t)$, *one has*

$$t^* + H(x^*) \preccurlyeq_* 0.$$

If, moreover, t belongs to $\operatorname{int} T_+$, *then* $t^* + H(x^*) = 0$.

3. The generalized Lax function

Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc proper function and assume that $\operatorname{epi} H$ is nonempty, w^*-closed and convex. The generalized Lax function is defined as a certain infimal convolution (denoted by $\square$) with respect to the x variable:

$$u_{\text{Lax}}(x,t) = \begin{cases} [g \,\square\, (t \circ H)^*](x) & \text{if } (x,t) \in X \times T_+ \\ +\infty & \text{if not.} \end{cases}$$

For any $(x,t) \in X \times T_+$ one has, by definition,

$$u_{\text{Lax}}(x,t) = \inf_{y \in X} [g(x-y) + (t \circ H)^*(y)]. \tag{3.1}$$

The infimal convolution defining u_{Lax} is said to be *exact* if the infimum in (3.1) is attained.

If no further assumptions are made, u_{Lax} is neither convex nor lsc. Observe that

$$u_{\text{Lax}}(.,0) = g \,\square\, \iota^*_{\operatorname{dom} H} \leq g,$$

which implies that u_{Lax} is not identically equal to $+\infty$. But u_{Lax} may take the value $-\infty$. As in the scalar case Imbert (2001), the generalized Lax function can be expressed as an infimal convolution of two functions defined on $X \times T$ by using the following function

$$G(x,t) = \begin{cases} g(x) & \text{if } t = 0, \\ +\infty & \text{if not.} \end{cases}$$

PROPOSITION 4 $u_{\mathrm{Lax}} = G \,\square\, \iota^*_{\widehat{\mathrm{epi}\,H}}$.

COROLLARY 3 *If g is convex, so is u_{Lax}.*

We know from Theorem 1 that u_{Hopf} is a subsolution of (1.1). In order to prove that so is u_{Lax}, a regularity condition is required. As in Imbert (2001), the generalized Lax function is said to be *regular* if it is lsc, proper and if the infimal convolution in (3.1) is exact when finite. Such a condition holds true under assumptions of Proposition 5 and in Examples 5.1 and 5.2 below. In the scalar case, several sufficient conditions can be found in Penot and Volle (2000), Prop. 3.1.

THEOREM 2 *Let g be lsc and proper and assume that* $\mathrm{epi}\,H$ *is nonempty, w^*-closed and convex. Moreover, assume that u_{Lax} is regular. Then it is a subsolution of (1.1). If, moreover,* $\mathrm{dom}\,H = X^*$ *or if* $\mathrm{dom}\,g^* \subset \mathrm{dom}\,H$ *and $g \in \Gamma_0(X)$, then u_{Lax} is a solution of (1.1).*

Proof. Let $(x^*, t^*) \in \partial_F u_{\mathrm{Lax}}(x,t)$. As u_{Lax} is regular, the infimal convolution in (3.1) is exact. It therefore follows from Proposition 4 and the well-known subdifferential calculus rule (see e.g. Lemma 5 in Imbert, 2001) that there exists $y \in X$ such that

$$(x^*, t^*) \in \partial_F G(x-y, 0) \cap \partial_F \iota^*_{\widehat{\mathrm{epi}\,H}}(y,t).$$

Since $\mathrm{epi}\,H$ is convex, $(x^*, t^*) \in \partial \iota^*_{\widehat{\mathrm{epi}\,H}}(y,t)$ and by Corollary 2, one has $H(x^*) \preccurlyeq_* -t^*$, that is to say u_{Lax} is a subsolution. If, moreover, t belongs to $\mathrm{int}\, T_+$, then Corollary 2 implies that $H(x^*) = -t^*$; it follows that u_{Lax} is a solution of (1.1) provided that $g \,\square\, \iota^*_{\mathrm{dom}\,H} \geq g$ holds true. Such an inequality is verified if $\mathrm{dom}\,H = X^*$ or if $\mathrm{dom}\,g^* \subset \mathrm{dom}\,H$ and $g \in \Gamma_0(X)$. ■

The next result sheds light on an interesting link between the functions u_{Hopf} and u_{Lax}.

THEOREM 3 *Let g be lsc and proper and suppose that* $\mathrm{epi}\,H$ *is w^*-closed and convex. Assume moreover that*

$$\mathrm{dom}\,g^* \cap \mathrm{dom}\,H \neq \emptyset.$$

Then u_{Lax} is proper and $u^*_{\mathrm{Lax}} = u^*_{\mathrm{Hopf}}$ so that u_{Hopf} is the lsc convex hull of u_{Lax}.

If, moreover, g is convex, then u_{Hopf} is the lsc hull of u_{Lax}.

Proof. Observe that $G^* = \Phi$ (see (2.5)). Using Propositions 4 and 2, we get $u^*_{\mathrm{Lax}} = \Phi + \iota_{\widehat{\mathrm{epi}\,H}} = u^*_{\mathrm{Hopf}}$. We then obtain $u^{**}_{\mathrm{Lax}} = u_{\mathrm{Hopf}}$ and since u_{Hopf} is proper, u_{Lax} does not take the value $-\infty$. If g is convex, u_{Lax} is also convex and the lsc hull of u_{Lax} coincides with $u^{**}_{\mathrm{Lax}} = u_{\mathrm{Hopf}}$. ■

We just have seen that when g is convex u_{Lax} and u_{Hopf} are very close. Let us give a condition under which they coincide.

PROPOSITION 5 *Assume that* X, T *are reflexive spaces, that* $g \in \Gamma_0(X)$ *and that* $\operatorname{cone}(\operatorname{dom} g^* - \operatorname{dom} H)$ *is a closed linear space. Then* u_{Lax} *is regular and it coincides with* u_{Hopf}.

Proof. By Attouch–Brezis Theorem, Attouch and Brezis (1986), one has

$$u_{\mathrm{Hopf}} = (\Phi + \iota_{\widehat{\operatorname{epi} H}})^* = G \,\square\, \iota^*_{\widehat{\operatorname{epi} H}} = u_{\mathrm{Lax}}$$

whenever $\operatorname{cone}(\operatorname{dom}\Phi - \widehat{\operatorname{epi}} H)$ is a closed linear space. Moreover, the infimal convolution $G \,\square\, \iota^*_{\widehat{\operatorname{epi} H}}$ is exact. Looking at the definition of Φ, (2.5), one can see that $\operatorname{dom}\Phi = \operatorname{dom} g^* \times T^*$ so that $\operatorname{dom}\Phi - \widehat{\operatorname{epi}} H = (\operatorname{dom} g^* - \operatorname{dom} H) \times T^*$ and the required condition holds. ∎

4. Bounds for subsolutions and supersolutions

In this section, we prove that any lsc subsolution of (1.1) is lower than or equal to u_{Lax} and that any weakly lsc supersolution is greater than or equal to u_{Hopf}. As in the scalar case, proofs are based on Clarke–Ledyaev's mean value inequalities. To avoid theoretical complications, we assume in this section that X and T are Hilbert spaces (see Borwein and Zhu 1996, Penot and Volle, 2000, for possible extensions to more general spaces). Under appropriate assumptions we obtain that u_{Hopf} is the unique solution of (1.1). Unless specified otherwise g is just an lsc proper function defined on X and $H : \operatorname{dom} H \subset X^* \to T^*$ is just a mapping. In the following, B denotes the unit ball of any space (X, T, $X \times T$ *etc.*).

PROPOSITION 6 *Let* u *be a lsc subsolution of (1.1); then* $u \leq u_{\mathrm{Lax}}$.

Proof. According to (3.1) we have to prove that for any $x, y \in X$ and any $t \in T_+$, one has

$$u(x,t) \leq g(x-y) + (t \circ H)^*(y).$$

As $u(.,0) \leq g$ it suffices to prove that

$$u(x,t) \leq u(x-y,0) + (t \circ H)^*(y). \tag{4.1}$$

If $u(x-y,0) = +\infty$, it is clear. If not, choose $r < u(x,t) - u(x-y,0)$. By the multidirectional Mean Value Inequality due to Clarke and Ledyaev (Clarke, Ledyaev, Stern and Wolenski, 1997, p. 117), there exists a point $(z,s) \in [(x,t),(x-y,0)]+B$ and a subgradient $(x^*,t^*) \in \partial_F u(z,s)$ such that $r < \langle y, x^*\rangle + \langle t, t^*\rangle$. Using the fact that u is a subsolution, we know that $H(x^*) \preccurlyeq_* -t^*$, and since $t \in T_+$, we finally obtain

$$r < \langle y, x^*\rangle - \langle t, H(x^*)\rangle \leq (t \circ H)^*(y).$$

As $r < u(x,t) - u(x-y,0)$ is arbitrary, we get (4.1). ∎

THEOREM 4 *Let g be lsc and proper, let* $\operatorname{epi} H$ *be closed and convex and let* u_{Lax} *be regular. Then* u_{Lax} *is the greatest lsc subsolution of (1.1).*

Proof. Apply Theorem 2 and Proposition 6. ■

When g and $\operatorname{epi} H$ are convex, we obtain (see Imbert, 2001, for the scalar case):

THEOREM 5 *Assume that* $g \in \Gamma_0(X)$, *that* $\operatorname{epi} H$ *is closed and convex and that (2.3) holds. Then* u_{Hopf} *is the greatest lsc subsolution of (1.1).*

Proof. By Theorem 1, u_{Hopf} is a subsolution. Theorem 3 ensures that u_{Hopf} is the lsc hull of u_{Lax}. It then follows from Proposition 6 that u_{Hopf} is the greatest lsc subsolution of (1.1). ■

From Proposition 1, we know that u_{Hopf} is convex and lsc. It follows that u_{Hopf} is also weakly lsc. In the next result, we show that u_{Hopf} bounds from below any weakly lsc supersolution of (1.1). As in the scalar case, Imbert (2001), the proof relies on the Mean Value Inequality.

PROPOSITION 7 *Assume that H is Lipschitz continuous on its domain and that g is lsc and proper. Then for any weakly lsc supersolution u of (1.1) one has*

$$u_{\text{Hopf}} \leq u \text{ in } X \times \operatorname{int} T_+.$$

Proof. Let $y^* \in \operatorname{dom} g^* \cap \operatorname{dom} H$ and define

$$w(x,t) = u(x,t) - \langle x, y^* \rangle + g^*(y^*) + \langle t, H(y^*) \rangle.$$

Observe that w is weakly lsc and $w(.,0) \geq 0$. We have to prove that $w \geq 0$ in $X \times \operatorname{int} T_+$.

Assume the contrary: there exists $(\bar{x}, \bar{t}) \in X \times \operatorname{int} T_+$ such that $w(\bar{x}, \bar{t}) = -\alpha$ with $\alpha > 0$. For any $r > 0$ we claim that there exists $\underline{t}$ in the line $]0, \bar{t}[$ such that

$$w(x, \underline{t}) \geq -\frac{\alpha}{2} \text{ for all } x \in B(\bar{x}, r). \tag{4.2}$$

If such a $\underline{t}$ does not exist, then for any integer $n \geq 1$ there exists a point $x_n \in B(\bar{x}, r)$ such that $w(x_n, \frac{1}{n}\bar{t}) < -\frac{\alpha}{2}$. Considering a weakly convergent subsequence $x_p \rightharpoonup x$, we therefore obtain the following contradiction:

$$0 \leq w(x,0) \leq \liminf_{p \to +\infty} w\left(x_p, \frac{1}{p}\bar{t}\right) \leq -\frac{\alpha}{2}.$$

Let us set $Y := B(\bar{x}, r) \times \{\underline{t}\}$. From (4.2) we get

$$\forall (x,t) \in Y, w(x,t) - w(\bar{x}, \bar{t}) \geq \frac{\alpha}{2}.$$

By the Mean Value Inequality (Clarke, Ledyaev, Stern and Wolenski, 1997, p. 117), for any $\epsilon > 0$, there exists a point $(x,t) \in [(\bar{x},\bar{t}), Y] + \epsilon B$ and a subgradient $(x^*,t^*) \in \partial_F w(x,t)$ such that

$$\langle x - \bar{x}, x^* \rangle + \langle \bar{t} - \underline{t}, t^* \rangle \geq \frac{\alpha}{3} \text{ for all } x \in B(\bar{x}, r).$$

Looking at the definition of w, we observe that $(x^* + y^*, t^* - H(y^*)) \in \partial_F u(x,t)$. Next, $\epsilon > 0$ is chosen small enough in order to ensure that $t \in \operatorname{int} T_+$ and, consequently, $H(x^* + y^*) = H(y^*) - t^*$. We then have

$$\langle x - \bar{x}, x^* \rangle + \langle \underline{t} - \bar{t}, H(x^* + y^*) - H(y^*) \rangle \geq \frac{\alpha}{3} \text{ for all } x \in B(\bar{x}, r).$$

If K denotes a Lipschitz constant of H, the previous inequality yields

$$-r|x^*| + K|\bar{t}||x^*| \geq \frac{\alpha}{3}.$$

A contradiction is obtained by choosing $r = K|\bar{t}|$. Hence $w(\bar{x},\bar{t}) \geq 0$ and $u_{\text{Hopf}} \leq u$ in $X \times \operatorname{int} T_+$. ■

From Theorems 1 and 5 and Proposition 7, we obtain the following uniqueness result.

THEOREM 6 *Assume that H is Lipschitz continuous on its domain, that* $\operatorname{epi} H$ *is closed and convex and that $g \in \Gamma_0(X)$ with* $\operatorname{dom} g^* \subset \operatorname{dom} H$. *Then u_{Hopf} is a weakly lsc solution of (1.1) and any weakly lsc solution of the vectorial Hamilton–Jacobi equation (1.1) coincides with u_{Hopf} on* $(X \times \operatorname{int} T_+) \cup X \times \{0\}$.

5. Examples

5.1. Multitime Hamilton–Jacobi equations

In order to apply the results of the previous sections to multitime Hamilton–Jacobi equations introduced by Lions and Rochet (1986), we consider the space $X = \mathbb{R}^n$, the two convex cones $T = \mathbb{R}^n = T^*$, $T_+ = \mathbb{R}^n_+ = T^*_+$ and the n functions $H_1, \ldots, H_n : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. The corresponding multitime Hamilton–Jacobi equation is

$$\begin{aligned} &\frac{\partial u}{\partial t_i} + H_i(D_x u) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+, \ 1 \leq i \leq n, \\ &u(x, 0) = g(x) \text{ in } \mathbb{R}^n. \end{aligned}$$

Such a system may be written as in (1.1) by defining the mapping H as follows:

$$H(x^*) = (H_1(x^*), \ldots, H_n(x^*)) \text{ for all } x^* \in \cap_{i=1}^n \operatorname{dom} H_i.$$

We then have for any $x \in \mathbb{R}^n$ and any $t = (t_1, \ldots, t_n) \in \mathbb{R}^n_+$:

$$u_{\mathrm{Hopf}}(x,t) = \Big(g^* \dot{+} \sum_{i=1}^{n} t_i H_i\Big)^*(x)$$

$$u_{\mathrm{Lax}}(x,t) = g \,\Box\, \Big(\sum_{i=1}^{n} t_i H_i\Big)^*(x)$$

Observe that if $H_i \in \Gamma_0(\mathbb{R}^n)$ for $i = 1, \ldots, n$, then $\operatorname{epi} H$ is closed and convex.

5.2. Linear vectorial Hamilton–Jacobi equations

Assume that X, T are Hilbert spaces, that $A : X \to T$ is continuous and linear and denote by $A^* : T \to X$ the associated transposed linear mapping. Let also g be a lsc proper function defined on X and consider the following linear vectorial Hamilton–Jacobi equation

$$\begin{array}{ll} D_t u(x,t) + A(D_x u(x,t)) = 0 & \text{in } X \times T, \\ u(x,0) = g(x) & \text{in } X. \end{array} \tag{5.1}$$

Here $H = A$ is continuous and linear so that its graph is a closed linear space. Choosing $T_+ = T$ one has $T_+^* = \{0\}$ so that the epigraph of H with respect to T_+^* coincides with the graph of A and one has

$$(t \circ H)^*(y) = \begin{cases} 0 & \text{if } A^*(t) = y \\ +\infty & \text{if not.} \end{cases}$$

It follows that $u_{\mathrm{Lax}}(x,t) = g(x - A^*(t))$ for all $(x,t) \in X \times T$ and it is regular. By Theorem 2 it is an lsc solution of the linear vectorial Hamilton–Jacobi equation (5.1). By Theorem 5 it is the greatest lsc subsolution of (5.1). The Hopf function is given by $u_{\mathrm{Hopf}}(x,t) = g^{**}(x - A^*(t))$. If $g \in \Gamma_0(X)$ then $u_{\mathrm{Hopf}}(x,t) = g(x - A^*(t))$ is the unique weakly lsc solution of (5.1) (see Theorem 6).

5.3. Schur vectorial order

Let us consider the Schur vectorial order on $\mathbb{R}^n$ which is associated with the nonnegative convex cone

$$S = \Big\{ y \in \mathbb{R}^n : \sum_{i=1}^{k} y_i \geq 0, 1 \leq k < n, \sum_{i=1}^{n} y_i = 0 \Big\}.$$

Given $a, b \in \mathbb{R}^n$, $a \leq_S b$ means $b - a \in S$. The nonnegative polar cone of S is

$$\mathbb{R}^n_{\geq} = \{ t \in \mathbb{R}^n : t_1 \geq \ldots \geq t_n \}.$$

Given $x \in \mathbb{R}^n$, we denote by $[x]$ the element of $\mathbb{R}^n$ whose components are those of x arranged in nondecreasing order. It turns out that the mapping

$$[\,] : \mathbb{R}^n \to \mathbb{R}^n, \;\; x \mapsto [x]$$

is S-convex (in fact sublinear; see, for instance, Borwein and Lewis, 2000, p. 26). The corresponding vectorial Hamilton–Jacobi equation is

$$\begin{array}{ll} D_t u(x,t) + [D_x u(x,t)] = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}^n_{\geq} \\ u(x,0) = g(x) & \text{in } \mathbb{R}^n, \end{array} \tag{5.2}$$

where g is a lsc proper function defined on $\mathbb{R}^n$. Denoting by $[x]_i$ the i^{th} greatest component of x, one has for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}^n_{\geq}$:

$$u_{\text{Hopf}}(x,t) = \Big(g^* + \sum_{i=1}^{n} t_i[\,]_i\Big)^*(x).$$

In order to explicit the Lax function we need a lemma. We denote by $\mathcal{Q}$ the compact convex set of $n \times n$ bistochastic matrices. Let us first recall the Hardy–Littlewood–Polya Theorem (see Berge, 1959, p. 191):

$$\forall a, b \in \mathbb{R}^n_{\geq} : \ a \leq_S b \Leftrightarrow \exists Q \in \mathcal{Q} : a = Qb. \tag{5.3}$$

LEMMA 1 *For any $(y,t) \in \mathbb{R}^n \times \mathbb{R}^n_{\geq}$ one has:*

$$\langle t, [y] \rangle = \sup\{\langle x, y \rangle : (x, Q) \in \mathbb{R}^n \times \mathcal{Q}, x = Qt\}.$$

Proof. Let $(x,Q) \in \mathbb{R}^n \times \mathcal{Q}$ with $x = Qt$. There exists a permutation matrix P such that $[x] = Px$ and we have $[x] = (PQ)t$ with PQ bistochastic. By (5.3) it follows that $[x] \leq_S t$. Since $[y] \in \mathbb{R}^n_{\geq}$ one has: $\langle [x], [y] \rangle \leq \langle t, [y] \rangle$. Now it is known (see Borwein and Lewis, 2000, p. 10) that $\langle x, y \rangle \leq \langle [x], [y] \rangle$. Therefore the inequality $\geq$ holds in Lemma 1.

Conversely, there is a permutation matrix M such that $[y] = My$; taking $x = M^{-1}t$ one has $\langle x, y \rangle = \langle M^{-1}t, y \rangle = \langle t, My \rangle$ so that the inequality $\leq$ holds in Lemma 1. ■

As the set $\cup_{Q \in \mathcal{Q}} Qt$ is compact and convex, it follows from Lemma 1 that the Legendre–Fenchel conjugate of the support function $t \circ [\,]$ coincides with the indicator function of this set. The Lax function can be written under the following form:

PROPOSITION 8 *For any $(x,t) \in \mathbb{R}^n \times \mathbb{R}^n_{\geq}$:*

$$u_{\text{Lax}}(x,t) = \min_{Q \in \mathcal{Q}} g(x - Qt).$$

The Lax function is regular; it is therefore the greatest lsc subsolution of (5.2).

5.4. Vectorial Hamilton–Jacobi equations in matrix spaces

In this subsection, X is the Euclidian space $\mathbb{S}_n$ of $n \times n$ real symmetric matrices equipped with the scalar product $\langle\!\langle M, N\rangle\!\rangle = \operatorname{trace}(MN)$ and the two cones T and T^* coincide with the finite dimensional space $\mathbb{R}^n$ equipped with the canonical scalar product $\langle .,.\rangle$. Let us consider the spectral mapping $H = \lambda$ that associates with any $N \in \mathbb{S}_n$ its eigenvalues $\lambda(N) = (\lambda_1(N), \ldots, \lambda_n(N))$ in such a way that $\lambda_1(N) \geq \ldots \geq \lambda_n(N)$. Observe that $\lambda(\mathbb{S}_n) = \mathbb{R}^n_{\geq}$. An important property of the mapping $\lambda = H : \mathbb{S}_n \to \mathbb{R}^n$ is that it is continuous and sublinear with respect to S (Borwein and Lewis, 2000, pp. 10, 108). In particular

$$\operatorname{epi}\lambda = \{(N,y) \in \mathbb{S}_n \times \mathbb{R}^n : y - \lambda(N) \in S\}$$

is a closed convex cone. Let us consider the underlying Hamilton–Jacobi equation

$$\begin{array}{ll} D_t u(M,t) + \lambda(D_M u(M,t)) = 0 & \text{in } \mathbb{S}_n \times \mathbb{R}^n_{\geq}, \\ u(M,0) = g(M) & \text{in } \mathbb{S}_n, \end{array} \tag{5.4}$$

where g is a lsc proper function defined on $\mathbb{S}_n$.

The Hopf function associated with (5.4) turns out to be

$$u_{\mathrm{Hopf}}(M,t) = \Big(g^* + \sum_{i=1}^{n} \lambda_i t_i\Big)^*(M)$$

for any $M \in \mathbb{S}_n$ and any $t = (t_1, \ldots, t_n) \in \mathbb{R}^n_{\geq}$.

In order to make explicit the Lax function, we need a lemma.

Lemma 2 *For any $(M,t) \in \mathbb{S}_n \times \mathbb{R}^n_{\geq}$ one has*

$$(t \circ \lambda)(M) = \sup\{\langle\!\langle M, N\rangle\!\rangle : (N,Q) \in \mathbb{S}_n \times \mathcal{Q}, \lambda(N) = Qt\}.$$

Proof. Let $(N,Q) \in \mathbb{S}_n \times \mathcal{Q}$ with $\lambda(N) = Qt$. From (5.3) one has $\lambda(N) \leq_S t$. Since $\lambda(M) \in \mathbb{R}^n_{\geq}$ it follows that $\langle \lambda(N), \lambda(M)\rangle \leq \langle t, \lambda(M)\rangle$. Since $\langle\!\langle M, N\rangle\!\rangle \leq \langle \lambda(M), \lambda(N)\rangle$ (see Borwein and Lewis, 2000, pp. 10, for instance) we get the inequality $\geq$ in Lemma 2.

Conversely, there exists an orthonormal matrix P such that $P^{-1}MP = \operatorname{diag}\lambda(M)$, where, for a given vector $y \in \mathbb{R}^n$, $\operatorname{diag}(y)$ denotes the diagonal matrix whose entries are $y_1, \ldots, y_n$. One has

$$\begin{aligned} \langle\!\langle M, \operatorname{diag} t\rangle\!\rangle &= \langle\!\langle P \operatorname{diag}\lambda(M)P^{-1}, P \operatorname{diag} t\, P^{-1}\rangle\!\rangle \\ &= \langle \lambda(M), t\rangle = (t \circ \lambda)(M). \end{aligned}$$

The proof of the lemma is therefore achieved. ∎

As the set $\{N \in \mathbb{S}_n : \exists Q \in \mathcal{Q}, \lambda(N) = Qt\}$ is closed and convex, it follows from Lemma 2 that the Legendre-Fenchel conjugate of the support function $t \circ \lambda$ coincides with the indicator function of this set, so that the Lax function can be expressed as follows.

PROPOSITION 9 $\forall (M,t) \in \mathbb{S}_n \times \mathbb{R}^n_{\geq}$,

$$u_{\mathrm{Lax}}(M,t) = \inf\{g(M-N) : (N,Q) \in \mathbb{S}_n \times \mathcal{Q}, \lambda(N) = Qt\}.$$

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