

OPTIMAL CONTROL UNDER STOCHASTIC TARGET CONSTRAINTS*

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Abstract. We study a class of Markovian optimal stochastic control problems in which the controlled process Z^ν is constrained to satisfy an almost sure constraint $Z^\nu(T) \in G \subset \mathbb{R}^{d+1}$ \mathbb{P} -a.s. at some final time $T > 0$. When the set is of the form $G := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : g(x, y) \geq 0\}$, with g nondecreasing in y , we provide a Hamilton–Jacobi–Bellman characterization of the associated value function. It gives rise to a state constraint problem, where the constraint can be expressed in terms of an auxiliary value function w which characterizes the set $D := \{(t, Z^\nu(t)) \in [0, T] \times \mathbb{R}^{d+1} : Z^\nu(T) \in G \text{ a.s. for some } \nu\}$. Contrary to standard state constraint problems, the domain D is not given a priori and we do not need to impose conditions on its boundary. It is naturally incorporated in the auxiliary value function w , which itself is a viscosity solution of a nonlinear parabolic PDE. Applying ideas recently developed in Bouchard, Elie, and Touzi [*SIAM J. Control Optim.*, 48 (2009), pp. 3123–3150], our general result also allows us to consider optimal control problems with moment constraints of the form $\mathbb{E}[g(Z^\nu(T))] \geq 0$ or $\mathbb{P}[g(Z^\nu(T)) \geq 0] \geq p$.

Key words. optimal control, state constraint problems, stochastic target problem, discontinuous viscosity solutions

AMS subject classifications. Primary, 93E20, 49L25; Secondary, 60J60

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1. Introduction. The aim of this paper is to study stochastic control problems under stochastic target constraint of the form

$$(1.1) \quad V(t, z) := \sup \left\{ \mathbb{E} \left[f(Z_{t,z}^\nu(T)) \right], \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P}\text{-a.s.} \right\},$$

where the controlled process $Z_{t,z}^\nu$ is the strong solution of a stochastic differential equation,

$$Z_{t,z}^\nu(s) = z + \int_t^s \mu_Z(Z_{t,z}^\nu(r), \nu_r) dr + \int_t^s \sigma_Z(Z_{t,z}^\nu(r), \nu_r) dW_r, \quad t \leq s \leq T,$$

for some d -dimensional Brownian motion W , and the set of controls \mathcal{U} is the collection of U -valued square integrable progressively measurable processes (for the filtration generated by W) for some closed set $U \subset \mathbb{R}^d$.

Such problems naturally appear in economics where a controller tries to maximize (or minimize) a criterion under some almost sure constraint on the final output. It is typically the case in finance or insurance where a manager tries to maximize the utility (or minimize the expected value of a loss function) of the terminal value of his portfolio under some strong no-bankruptcy-type condition; see, e.g., [16], [7], and the references therein. See also the example of application in section 4 below.

From the mathematical point of view, (1.1) can be seen as a state constraint problem. However, it is at first glance somehow “nonstandard” because the almost

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sure constraint is imposed only at $t = T$ and not on $[0, T]$ as usual; compare with [9], [13], [14], [17], and [18].

Our first observation is that it can actually be converted into a “classical” state constraint problem:

(1.2)

$$V(t, z) = \sup \left\{ \mathbb{E} \left[f(Z_{t,z}^\nu(T)) \right], \nu \in \mathcal{U} \text{ s.t. } (s, Z_{t,z}^\nu(s)) \in D \text{ } \mathbb{P}\text{-a.s. } \forall t \leq s \leq T \right\},$$

where

$$D := \{ (t, z) \in [0, T] \times \mathbb{R}^{d+1} : g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P}\text{-a.s. for some } \nu \in \mathcal{U} \}$$

is the corresponding *viability domain* in the terminology of [1]. This is indeed an immediate consequence of the so-called *geometric dynamic programming principle* introduced by Soner and Touzi [21] and [22] in the context of stochastic target problems:

$$g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P}\text{-a.s.} \iff (s, Z_{t,z}^\nu(s)) \in D \text{ } \mathbb{P}\text{-a.s.} \quad \forall t \leq s \leq T.$$

There exists a huge literature on state constraint problems. For deterministic control problems, Soner [20] first derived the Bellman equation under the appropriate form in order to characterize the value function as the unique viscosity solution. The main issue is to deal with the boundary of the domain. One of the important contributions of [20] is to show that the value function is the unique viscosity solution of the Bellman equation which is a subsolution “up to the boundary.” Hence, loosely speaking, Soner exhibited a boundary condition ensuring the uniqueness of the viscosity solution. A rather complete study for first order equations can be found in [9]. See also [13]. For stochastic control problems, the study of viscosity solutions with such a boundary condition was initiated in [18], where the volatility is the identity matrix. The results of [18] have been generalized since then; see, for instance, [17], [2], and [14].

The main difference with standard state constraint problems is that, in this paper, the set D is not given a priori but is defined implicitly by a stochastic target problem. We should also emphasize the fact that we focus here on the derivation of the PDEs and not on the study of the PDEs in themselves (comparison, existence of regular solutions, etc.). The last point, which is obviously very important, is left for further research.

In this paper, we shall restrict ourselves to the case where $(x, y) \in \mathbb{R}^d \times \mathbb{R} \mapsto g(x, y)$ is nondecreasing in y , and $Z_{t,z}^\nu$ is of the form $(X_{t,x}^\nu, Y_{t,x,y}^\nu)$, where $(X_{t,x}^\nu, Y_{t,x,y}^\nu)$ takes values in $\mathbb{R}^d \times \mathbb{R}$ and follows the dynamics

$$\begin{aligned} X_{t,x}^\nu(s) &= x + \int_t^s \mu_X(X_{t,x}^\nu(r), \nu_r) dr + \int_t^s \sigma_X(X_{t,x}^\nu(r), \nu_r) dW_r, \\ (1.3) \quad Y_{t,x,y}^\nu(s) &= y + \int_t^s \mu_Y(Z_{t,x,y}^\nu(r), \nu_r) dr + \int_t^s \sigma_Y(Z_{t,x,y}^\nu(r), \nu_r)^\top dW_r. \end{aligned}$$

In this case, the set $\{y \in \mathbb{R} : (t, x, y) \in D\}$ is a half-space for fixed (t, x) . This allows us to characterize the (closure of the) set D in terms of the auxiliary value function

$$w(t, x) := \inf \{ y \in \mathbb{R} : (t, x, y) \in D \}.$$

Assuming that the infimum is always achieved in the definition of w , (1.2) can then be written equivalently as

$$V(t, x, y) = \sup\{\mathbb{E} [f(Z_{t,x,y}^\nu(T))], \nu \in \mathcal{U} \text{ s.t. } Y_{t,x,y}^\nu(s) \geq w(s, X_{t,x}^\nu(s)) \mathbb{P}\text{-a.s. } \forall t \leq s \leq T\} .$$

We should immediately observe that more general situations could be discussed by following the approach of [21], which consists in replacing w by $\bar{w}(t, z) := 1 - \mathbf{1}_{(t,z) \in D}$, and working with \bar{w} instead of w .

As usual in state constraints problems, the constraint is not binding until one reaches the boundary. We actually show that V solves (in the discontinuous viscosity sense) the usual Hamilton–Jacobi–Bellman equation

$$(1.4) \quad - \sup_{u \in U} \mathcal{L}_Z^u V = 0, \text{ where } \mathcal{L}_Z^u V := \partial_t V + \langle \mu_Z(\cdot, u), DV \rangle + \frac{1}{2} \text{Tr} [\sigma_Z \sigma_Z^\top(\cdot, u) D^2 V]$$

in the interior of the domain D with the appropriate boundary condition at $t = T$:

$$(1.5) \quad V(T-, \cdot) = f .$$

On the other hand, assuming for a while that w is smooth (and that the infimum is always achieved in its definition), any admissible control ν should be such that $dY_{t,x,y}^\nu(s) \geq dw(s, X_{t,x}^\nu(s))$ when $(s, X_{t,x}^\nu(s), Y_{t,x,y}^\nu(s)) \in \partial_Z D := \{(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} : y = w(t, x)\}$, in order to avoid crossing the boundary of D . Applying Itô’s lemma to w , this shows that ν_s should satisfy

$$\sigma_Y(Z_{t,x,y}^\nu(s), \nu_s) = \sigma_X(X_{t,x}^\nu(s), \nu_s)^\top Dw(s, X_{t,x}^\nu(s))$$

and

$$\mu_Y(Z_{t,x,y}^\nu(s), \nu_s) \geq \mathcal{L}_X^{\nu_s} w(s, X_{t,x}^\nu(s)) ,$$

where \mathcal{L}_X^u denotes the Dynkin operator of X for the control u . This formally shows that V solves the constrained Hamilton–Jacobi–Bellman equation

$$(1.6) \quad - \sup_{u \in U(t,x,y)} \mathcal{L}_Z^u V = 0 ,$$

where $U(\cdot) := \{u \in U : \sigma_Y(\cdot, u) = \sigma_X(\cdot, u)^\top Dw, \mu_Y(\cdot, u) - \mathcal{L}_X^u w \geq 0\}$

on the boundary $\partial_Z D$, where, by Soner and Touzi [22] and Bouchard, Elie, and Touzi [5], the auxiliary value function w is a (discontinuous viscosity) solution of

$$(1.7) \quad \sup_{u \in U, \sigma_Y(\cdot, u) = \sigma_X(\cdot, u)^\top Dw} (\mu_Y(\cdot, w, u) - \mathcal{L}_X^u w) = 0, g(\cdot, w(T-, \cdot)) = 0 .$$

This implies that no a priori assumption has to be imposed on the boundary on D and the coefficients μ_Z and σ_Z in order to ensure that $Z_{t,z}^\nu$ can actually be reflected on the boundary. It is automatically incorporated in the auxiliary value function through its PDE characterization (1.7). This is also a main difference with the literature on state constraints where either strong conditions are imposed on the boundary, so as to ensure that the constrained process can be reflected, or the solution explodes on the boundary; see the above-quoted papers.

From the numerical point of view, the characterization (1.7) also allows one to compute the value function V by first solving (1.7) to compute w and then using w to solve (1.4)–(1.6).

The main difficulties come from the following points:

1. The stochastic control problem (1.1) being nonstandard, we first need to establish a dynamic programming principle for optimal control under stochastic constraints. This is done by appealing to the *geometric dynamic principle* of Soner and Touzi [21].
2. The auxiliary value function w is in general not smooth. To give a sense to (1.6), we therefore need to consider a weak formulation which relies on test functions for w . This essentially amounts to considering a proximal normal vector at the singular points of ∂D ; see [10].
3. The set U in which the controls take their values is a priori not compact, which makes (1.4) and (1.6) discontinuous. As usual, this is overcome by considering the lower semicontinuous and upper semicontinuous envelopes of the corresponding operators; see [11].

We should also note that, in classical state constraint problems, the subsolution property on the spacial boundary is usually not fully specified; see the above-quoted papers. One can typically show only that the PDE (1.4) propagates up to the boundary. In the case where w is continuous, we actually prove that V is a subsolution of the constraint Hamilton–Jacobi–Bellman equation (1.6) on $\partial_Z D$. Under additional assumptions on the coefficients and w , this allows us to provide a PDE characterization for $\mathcal{V}(t, x) := V(t, x, w(t, x))$ and allows us to replace the boundary condition (1.6) by a simple Dirichlet condition $V(t, x, y) = \mathcal{V}(t, x)$ on $\{y = w(t, x)\}$. This simplifies the numerical resolution of the problem: first solve (1.7) to compute w and therefore D , then solve the PDE associated with \mathcal{V} , and finally solve the Hamilton–Jacobi–Bellman equation (1.4) in D with the boundary conditions (1.5) at $t = T$ and $V(t, x, y) = \mathcal{V}(t, x)$ on $\partial_Z D$.

Many extensions of this work could be considered. First, jumps could be introduced without much difficulty; see [4] and [19] for stochastic target problems with jumps. One could also consider more stringent conditions of the form $g(Z_{t,z}^\nu(s)) \geq 0$ \mathbb{P} -a.s. for all $t \leq s \leq T$ by applying the American version of the geometric dynamic programming of [6], or moments constraints of the form $\mathbb{E}[g(Z_{t,z}^\nu(T))] \geq 0$ \mathbb{P} -a.s. or $\mathbb{P}[g(Z_{t,z}^\nu(T)) \geq 0] \geq p$ by following the ideas introduced in [5]. Such extensions are rather immediate and will be discussed in section 5.

The rest of the paper is organized as follows. The problem and its link with standard state constraints problems are presented in section 2. Section 3 contains our main results. The possible immediate extensions and an example of application are discussed in sections 4 and 5. The proofs are collected in section 6. Finally, the appendix contains the proof of a version of the geometric dynamic programming principle we shall use in this paper.

Notation. For any $\kappa \in \mathbb{N}$, we shall use the following notation. Any element of \mathbb{R}^κ is viewed as a column vector. The Euclidean norm of a vector or a matrix is denoted by $|\cdot|$, $^\top$ stands for transposition, and $\langle \cdot, \cdot \rangle$ denotes the natural scalar product. We denote by \mathbb{M}^κ (resp., \mathbb{S}^κ) the set of κ -dimensional (resp., symmetric) square matrices. Given a smooth function $\varphi : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^\kappa \mapsto \varphi(t, x) \in \mathbb{R}$, we denote by $\partial_t \varphi$ its derivative with respect to its first variable and by $D\varphi$ and $D^2\varphi$ its Jacobian and Hessian matrix with respect to the second one. For a set $\mathcal{O} \subset \mathbb{R}^\kappa$, $\text{int}(\mathcal{O})$ denotes its interior, $\text{cl}(\mathcal{O})$ its closure, and $\partial\mathcal{O}$ its boundary. If $B = [s, t] \times \mathcal{O}$ for $s \leq t$ and $\mathcal{O} \subset \mathbb{R}^\kappa$, we write $\partial_p B := ([s, t] \times \partial\mathcal{O}) \cup (\{t\} \times \text{cl}(\mathcal{O}))$ for its parabolic boundary.

Given $r > 0$ and $x \in \mathbb{R}^k$, $B_r(x)$ denotes the open ball of radius r and center x . In the following, the variable $z \in \mathbb{R}^{d+1}$ will often be understood as a couple $(x, y) \in \mathbb{R}^d \times \mathbb{R}$.

2. The optimal control problem under stochastic target constraint.

2.1. Problem formulation. Let $T > 0$ be the finite time horizon and let Ω denote the space of \mathbb{R}^d -valued continuous functions $(\omega_t)_{t \leq T}$ on $[0, T]$, $d \geq 1$, endowed with the Wiener measure \mathbb{P} . We denote by W the coordinate mapping, i.e., $(W(\omega)_t)_{t \leq T} = (\omega_t)_{t \leq T}$ for $\omega \in \Omega$, so that W is a d -dimensional Brownian motion on the canonical filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where \mathcal{F} is the Borel tribe of Ω and $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ is the \mathbb{P} -augmentation of the right-continuous filtration generated by W .

Let \mathcal{U} be the collection of progressively measurable processes ν in $L^2([0, T] \times \Omega)$, with values in a given closed subset U of \mathbb{R}^d .

For $t \in [0, T]$, $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}$, and $\nu \in \mathcal{U}$, the controlled process $Z_{t,z}^\nu := (X_{t,x}^\nu, Y_{t,x,y}^\nu)$ is defined as the $\mathbb{R}^d \times \mathbb{R}$ -valued unique strong solution of the stochastic differential equation (1.3), where $(\mu_X, \sigma_X) : (x, u) \in \mathbb{R}^d \times U \mapsto \mathbb{R}^d \times \mathbb{M}^d$ and $(\mu_Y, \sigma_Y) : (z, u) \in \mathbb{R}^d \times \mathbb{R} \times U \mapsto \mathbb{R} \times \mathbb{R}^d$ are assumed to be Lipschitz continuous. Note that this implies that $\sup_{t \leq s \leq T} |Z_{t,z}^\nu|$ is bounded in $L^2(\Omega)$ for any $\nu \in \mathcal{U}$.

For later use, it is convenient to consider $\mu_Z : \mathbb{R}^{d+1} \times U \rightarrow \mathbb{R}^{d+1}$ and $\sigma_Z : \mathbb{R}^{d+1} \times U \rightarrow \mathbb{M}^{d+1,d}$ defined, for $z = (x, y) \in \mathbb{R}^{d+1}$, as

$$\mu_Z(z, u) := \begin{pmatrix} \mu_X(x, u) \\ \mu_Y(z, u) \end{pmatrix} \sigma_Z(z, u) := \begin{bmatrix} \sigma_X(x, u) \\ \sigma_Y(z, u)^\top \end{bmatrix} .$$

The aim of this paper is to provide a PDE characterization of the value function of the optimal control problem under stochastic target constraint

$$(t, z) \mapsto \sup \{ \mathbb{E} [f(Z_{t,z}^\nu(T))] , \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P}\text{-a.s.} \} ,$$

which means that we restrict ourselves to controls satisfying the stochastic target constraints $g(Z_{t,z}^\nu(T)) \geq 0$ \mathbb{P} -a.s.

In order to give a sense to the above expression, we shall assume throughout this paper that $f, g : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ are two locally bounded Borel-measurable maps, and that f has quadratic growth, which ensures that the above expectation is well defined for any $\nu \in \mathcal{U}$.

In order to simplify the proof of the dynamic programming principle (see section 6.1 and the appendix), we shall restrict ourselves at time t to the subset \mathcal{U}^t of controls defined as follows:

$$\mathcal{U}^t := \{ \nu \in \mathcal{U} : \nu \text{ independent of } \mathcal{F}_t \} .$$

Our problem is thus formulated as

$$(2.1) \quad V(t, z) := \sup_{\nu \in \mathcal{U}_{t,z}} J(t, z; \nu) ,$$

where

$$J(t, z; \nu) := \mathbb{E} [f(Z_{t,z}^\nu(T))] \quad \text{for } \nu \in \mathcal{U} ,$$

$$\mathcal{U}_{t,z} := \{ \nu \in \mathcal{U}^t : g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P}\text{-a.s.} \} .$$

Remark 2.1. Note that this restriction is natural and consistent with the definition of $V(0, \cdot)$ for which no past is taken into account. It is also commonly used;

compare, e.g., with Chapter 4.3 in [23]. As already mentioned above, it simplifies the proof of the dynamic programming principle, which is the keystone in the derivation of the PDEs satisfied by V . We shall, however, see in Remark 6.1 that the restriction to \mathcal{U}^t is indeed not important, in the sense that allowing a dependence on \mathcal{F}_t in the definition of the set of controls would not change the value function.

For the sake of simplicity, we shall consider only the case where, for every fixed x ,

$$(2.2) \quad y \longmapsto g(x, y) \text{ is nondecreasing and right-continuous.}$$

Remark 2.2. It follows from assumption (2.2) that $\mathcal{U}_{t,x,y} \supset \mathcal{U}_{t,x,y'}$ for $y \geq y'$.

Remark 2.3. We have imposed above strong Lipschitz continuity assumptions and integrability conditions on the coefficients and the set of controls in order to avoid additional technical difficulties. It will be clear from the proofs that these conditions can be relaxed in particular situations whenever the quantities introduced below are well defined.

2.2. Dynamic programming for stochastic target problems and interpretation as a state constraint problem. Note that the above problem is well-posed only on the domain

$$D := \{(t, z) \in [0, T] \times \mathbb{R}^{d+1} : \mathcal{U}_{t,z} \neq \emptyset\},$$

and that, at least formally, the control problem (2.1) is equivalent to the state constraint problem

$$\sup \left\{ \mathbb{E} \left[f(Z_{t,z}^\nu(T)) \right], \nu \in \mathcal{U} \text{ s.t. } Z_{t,z}^\nu(s) \in D \text{ } \mathbb{P}\text{-a.s. } \forall t \leq s \leq T \right\} .$$

This can be made rigorous by appealing to the *geometric dynamic programming principle* of Soner and Touzi [21].

THEOREM 2.1. *For any $(t, z) \in [0, T] \times \mathbb{R}^{d+1}$ and $\nu \in \mathcal{U}^t$, we have*

$$\exists \tilde{\nu} \in \mathcal{U}_{t,z} \text{ s.t. } \nu = \tilde{\nu} \text{ on } [t, \theta] \iff (\theta, Z_{t,z}^\nu(\theta)) \in D \text{ } \mathbb{P}\text{-a.s.}$$

for all $[t, T]$ -valued stopping times θ .

The proof of this result is provided in [21] when the set of controls is the whole set \mathcal{U} instead of \mathcal{U}^t . We explain in the appendix how to modify their arguments in our framework.

This result can be reformulated in terms of the auxiliary value function w defined as follows:

$$(2.3) \quad w(t, x) := \inf \{y \in \mathbb{R} : (t, x, y) \in D\} ,$$

which, by Remark 2.2, characterizes (the closure of) D .

COROLLARY 2.1. *For any $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$, $\nu \in \mathcal{U}^t$, and $[t, T]$ -valued stopping time θ , we have the following:*

1. *If $Y_{t,x,y}^\nu(\theta) > w(\theta, X_{t,x}^\nu(\theta))$ \mathbb{P} -a.s., then there exists a control $\tilde{\nu} \in \mathcal{U}_{t,x,y}$ such that $\nu = \tilde{\nu}$ on $[t, \theta]$.*
2. *If there exists $\tilde{\nu} \in \mathcal{U}_{t,x,y}$ such that $\nu = \tilde{\nu}$ on $[t, \theta]$, then $Y_{t,x,y}^\nu(\theta) \geq w(\theta, X_{t,x}^\nu(\theta))$ \mathbb{P} -a.s.*

Note that the latter result allows us to provide a PDE characterization of the auxiliary value function; see (1.7). A rigorous version is reported in Theorem 3.2 under strong smoothness assumptions. The nonsmooth case is studied in [22] when U is bounded and in [5] when U is not bounded, and we refer to these papers for the exact formulation in more general situations.

From now on, we shall work under the following assumption.

Standing assumption. w is continuous on $[0, T] \times \mathbb{R}^d$ and admits a continuous extension \hat{w} on $[0, T] \times \mathbb{R}^d$ such that $g(\cdot, \hat{w}(T, \cdot)) \geq 0$ on \mathbb{R}^d .

In the following, we shall write $w(T, \cdot)$ for $\hat{w}(T, \cdot)$ for ease of notation.

Note that the previous assumption implies that $\text{cl}(D)$ can be written as $\text{int}_p(D) \cup \partial_p D$ with

$$(2.4) \quad \begin{cases} \text{int}_p(D) &= \{(t, x, y) \in [0, T] \times \mathbb{R}^{d+1} : y > w(t, x)\} , \\ \partial_p D &= \partial_Z D \cup \partial_T D , \end{cases}$$

where

$$(2.5) \quad \begin{aligned} \partial_Z D &:= \partial D \cap ([0, T] \times \mathbb{R}^{d+1}) = \{(t, x, y) \in [0, T] \times \mathbb{R}^{d+1} : y = w(t, x)\} , \\ \partial_T D &:= \partial D \cap (\{T\} \times \mathbb{R}^{d+1}) = \{(t, x, y) \in \{T\} \times \mathbb{R}^{d+1} : y \geq w(t, x)\} . \end{aligned}$$

3. Viscosity characterization of the value function. Before stating our main result, let us start with a formal discussion.

3.1. Formal discussion.

In the interior of the domain. First, it follows from Theorem 2.1 that the constraint is not binding in the interior of the domain. We can thus expect V to be a viscosity solution on $\text{int}_p(D)$ of the usual Hamilton–Jacobi–Bellman equation

$$-\partial_t \varphi(t, z) + H(z, D\varphi(t, z), D^2\varphi(t, z)) = 0 ,$$

where, for $(z, q, A) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \times \mathbb{S}^{d+1}$,

$$(3.1) \quad \begin{cases} H(z, q, A) &:= \inf_{u \in U} H^u(z, q, A) , \\ H^u(z, q, A) &:= -\langle \mu_Z(z, u), q \rangle - \frac{1}{2} \text{Tr} [(\sigma_Z \sigma_Z^\top)(z, u)A] . \end{cases}$$

However, since U may not be bounded, the above operator is not necessarily continuous and we shall have to relax it and consider its lower semicontinuous and upper semicontinuous envelopes H_* and H^* on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d+1} \times \mathbb{S}^{d+1}$.

On the time boundary. From the definition of V , one could also expect that $V_*(T, z) \geq f_*(z)$ and $V^*(T, z) \leq f^*(z)$. This will indeed be satisfied, and the proof is actually standard. The only difficulty comes from the fact that U may be unbounded.

On the spacial boundary. We now discuss the boundary condition on $\partial_Z D$. First, note that Theorem 2.1 implies that the process $Z_{t,z}^\nu$ should never cross the boundary $\partial_Z D$ whenever $\nu \in \mathcal{U}_{t,z}$. In view of (2.5), this implies that at the limit, when $y \rightarrow w(t, x)$, the process $Y_{t,x,y}^\nu - w(\cdot, X_{t,x}^\nu)$ should have a nonnegative drift and a zero volatility. This means that, at a formal level, the control ν should satisfy

$$\begin{aligned} \sigma_Y(x, y, \nu_t) - \sigma_X(x, \nu_t)^\top Dw(t, x) &= 0 , \\ \mu_Y(x, y, \nu_t) - \mathcal{L}_X^\nu w(t, x) &\geq 0 , \end{aligned}$$

where \mathcal{L}_X^u denotes the Dynkin operator of X ; precisely, for a smooth function $\varphi \in C^{1,2}$,

$$\mathcal{L}_X^u \varphi(t, x) := \partial_t \varphi(t, x) + \langle \mu_X(x, u), D\varphi(t, x) \rangle + \frac{1}{2} \text{Tr} [(\sigma_X \sigma_X^\top)(x, u) D^2 \varphi(t, x)] .$$

Hence, V should satisfy on $\partial_Z D$ the following equation:

$$-\partial_t \varphi(t, z) + H_{\text{int}}(z, D\varphi(t, z), D^2 \varphi(t, z)) = 0 ,$$

where

$$H_{\text{int}}(t, x, y, q, A) = \inf_{u \in U_{\text{int}}(t, x, y, w)} H^u(x, y, q, A)$$

and

$$U_{\text{int}}(t, x, y, w) = \{u \in U : \sigma_Y(x, y, u) - \sigma_X(x, u)^\top Dw(t, x) = 0, \mu_Y(x, y, u) - \mathcal{L}_X^u w(t, x) \geq 0\}$$

corresponds to controls driving the process *inside* the domain. Remark that $H_{\text{int}} \geq H$ since $U_{\text{int}} \subset U$.

Since w may not be smooth, we need to use the notion of test functions to give a precise meaning to the above expression. We therefore introduce the set $\mathcal{W}^*(t, x)$ defined as follows:

$$\mathcal{W}^*(t, x) = \{\phi \in C^{1,2}([0, T] \times \mathbb{R}^d) \text{ s.t. } (w - \phi) < (w - \phi)(t, x) = 0 \text{ in a neighborhood of } (t, x)\}.$$

The set $\mathcal{W}_*(t, x)$ is defined analogously:

$$\mathcal{W}_*(t, x) = \{\phi \in C^{1,2}([0, T] \times \mathbb{R}^d) \text{ s.t. } (w - \phi) > (w - \phi)(t, x) = 0 \text{ in a neighborhood of } (t, x)\}.$$

Our operators are then defined as follows.

0. *Relaxation.* As in [5], we first need to relax the constraints in U_{int} . For $\delta > 0$, $\gamma \in \mathbb{R}$, and a smooth function ϕ , we therefore define

$$\begin{aligned} N^u(x, y, q) &:= \sigma_Y(x, y, u) - \sigma_X(x, u)^\top q , \\ \mathcal{N}_\delta(x, y, q) &:= \{u \in U : |N^u(x, y, q)| \leq \delta\} , \\ U_{\delta, \gamma}(t, x, y, \phi) &:= \{u \in \mathcal{N}_\delta(x, y, D\phi(t, x)) : \mu_Y(x, y, u) - \mathcal{L}_X^u \phi(t, x) \geq \gamma\} \end{aligned}$$

and

$$F_{\delta, \gamma}^\phi(t, z, q, A) := \inf_{u \in U_{\delta, \gamma}(t, z, \phi)} \{-\langle \mu_Z(z, u), q \rangle - \text{Tr} [(\sigma_Z \sigma_Z^\top)(z, u) A]\}$$

with the convention $\inf \emptyset = \infty$. The set $U_{\delta, \gamma}(\cdot, \phi)$ and the operator $F_{\delta, \gamma}^\phi$ will correspond to the relaxed versions of U_{int} and H_{int} associated with a test function ϕ for w .

Remark 3.1. Remark that the fact that $U_{\delta, \gamma}(t, z, \phi) \subset U$ implies, in particular, that, for all (t, z, q, A) , we have

$$(3.2) \quad F_{\delta, \gamma}^\phi(t, z, q, A) \geq H(z, q, A)$$

for any smooth function ϕ .

1. *Supersolution.* We first relax the supersolution property by classically considering the upper half-relaxed limit $F^{\phi*}$ of the family of functions $(F_{\delta,\gamma}^\phi)_{\delta,\gamma}$. We recall that it is defined as follows:

$$F^{\phi*}(t, z, q, A) := \limsup_{\substack{(t', z', q', A') \rightarrow (t, z, q, A) \\ (\delta', \gamma') \rightarrow 0}} F_{\delta', \gamma'}^\phi(t', z', q', A'),$$

for $\phi \in \mathcal{W}^*(t, x)$ and $(t, z, q, A) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d+1} \times \mathbb{S}^{d+1}$.

Note that, when $w \in C^{1,2}$, $\phi \in \mathcal{W}^*(t, x)$ implies that $\partial_t w(t, x) \leq \partial_t \phi(t, x)$, $Dw(t, x) = D\phi(t, x)$, and $D^2 w(t, x) \leq D^2 \phi(t, x)$, which leads to $U_{\delta,\gamma}(\cdot, \phi) \subset U_{\delta,\gamma}(\cdot, w)$, and therefore

$$(3.3) \quad \text{if } w \in C^{1,2}, \text{ then } \forall \phi \in \mathcal{W}^*(t, x), \quad F_{\delta,\gamma}^\phi(t, z, q, A) \geq F_{\delta,\gamma}^w(t, z, q, A).$$

Also note that

$$(3.4) \quad F^{\phi*}(t, z, q, A) := \limsup_{\substack{(t', z', q', A') \rightarrow (t, z, q, A) \\ \gamma' \downarrow 0}} F_{0,\gamma'}^\phi(t', z', q', A'),$$

since $U_{0,\gamma'} \subset U_{\delta,\gamma}$ for $\delta \geq 0$ and $\gamma \leq \gamma'$. Moreover, it follows from (3.2) that $F^{\phi*}$ is the upper semicontinuous envelope of

$$\bar{\mathcal{H}}^\phi(t, z, q, A) := \begin{cases} H^*(z, q, A) & \text{if } (t, z) \in \text{int}_p(D), \\ F^{\phi*}(t, z, q, A) & \text{if } (t, z) \in \partial_p D \end{cases}$$

on $\partial_p D$, which is consistent with Definition 7.4 in [11].

2. *Subsolution.* In order to relax the subsolution property, we consider the lower half-relaxed limit F_*^ϕ of $(F_{\delta,\gamma}^\phi)_{\delta,\gamma}$. We recall that

$$(3.5) \quad F_*^\phi(t, z, q, A) := \liminf_{\substack{(t', z', q', A') \rightarrow (t, z, q, A) \\ (\delta', \gamma') \rightarrow 0}} F_{\delta', \gamma'}^\phi(t', z', q', A'),$$

for $\phi \in \mathcal{W}^*(t, x)$ and $(t, z, q, A) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d+1} \times \mathbb{S}^{d+1}$.

As above, we can note that, when $w \in C^{1,2}$, $\phi \in \mathcal{W}_*(t, x)$ implies that $\partial_t w(t, x) \geq \partial_t \phi(t, x)$, $Dw(t, x) = D\phi(t, x)$, and $D^2 w(t, x) \geq D^2 \phi(t, x)$, which leads to the following:

$$(3.6) \quad \text{if } w \in C^{1,2}, \text{ then } \forall \phi \in \mathcal{W}_*(t, x), \quad F_{\delta,\gamma}^\phi(t, z, q, A) \leq F_{\delta,\gamma}^w(t, z, q, A).$$

However, (3.2) implies that $F_*^\phi \geq H_*$. Hence,

$$\underline{\mathcal{H}}^\phi(z, q, A) := \begin{cases} H_*(z, q, A) & \text{if } (t, z) \in \text{int}_p(D), \\ F_*^\phi(t, z, q, A) & \text{if } (t, z) \in \partial_p D \end{cases}$$

may not be lower semicontinuous. Since, according to [11], the subsolution property has to be stated in terms of the lower semicontinuous envelope $\underline{\mathcal{H}}_*^\phi$ of $\underline{\mathcal{H}}^\phi$, we can not a priori state the boundary condition in terms of F_*^ϕ . This will be possible only under additional assumptions on the coefficients; see Proposition 3.1.

To alleviate notation, we simply write $F^{\phi*}\varphi(t, z)$ for $F^{\phi*}(t, z, D\varphi(t, z), D^2\varphi(t, z))$ below, and similarly for $F_*^\phi\varphi(t, z)$, $H_*\varphi(t, z)$, $H^*\varphi(t, z)$, $\bar{\mathcal{H}}^\phi\varphi(t, z)$, $\underline{\mathcal{H}}^\phi\varphi(t, z)$, and $\underline{\mathcal{H}}_*^\phi\varphi(t, z)$.

Finally, since the value function V may not be smooth, we can provide only a PDE characterization in the discontinuous viscosity sense. We therefore introduce its semicontinuous envelopes: for all $(t, z) \in \text{cl}(D)$,

$$V_*(t, z) := \liminf_{(t', z') \in \text{int}_p(D) \rightarrow (t, z)} V(t', z'), V^*(t, z) := \limsup_{(t', z') \in \text{int}_p(D) \rightarrow (t, z)} V(t', z') .$$

3.2. Main results. We shall appeal to the following weak version of the continuity assumption on \mathcal{N}_0 made in [5] (it will be strengthened when discussing the subsolution property on the boundary; see (ii) of Assumption 3.3 below).

Assumption 3.1 (continuity of $\mathcal{N}_0(x, y, q)$). Let ψ be a Lipschitz continuous function on $[0, T] \times \mathbb{R}^d$ and let \mathcal{O} be some open subset of $[0, T] \times \mathbb{R}^{d+1}$ such that $\mathcal{N}_0(x, y, \psi(t, x)) \neq \emptyset$ for all $(t, x, y) \in \mathcal{O}$. Then, for every $\varepsilon > 0$, $(t_0, x_0, y_0) \in \mathcal{O}$ and $u_0 \in \mathcal{N}_0(x_0, y_0, \psi(t_0, x_0))$, there exist an open neighborhood \mathcal{O}' of (t_0, x_0, y_0) and a locally Lipschitz continuous map \hat{u} defined on \mathcal{O}' such that $|\hat{u}(t_0, x_0, y_0) - u_0| \leq \varepsilon$ and $\hat{u}(t, x, y) \in \mathcal{N}_0(x, y, \psi(t, x))$ on \mathcal{O}' .

Remark 3.2. As we shall see in the statement of the following theorem (and as will be clear from its proof), Assumption 3.1 is required only to get the supersolution property on $\partial_Z D$.

We can now state the main result of this section.

THEOREM 3.1 (PDE characterization of the value function). *The following hold:*

1. *If Assumption 3.1 holds, then the function V_* is a viscosity supersolution on $\text{cl}(D)$ of*

$$(3.7) \quad \left. \begin{aligned} &(-\partial_t \varphi + H^* \varphi)(t, x, y) \geq 0 \quad \text{if } (t, x, y) \in \text{int}_p(D) \\ &(-\partial_t \varphi + F^{\phi_*} \varphi)(t, x, y) \geq 0 \quad \text{if } (t, x, y) \in \partial_Z D \end{aligned} \right\},$$

$$(3.8) \quad \left. \begin{aligned} &\varphi(T, x, y) \geq f_*(x, y) \quad \text{if } (t, x, y) \in \partial_T D \\ &y > w(T, x), H^* \varphi(T, x, y) < \infty \end{aligned} \right\}.$$

2. *The function V^* is a viscosity subsolution on $\text{cl}(D)$ of*

$$(3.9) \quad (-\partial_t \varphi + H_* \varphi)(t, x, y) \leq 0 \quad \text{if } (t, x, y) \in \text{int}_p(D) \cup \partial_Z D ,$$

$$(3.10) \quad \varphi(T, x, y) \leq f^*(x, y) \quad \text{if } (t, x, y) \in \partial_T D \quad \text{and} \quad H_* \varphi(T, x, y) > -\infty .$$

The proof will be provided in section 6.

Remark 3.3. Note that the above theorem does not provide the boundary condition for V_* on the corner $\{(T, x, y) : y = w(T, x)\}$. However, particular cases can be studied in more details. For instance, if U is bounded, one easily checks, by using the Lipschitz continuity of μ_Z and σ_Z , that $Z_{t_n, z_n}^{\nu_n}(T) \rightarrow z_0$ in $L^4(\Omega)$ for any sequence $(t_n, z_n, \nu_n)_n \in D \times \mathcal{U}$ such that $\nu_n \in \mathcal{U}_{t_n, z_n}$ for all n and $(t_n, z_n) \rightarrow (T, z_0)$. Since, by definition, $V(t_n, z_n) \geq \mathbb{E}[f(Z_{t_n, z_n}^{\nu_n}(T))]$, the quadratic growth assumption on f implies $\liminf_{n \rightarrow \infty} V(t_n, z_n) \geq f_*(z_0)$. A more general result will be stated in Proposition 3.2 when w is smooth.

To conclude this section, we now provide an equivalence result between subsolutions of $-\partial_t \varphi + H_* \varphi = 0$ and subsolutions of $-\partial_t \varphi + F_*^\phi \varphi = 0$ for any $\phi \in \mathcal{W}_*$, under the following additional assumption.

Assumption 3.2. One of the following holds:

- U is bounded.

- $|\sigma_Y(x, y, u)| \rightarrow +\infty$ as $|u| \rightarrow +\infty$ and
(3.11)

$$\limsup_{|u| \rightarrow \infty, u \in U} \frac{|\mu_Y(x, y, u)| + |\mu_X(x, u)| + |\sigma_X(x, u)|^2}{|\sigma_Y(x, y, u)|^2} = 0 \quad \forall (x, y) \in \mathbb{R}^{d+1},$$

and the convergence is local uniform with respect to (x, y) .

Remark 3.4. Note that the above assumption implies that $N_\delta(x, y, q)$ is compact for all $(x, y, q) \in \mathbb{R}^{d+1} \times \mathbb{R}^d$.

Remark 3.5. This condition will be satisfied in the example of application of section 4.

PROPOSITION 3.1. *Let Assumption 3.2 hold. Let v be an upper semicontinuous function. Assume that for all smooth functions φ and $(t, x, y) \in \partial_Z D$ such that (t, x, y) is a strict maximum point on $\text{cl}(D)$ of $v - \varphi$ we have*

$$-\partial_t \varphi(t, x, y) + H_* \varphi(t, x, y) \leq 0.$$

Then, for all smooth functions φ and $(t, x, y) \in \partial_Z D$ such that (t, x, y) is a strict maximum point on $\text{cl}(D)$ of $v - \varphi$ and all $\phi \in \mathcal{W}_(t, x)$, we have*

$$-\partial_t \varphi(t, x, y) + F_*^\phi \varphi(t, x, y) \leq 0.$$

The proof will be provided in section 6.3.

This allows us to rewrite the spacial boundary condition for V^* in terms of the more natural constrained operator F_*^ϕ instead of H_* .

COROLLARY 3.1 (boundary conditions in the classical sense). *Let Assumption 3.2 hold. Then, V^* is a viscosity subsolution on $\text{cl}(D)$ of*

$$(3.12) \quad \forall \phi \in \mathcal{W}_*(t, x), \quad \begin{aligned} &(-\partial_t \varphi + H_* \varphi)(t, x, y) \leq 0 \quad \text{if } (t, x, y) \in \text{int}_p(D) \\ &(-\partial_t \varphi + F_*^\phi \varphi)(t, x, y) \leq 0 \quad \text{if } (t, x, y) \in \partial_Z D. \end{aligned}$$

3.3. Dirichlet boundary condition when w is smooth. Recall that the boundary $\partial_Z D$ corresponds to the point (t, x, y) such that $y = w(t, x)$. It is therefore tempting to rewrite the boundary condition on $\partial_Z D$ for V as $V(t, x, y) = \mathcal{V}(t, x) := V(t, x, w(t, x))$ and try to obtain a PDE characterization for \mathcal{V} by performing a simple change of variables.

In this section, we shall assume that w is smooth and that $\mathcal{N}_0(z, p)$ admits a unique argument which is continuous in (z, p) . This assumption is made precise as follows.

Assumption 3.3.

- (i) The map w is $C^{1,2}([0, T] \times \mathbb{R}^d)$.
- (ii) There exists a locally Lipschitz map $\tilde{u} : \mathbb{R}^{d+1} \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\mathcal{N}_0(z, p) \neq \emptyset \Rightarrow \mathcal{N}_0(z, p) = \{\tilde{u}(z, p)\}.$$

Remark 3.6. Note that Assumption 3.1 is a consequence of Assumption 3.3.

Under the above assumption, it follows from [5] and [22] that w is a strong solution of (3.13).

THEOREM 3.2 (see [5] and [22]). *Let Assumption 3.3 hold. Then, $\mathcal{N}_0(\cdot, w, Dw) \neq \emptyset$ on $[0, T] \times \mathbb{R}^d$ and w satisfies*

$$(3.13) \quad 0 = \min \left\{ \mu_Y(\cdot, w, \hat{u}) - \mathcal{L}_X^{\hat{u}} w, \mathbf{1}_{\hat{u} \in \text{int}(U)} \right\} \quad \text{on } [0, T] \times \mathbb{R}^d,$$

where

$$\hat{u} := \check{u}(\cdot, w, Dw) .$$

Moreover, for all $x \in \mathbb{R}^d$ and smooth functions ϕ such that x reaches a local minimum of $w(T, \cdot) - \phi$, the set $\mathcal{N}_0(x, w(T, x), D\phi(x))$ is nonempty.

Remark 3.7. Theorem 3.2 is proved in [5] and [22] when the set of controls is \mathcal{U} in place of \mathcal{U}^t as a consequence of the geometric dynamic programming principle of Soner and Touzi [21]. Even if the formulation is slightly different here, the proofs of the above papers can be entirely reproduced for \mathcal{U}^t in place of \mathcal{U} by appealing to the version of geometric dynamic programming principle stated in Theorem 2.1 and Corollary 2.1.

Also observe that the first assertion in Assumption 3.3 combined with the discussion of section 3.1 (see (3.3) and (3.6)) implies that the boundary condition on $\partial_Z D$ can be simplified and written in terms of $F_{\delta, \gamma}^w$. Using this simplification and computing the derivatives of \mathcal{V} in terms of the derivatives of V and w , we deduce from Theorem 3.1, Corollary 3.1, and Theorem 3.2 that, at least at a formal level, \mathcal{V} is a solution of (3.14)–(3.15).

To be more precise, the viscosity solution property will be stated in terms of

$$\mathcal{V}_*(t, x) := V_*(t, x, w(t, x)) \quad \text{and} \quad \mathcal{V}^*(t, x) := V^*(t, x, w(t, x))$$

for $(t, x) \in [0, T] \times \mathbb{R}^d$. Since w is continuous, \mathcal{V}_* (resp., \mathcal{V}^*) is lower semicontinuous (resp., upper semicontinuous).

Our proof of the subsolution property will also require the following additional technical assumption.

Assumption 3.4. The condition (3.11) holds. Moreover, for all sequences $(t_n, x_n)_n$ in $[0, T] \times \mathbb{R}^d$ such that $(t_n, x_n) \rightarrow (t_0, x_0) \in [0, T] \times \mathbb{R}^d$ such that $\hat{u}(t_0, x_0) \in \text{int}(U)$, we have

$$\liminf_{n \rightarrow \infty} \inf_{u \in U} [n (\mathcal{L}_X^u w(t_n, x_n) - \mu_Y(x_n, w(t_n, x_n), u)) + n^2 |N^u(x_n, w(t_n, x_n), Dw(t_n, x_n))|^2] > -\infty .$$

We can now state the main result of this section.

PROPOSITION 3.2. *Let Assumption 3.3 hold, and assume that V and w have polynomial growth. Then \mathcal{V}_* is a viscosity supersolution on $[0, T] \times \mathbb{R}^d$ of*

$$(3.14) \quad \begin{cases} (-\mathcal{L}_X^{\hat{u}} \varphi) \mathbf{1}_{\hat{u} \in \text{int}(U)} = 0 & \text{on } [0, T] \times \mathbb{R}^d, \\ \varphi(T, \cdot) \geq f_*(\cdot, w(T, \cdot)) & \text{if } \limsup_{\substack{(t', x') \rightarrow (T, \cdot) \\ t' < T}} |\hat{u}(t', x')| < \infty \text{ and } \mathcal{W}^*(T, \cdot) \neq \emptyset . \end{cases}$$

Moreover, if, in addition, Assumption 3.4 holds, then \mathcal{V}^* is a viscosity subsolution on $[0, T] \times \mathbb{R}^d$ of

$$(3.15) \quad \begin{cases} (-\mathcal{L}_X^{\hat{u}} \varphi) \mathbf{1}_{\hat{u} \in \text{int}(U)} = 0 & \text{in } [0, T] \times \mathbb{R}^d, \\ \varphi(T, \cdot) \leq f^*(\cdot, w(T, \cdot)) & \text{if } \mathcal{W}_*(T, \cdot) \neq \emptyset . \end{cases}$$

See section 6.4 for the proof.

Remark 3.8. (i) Note that $\mathbf{1}_{\hat{u} \in \text{int}(U)} = 1$ if $U = \mathbb{R}^d$. Even if $U \neq \mathbb{R}^d$, the PDE simplifies whenever w solves

$$(3.16) \quad \mu_Y(\cdot, w, \hat{u}) - \mathcal{L}_X^{\hat{u}} w = 0 \quad \text{on } [0, T] \times \mathbb{R}^d .$$

It will be clear from the proof of Proposition 3.2 that in this case \mathcal{V}_* is a viscosity supersolution on $[0, T) \times \mathbb{R}^d$ of $-\mathcal{L}_X^{\hat{u}}\varphi = 0$. Similarly, \mathcal{V}^* is a viscosity subsolution of the same equation if Assumption 3.4 holds without the condition $\hat{u}(t_0, x_0) \in \text{int}(U)$. We refer the reader to [8] for situations in mathematical finance where w is actually a viscosity solution of the above equation, although $U \neq \mathbb{R}^d$.

(ii) Assumption 3.4 will be satisfied in our example of application in section 4. Note that it also holds in our general framework when μ_Z and $|\sigma_X|^2$ are locally Lipschitz in u , locally uniformly in the other variables, and

$$|N^u(x, y, q) - N^{u'}(x, y, q)|^2 \geq \rho_1(x, y, q, u, u')|u - u'|^2 - \rho_2(x, y, q, u, u'),$$

for some locally bounded maps ρ_1, ρ_2 such that $\rho_1 > 0$ locally uniformly. Indeed, in this case, we have $\hat{u}(t_n, x_n) \rightarrow \hat{u}(t_0, x_0) \in \text{int}(U)$ and Theorem 3.2 implies that

$$\begin{aligned} & n(\mathcal{L}_X^u w(t_n, x_n) - \mu_Y(x_n, w(t_n, x_n), u)) + n^2|N^u(x_n, w(t_n, x_n), Dw(t_n, x_n))|^2 \\ &= n(\mathcal{L}_X^u w(t_n, x_n) - \mu_Y(x_n, w(t_n, x_n), u)) \\ &\quad - n\left(\mathcal{L}_X^{\hat{u}(t_n, x_n)} w(t_n, x_n) - \mu_Y(x_n, w(t_n, x_n), \hat{u}(t_n, x_n))\right) \\ &\quad + n^2|N^u(x_n, w(t_n, x_n), Dw(t_n, x_n)) - N^{\hat{u}(t_n, x_n)}(x_n, w(t_n, x_n), Dw(t_n, x_n))|^2 \\ &\geq -nC_1|u - \hat{u}(t_n, x_n)| + n^2C_2|u - \hat{u}(t_n, x_n)|^2 - C_3 \geq -\frac{C_1^2}{4C_2} - C_3, \end{aligned}$$

for some constant $C_1, C_2, C_3 > 0$. In view of (i), it also holds without the condition $\hat{u}(t_0, x_0) \in \text{int}(U)$ when w solves (3.16).

(iii) If the map \hat{u} is locally bounded, then the condition $\limsup\{|\hat{u}(t', x')|, (t', x') \rightarrow (T, \cdot), t' < T\} < \infty$ holds when $x \mapsto \limsup\{|Dw(t', x')|, (t', x') \rightarrow (T, x), t' < T\}$ is locally bounded.

Under the above assumptions, we can then replace the boundary condition of Theorem 3.1 and Corollary 3.1 by a simple Dirichlet condition $V(t, x, y) = \mathcal{V}(t, x)$ on $\partial_Z D$, where \mathcal{V} can be computed by solving $-\mathcal{L}_X^{\hat{u}}\varphi = 0$ on $[0, T) \times \mathbb{R}^d$ with the corresponding terminal condition at $t = T$.

Obviously, the construction of the associated numerical schemes requires at least comparison results for the corresponding PDEs which are a priori highly nonlinear and not continuous. We leave this important point for further research.

4. Example of application in mathematical finance. In order to illustrate our results, we present in detail in this section an application to the problem of terminal wealth maximization under a superreplication constraint.

4.1. The optimization problem. Namely, we consider a financial market consisting of a riskless asset, whose price process is normalized to 1 for simplicity, and one risky asset whose dynamics is given by

$$X_{t,x}(s) = x + \int_t^s X_{t,x}(u)\mu(X_{t,x}(u))du + \int_t^s X_{t,x}(u)\sigma(X_{t,x}(u))dW_u,$$

where $x \in [0, \infty) \mapsto (x\mu(x), x\sigma(x))$ is Lipschitz continuous and

$$(4.1) \quad \sigma \geq \varepsilon \quad \text{and} \quad |\mu/\sigma| \leq \varepsilon^{-1} \quad \text{for some real constant } \varepsilon > 0.$$

The control process ν corresponding to the amount of money invested in the risky asset is valued in $U = \mathbb{R}$. Under the self-financing condition, the value of the

corresponding portfolio Y^ν is given by

$$Y_{t,x,y}^\nu(s) = y + \int_t^s \nu_u \frac{dX_{t,x}(u)}{X_{t,x}(u)} = y + \int_t^s \nu_u \mu(X_{t,x}(u)) du + \int_t^s \nu_u \sigma(X_{t,x}(u)) dW_u.$$

We consider a fund manager whose preferences are given by a bounded from above, C^2 , increasing, strictly concave utility function U on \mathbb{R}_+ satisfying the classical Inada conditions

$$(4.2) \quad U'(0+) = +\infty \quad \text{and} \quad U'(+\infty) = 0.$$

We assume that his objective is to maximize his expected utility of wealth under the constraint that it performs better than the benchmark (or superhedge some European option's payoff) $\psi(X_{t,x}(T))$, with ψ a continuous function with polynomial growth.

The corresponding optimal control problem under stochastic constraint is given by

$$V(t, x, y) := \sup_{\nu \in \mathcal{U}_{t,x,y}} \mathbb{E} [U(Y_{t,x,y}^\nu(T))]$$

with

$$\mathcal{U}_{t,x,y} := \{ \nu \in \mathcal{U}^t, Y_{t,x,y}^\nu(T) \geq \psi(X_{t,x}(T)) \quad \mathbb{P}\text{-a.s.} \}.$$

4.2. The PDE characterization. Note that, under the condition (4.1), we simply have

$$(4.3) \quad \mathcal{N}_\delta(x, y, q) = \{ u \in \mathbb{R} : |u\sigma(x) - x\sigma(x)q| \leq \delta \} = \{ xq + \delta\zeta/\sigma(x), |\zeta| \leq 1 \}.$$

Moreover, the function w is given by

$$(4.4) \quad w(t, x) = \mathbb{E}^{\mathbb{Q}_{t,x}} [\psi(X_{t,x}(T))],$$

where $\mathbb{Q}_{t,x} \sim \mathbb{P}$ is defined by

$$\frac{d\mathbb{Q}_{t,x}}{d\mathbb{P}} = \exp \left(-\frac{1}{2} \int_t^T |\lambda(X_{t,x}(s))|^2 ds - \int_t^T \lambda(X_{t,x}(s)) dW_s \right) \quad \text{with } \lambda := \mu/\sigma$$

(see, e.g., [15]) and is a continuous viscosity solution on $[0, T] \times (0, \infty)$ of

$$\begin{aligned} 0 &= -\partial_t \varphi(t, x) - \frac{1}{2} x^2 \sigma^2(x) D^2 \varphi(t, x) \\ &= \sup_{u \in \mathcal{N}_0(x, \varphi(t,x), D\varphi(t,x))} \left(u\mu(x) - \partial_t \varphi(t, x) - x\mu(x) D\varphi(t, x) - \frac{1}{2} x^2 \sigma^2(x) D^2 \varphi(t, x) \right) \end{aligned}$$

with the terminal condition $w(T, \cdot) = \psi$.

Also note that Assumptions 3.1 and 3.2 trivially hold in this context. It then follows that we can apply Theorem 3.1 and Corollary 3.1.

Moreover, if μ and σ are smooth enough, it follows from standard estimates that $w \in C^{1,2}([0, T] \times (0, \infty))$. If ψ is also Lipschitz continuous, then $|Dw|$ is locally bounded and so is $(t, x) \mapsto \hat{u}(t, x) = xDw(t, x)$; recall (4.3). The conditions of Assumption 3.4 are also satisfied; see (ii) of Remark 3.8. It then follows from Proposition 3.2 that \mathcal{V}_* and \mathcal{V}^* are viscosity super- and subsolutions on $[0, T] \times (0, \infty)$ of

$$0 = -\partial_t \varphi(t, x) - x\mu(x) D\varphi(t, x) - \frac{1}{2} x^2 \sigma^2(x) D^2 \varphi(t, x)$$

with the terminal condition, assuming that $\psi \in C^2$,

$$\varphi(T, x) = U(\psi(x)) .$$

Standard comparison results and the Feynman–Kac formula thus imply that

$$\mathcal{V}_*(t, x) = \mathcal{V}^*(t, x) = \mathbb{E} [U(\psi(X_{t,x}(T)))] =: \mathcal{V}(t, x) ,$$

which provides an “explicit” Dirichlet boundary condition for V on $\{y = w(t, x)\}$.

Summing up the above results, we obtain that V_* is a viscosity supersolution of

$$\begin{aligned} -\partial_t \varphi(t, x, y) + H^* \varphi(t, x, y) &\geq 0 && \text{if } t < T, y > w(t, x) , \\ \varphi(t, x, y) &\geq \mathcal{V}(t, x) && \text{if } t < T, y = w(t, x) , \\ \varphi(T, x, y) &\geq U(y) && \text{if } y \geq \psi(x) , \end{aligned}$$

and that V^* is a viscosity subsolution of

$$\begin{aligned} -\partial_t \varphi(t, x, y) + H_* \varphi(t, x, y) &\leq 0 && \text{if } t < T, y > w(t, x) , \\ \varphi(t, x, y) &\leq \mathcal{V}(t, x) && \text{if } t < T, y = w(t, x) , \\ \varphi(T, x, y) &\leq U(y) && \text{if } y \geq \psi(x) , \end{aligned}$$

where

$$H(x, y, q, A) := \inf_{u \in \mathbb{R}} \left(-\mu(x) \left\langle \begin{pmatrix} x \\ u \end{pmatrix}, q \right\rangle - \frac{1}{2} \sigma(x)^2 \text{Tr} \left[\begin{bmatrix} x^2 & xu \\ xu & u^2 \end{bmatrix} A \right] \right) .$$

4.3. Explicit resolution. In order to solve the above PDE explicitly, we proceed as in [5] and introduce the Fenchel–Legendre dual transform associated with V^* with respect to the y variable:

$$\tilde{V}^*(t, x, \gamma) := \sup_{y \in [w(t,x), \infty)} \{V^*(t, x, y) - y\gamma\} , \quad (t, x, \gamma) \in [0, T] \times (0, \infty) \times \mathbb{R} ,$$

where we use the usual extension $V^*(t, x, y) = -\infty$ if $y < w(t, x)$. Note that $\tilde{V}^*(t, x, \gamma) = \infty$ if $\gamma < 0$ since V is clearly nondecreasing in y . Moreover, since U is bounded from above, the supremum in the definition of \tilde{V}^* is achieved for any $\gamma > 0$, and \tilde{V}^* is upper semicontinuous on $[0, T] \times (0, \infty) \times (0, \infty)$.

The main interest in considering the dual transform of the value function is to get rid of the nonlinear terms in the above PDE. Indeed, by adapting the arguments of section 4 in [5], we easily obtain that \tilde{V}^* is a viscosity subsolution of $[0, T] \times (0, \infty) \times (0, \infty)$ of

$$\begin{aligned} -\mathcal{L}_X \varphi + x\sigma(x)\gamma\lambda(x)\frac{\partial^2}{\partial x \partial \gamma} \varphi - \frac{1}{2}\gamma^2\lambda(x)^2\frac{\partial^2}{\partial \gamma^2} \varphi &\leq 0 && \text{if } t < T, \frac{\partial}{\partial \gamma} \varphi(t, x, \gamma) < -w(t, x) , \\ \varphi(T, x, \gamma) &\leq U(I(\gamma) \vee \psi(x)) - \gamma[I(\gamma) \vee \psi(x)] && \text{if } t = T, \frac{\partial}{\partial \gamma} \varphi(T, x, \gamma) \leq -\psi(x) , \end{aligned}$$

where $I := (U')^{-1}$ and where we used (4.2) to derive the boundary condition. Let us now define

$$(4.5) \quad \begin{aligned} \bar{V}(t, x, \gamma) &:= \mathbb{E} [U[I(\Gamma_{t,x,\gamma}(T)) \vee \psi(X_{t,x}(T))]] \\ &\quad - \mathbb{E} [\Gamma_{t,x,\gamma}(T) \{I(\Gamma_{t,x,\gamma}(T)) \vee \psi(X_{t,x}(T))\}] , \end{aligned}$$

where $\Gamma_{t,x,\gamma}$ has the dynamics

$$\Gamma_{t,x,\gamma}(s) = \gamma + \int_t^s \lambda(X_{t,x}(u))\Gamma_{t,\gamma}(u)dW_u .$$

One easily checks that \bar{V} is a continuous viscosity solution of the above PDE satisfying $\bar{V}(T, x, \gamma) = U(I(\gamma) \vee \psi(x)) - \gamma[I(\gamma) \vee \psi(x)]$ and, recalling (4.4),

$$\begin{aligned} \frac{\partial}{\partial \gamma} \bar{V}(t, x, \gamma) &= -\mathbb{E} [\Gamma_{t,x,1}(T) \{I(\Gamma_{t,x,\gamma}(T)) \vee \psi(X_{t,x}(T))\}] \\ &\leq -\mathbb{E}^{\mathbb{Q}^{t,x}} [\psi(X_{t,x}(T))] = -w(t, x) . \end{aligned}$$

We then deduce from standard comparison results (see, e.g., [11]) that

$$(4.6) \quad \tilde{V}^*(t, x, \gamma) \leq \bar{V}(t, x, \gamma) .$$

This leads to

$$\begin{aligned} V^*(t, x, y) &\leq \inf_{\gamma > 0} \{ \bar{V}(t, x, \gamma) + y\gamma \} \\ &= \inf_{\gamma > 0} \{ \mathbb{E} [U[I(\Gamma_{t,x,\gamma}(T)) \vee \psi(X_{t,x}(T))]] \\ &\quad - \mathbb{E} [\Gamma_{t,x,\gamma}(T) \{I(\Gamma_{t,x,\gamma}(T)) \vee \psi(X_{t,x}(T))\}] + y\gamma \} . \end{aligned}$$

Computing the optimal argument $\hat{\gamma}(t, x, y)$ in the above expression (recall (4.2)), we obtain that

$$(4.7) \quad y := \mathbb{E}^{\mathbb{Q}^{t,x}} [I(\Gamma_{t,x,\hat{\gamma}(t,x,y)}(T)) \vee \psi(X_{t,x}(T))] ,$$

and therefore

$$V^*(t, x, y) \leq \bar{V}(t, x, \hat{\gamma}(t, x, y)) + y\hat{\gamma}(t, x, y) = \mathbb{E} [U[I(\Gamma_{t,x,\hat{\gamma}(t,x,y)}(T)) \vee \psi(X_{t,x}(T))]] .$$

Furthermore, (4.7), the martingale representation theorem and (4.1) ensure that we can find some \mathbb{P} -a.s.-square integrable predictable process $\hat{\nu}$, independent of \mathcal{F}_t , such that

$$Y_{t,y,x}^{\hat{\nu}}(T) = I(\Gamma_{t,\hat{\gamma}(t,x,y)}(T)) \vee \psi(X_{t,x}(T)) .$$

In particular, this implies $Y_{t,y}^{\hat{\nu}}(T) \geq \psi(X_{t,x}(T))$ so that $\nu \in \mathcal{U}_{t,x,y}$ if $I(\Gamma_{t,\hat{\gamma}(t,x,y)}(T)) \in L^{2+\varepsilon}(\Omega, \mathbb{Q})$ for some $\varepsilon > 0$; recall that ψ has polynomial growth so that $\psi(X_{t,x}(T)) \in L^p(\Omega, \mathbb{Q})$ for any $p \geq 1$. Therefore, under the above assumption, the value function is given by

$$(4.8) \quad \begin{aligned} V(t, x, y) &= \mathbb{E} [U[I(\Gamma_{t,x,\hat{\gamma}(t,x,y)}(T)) \vee \psi(X_{t,x}(T))]] \\ &\quad \text{with } y = \mathbb{E}^{\mathbb{Q}^{t,x}} [I(\Gamma_{t,x,\hat{\gamma}(t,x,y)}(T)) \vee \psi(X_{t,x}(T))] . \end{aligned}$$

This is the result obtained in [16] via duality methods in the case of a European guarantee.

Remark 4.1. The condition $I(\Gamma_{t,x,\hat{\gamma}(t,x,y)}(T)) \in L^{2+\varepsilon}(\Omega, \mathbb{Q})$ is satisfied for many examples of applications. This is, in particular, the case for $U(r) := r^p$ for $0 < p < 1$. In the case where it does not hold, one needs to relax the set of controls by allowing

ν to be only \mathbb{P} -a.s.-square integrable. The proofs of our general results can be easily adapted to this context in this example.

Remark 4.2. We have assumed that $\psi \in C^2$. If this is not the case, we can always consider sequences of smooth functions $(\underline{\psi}_n)_n$ and $(\bar{\psi}_n)_n$ such that $\underline{\psi}_n \leq \psi \leq \bar{\psi}_n$ and $\underline{\psi}_n, \bar{\psi}_n \rightarrow \psi$ pointwise. Letting $\underline{V}_n, \bar{V}_n$ be the value functions associated with $\underline{\psi}_n, \bar{\psi}_n$ instead of ψ , we clearly have $\underline{V}_n \leq V \leq \bar{V}_n$. Letting $n \rightarrow \infty$ and using (4.8) applied to $\underline{V}_n, \bar{V}_n$ shows that (4.8) holds for V too.

5. Possible extensions.

5.1. Constraints in moment or probability. In this section, we explain how the results of the section 3 can be adapted to stochastic control problems under moment constraints of the following form:

$$(5.1) \quad \tilde{V}(t, z; p) := \sup_{\nu \in \tilde{\mathcal{U}}_{t,z,p}} \mathbb{E} [f(Z_{t,z}^\nu(T))] ,$$

where

$$(5.2) \quad \tilde{\mathcal{U}}_{t,z,p} := \{ \nu \in \mathcal{U}^t : \mathbb{E} [g(Z_{t,z}^\nu(T))] \geq p \} , \quad p \in \mathbb{R} .$$

The main idea consists in adding a new controlled process $P_{t,p}^\alpha$ as done in Bouchard, Elie, and Touzi [5] for stochastic control problems with controlled loss.

Namely, let \mathcal{A} denote the set of \mathbb{F} -progressively measurable \mathbb{R}^d -valued square integrable processes and denote by \mathcal{A}^t the subset of elements of \mathcal{A} which are independent of \mathcal{F}_t . With $\alpha \in \mathcal{A}^t$ and initial conditions $(t, p) \in [0, T] \times \mathbb{R}$, we associate the controlled process $P_{t,p}^\alpha$ defined for $t \leq s \leq T$ by

$$(5.3) \quad P_{t,p}^\alpha(s) = p + \int_t^s \alpha_r^\top dW_r,$$

and introduce the set $\tilde{\mathcal{U}}_{t,z,p}$ of controls of the form $(\nu, \alpha) \in \mathcal{U}^t \times \mathcal{A}^t$ such that

$$\bar{g}(Z_{t,z}^\nu(T), P_{t,p}^\alpha(T)) \geq 0 \text{ } \mathbb{P}\text{-a.s.},$$

where $\bar{g}(Z, P) := g(Z) - P$.

LEMMA 5.1. *Fix $(t, z, p) \in [0, T] \times \mathbb{R}^{d+1} \times \mathbb{R}$ and $\nu \in \mathcal{U}^t$ such that $g(Z_{t,z}^\nu(T)) \in L^2(\Omega)$. Then, $\nu \in \tilde{\mathcal{U}}_{t,z,p}$ if and only if there exists $\alpha \in \mathcal{A}^t$ such that $(\nu, \alpha) \in \tilde{\mathcal{U}}_{t,z,p}$.*

Proof. Fix $\nu \in \tilde{\mathcal{U}}_{t,z,p}$. Since $g(Z_{t,z}^\nu(T)) \in L^2(\Omega)$ and is independent of \mathcal{F}_t , we can find $\alpha \in \mathcal{A}^t$ such that $g(Z_{t,z}^\nu(T)) = \mathbb{E} [g(Z_{t,z}^\nu(T))] - p + P_{t,p}^\alpha(T)$. Since $\mathbb{E} [g(Z_{t,z}^\nu(T))] \geq p$, it follows that $\bar{g}(Z_{t,z}^\nu(T), P_{t,p}^\alpha(T)) \geq 0$. The converse assertion follows from the martingale property of $P_{t,p}^\alpha$ for all $(\nu, \alpha) \in \tilde{\mathcal{U}}_{t,z,p}$. \square

This readily implies that $\tilde{V} = \bar{V}$, where

$$(5.4) \quad \bar{V}(t, z, p) := \sup_{\nu \in \tilde{\mathcal{U}}_{t,z,p}} \mathbb{E} [f(Z_{t,z}^\nu(T))] .$$

PROPOSITION 5.1. *Fix $(t, z, p) \in [0, T] \times \mathbb{R}^{d+1} \times \mathbb{R}$, and assume that $g(Z_{t,z}^\nu(T)) \in L^2(\Omega)$ for all $\nu \in \mathcal{U}^t$. Then, $\tilde{V}(t, z; p) = \bar{V}(t, z, p)$.*

Remark 5.1. (i) If g has linear growth, then the condition $g(Z_{t,z}^\nu(T)) \in L^2(\Omega)$ is satisfied since the diffusion has Lipschitz continuous coefficients and, by definition of \mathcal{U} , ν is square integrable.

(ii) Note that if U is bounded, then the condition (3.11) holds for the augmented system $(Z_{t,z}^\nu, P_{t,p}^\alpha)$ when looking at $P_{t,p}^\alpha$ as a new Y component and the former $Z_{t,z}^\nu$ as a new X component:

$$\limsup_{|(u,\alpha)| \rightarrow \infty, (u,\alpha) \in U \times \mathbb{R}} \left\{ \frac{|\mu_Z(\cdot, u)| + |\sigma_Z(\cdot, u)|^2}{|\alpha|^2} \right\} = \limsup_{|\alpha| \rightarrow \infty} \sup_{u \in U} \left\{ \frac{|\mu_Z(\cdot, u)| + |\sigma_Z(\cdot, u)|^2}{|\alpha|^2} \right\} = 0,$$

on \mathbb{R}^{d+1} locally uniformly.

(iii) Under the integrability condition of Proposition 5.1, the PDE characterization of section 3 can thus be applied to the value function $\bar{V} = \tilde{V}$.

One can similarly apply the results of the preceding section to problems with constraint in probability of the form

$$\check{V}(t, z; p) := \sup_{\nu \in \check{\mathcal{U}}_{t,z,p}} \mathbb{E} [f(Z_{t,z}^\nu(T))] \quad \text{with } \check{\mathcal{U}}_{t,z,p} := \{ \nu \in \mathcal{U}^t : \mathbb{P} [g(Z_{t,z}^\nu(T)) \geq 0] \geq p \},$$

by observing that

$$\mathbb{P} [g(Z_{t,z}^\nu(T)) \geq 0] = \mathbb{E} \left[\mathbf{1}_{\{g(Z_{t,z}^\nu(T)) \geq 0\}} \right].$$

In this case, we obtain

$$\begin{aligned} \check{V}(t, z; p) = \sup \left\{ \mathbb{E} [f(Z_{t,z}^\nu(T))] , (\nu, \alpha) \in \mathcal{U}^t \times \mathcal{A}^t \right. \\ \left. \text{s.t. } \mathbf{1}_{\{g(Z_{t,z}^\nu(T)) \geq 0\}} - P_{t,p}^\alpha(T) \geq 0 \text{ } \mathbb{P}\text{-a.s.} \right\}. \end{aligned}$$

Remark 5.2. Following [5], we could restrict ourselves to controls α such that $P_{t,p}^\alpha$ evolves in $[0, 1]$, which is the natural domain for the p variable. Note that, in any case, one should also discuss the boundary conditions at $p = 0$ and $p = 1$, as in [5]. We leave this nontrivial point to further research.

5.2. Almost sure constraints on $[0, T]$. An important extension consists in considering almost sure constraints on the whole time interval $[0, T]$:

$$(5.5) \quad V^a(t, z) := \sup_{\nu \in \mathcal{U}_{t,z}^a} \mathbb{E} [f(Z_{t,z}^\nu(T))] ,$$

where

$$(5.6) \quad \mathcal{U}_{t,z}^a := \{ \nu \in \mathcal{U}^t : g(Z_{t,z}^\nu(s)) \geq 0 \ \forall s \in [t, T] \text{ } \mathbb{P}\text{-a.s.} \} .$$

This corresponds to a classical state constraint problem on the spacial domain

$$\mathcal{O} := \{ z \in \mathbb{R}^{d+1} : g(z) \geq 0 \} .$$

This case can be treated exactly as the one considered in the present paper, after replacing D and w by D^a and w^a defined by

$$\begin{aligned} D^a &:= \{ (t, z) \in [0, T] \times \mathbb{R}^{d+1} : g(Z_{t,z}^\nu(s)) \geq 0 \ \forall s \in [t, T] \text{ } \mathbb{P}\text{-a.s. for some } \nu \in \mathcal{U}^t \}, \\ w^a(t, x) &:= \inf \{ y \in \mathbb{R} : (t, x, y) \in D^a \} . \end{aligned}$$

This is due to the fact that the *geometric dynamic programming principle* of Theorem 2.1 and Corollary 2.1 can be extended to D^a and w^a ; see Bouchard and Nam [6].

Repeating the arguments of section 6 then allows one to extend to V^a the results of sections 3.2 and 3.3.

We should note that the PDE characterization of w^a was obtained in Bouchard and Nam [6] only for a particular situation which arises in finance. However, it would not be difficult to derive the PDE associated with w^a in our more general framework by combining the arguments of [5] and [6]. We would obtain the same characterization as in [5] but only on the domain $g_*(x, w^{a*}(t, x)) > 0$ for the subsolution property, and with the additional inequality $g^*(x, w_*^a(t, x)) \geq 0$ on the whole domain; i.e., w^a would be a discontinuous viscosity solution of

$$\min \left\{ \sup_{u \in \mathcal{N}_0(\cdot, w^a, Dw^a)} (\mu_Y(\cdot, w^a, u) - \mathcal{L}_X^u w^a), g(\cdot, w^a) \right\} = 0 \quad \text{in } [0, T] \times \mathbb{R}^d,$$

$$g(\cdot, w^a(T, \cdot)) = 0 \quad \text{in } \mathbb{R}^d.$$

As already argued in the introduction, it is important to notice that our approach would not require imposing strong conditions on \mathcal{O} and the coefficients in order to ensure that $Z_{t,z}^\nu$ can actually be reflected on the boundary of \mathcal{O} . The only important assumption is that D^a is nonempty. By definition of D^a and the *geometric dynamic programming principle*, this condition will be automatically satisfied on the boundary of $D^a \subset \mathcal{O}$.

We think that this approach is more natural for state constraint problems and hope that it will open the door to new results in this field. A precise study is left for further research.

6. Proof of the PDE characterizations.

6.1. Dynamic programming principle. As usual, we first need to provide a dynamic programming principle for our control problem (2.1).

THEOREM 6.1 (dynamic programming principle). *Fix $(t, z) \in \text{int}_p(D)$, and let $\{\theta^\nu, \nu \in \mathcal{U}\}$ be a family of stopping times with values in $[t, T]$. Then,*

$$(6.1) \quad \begin{aligned} V(t, z) &\leq \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E} [f^*(Z_{t,z}^\nu(\theta^\nu)) \mathbf{1}_{\theta^\nu=T} + V^*(\theta^\nu, Z_{t,z}^\nu(\theta^\nu)) \mathbf{1}_{\theta^\nu < T}], \\ V(t, z) &\geq \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E} [f_*(Z_{t,z}^\nu(\theta^\nu)) \mathbf{1}_{\theta^\nu=T} + V_*(\theta^\nu, Z_{t,z}^\nu(\theta^\nu)) \mathbf{1}_{\theta^\nu < T}], \end{aligned}$$

where V_* (resp., V^*) denotes the lower semicontinuous (resp., upper semicontinuous) envelope of V .

Proof. In this proof we denote by $\omega^r := (\omega_{t \wedge r})_{t \leq T}$ the stopped canonical path and by $\mathbf{T}_r(\omega) := (\omega_{s+r} - \omega_r)_{s \leq T-r}$ the shifted canonical path, $r \in [0, T]$. For ease of notation, we omit the dependence of θ^ν with ν and simply write θ .

1. We start with the first inequality. To see that it holds, note that, by the flow property, for any $\nu \in \mathcal{U}_{t,z}$,

$$\begin{aligned} \mathbb{E} [f(Z_{t,z}^\nu(T))] &= \int \int f \left(Z_{\theta(\omega), Z_{t,z}^\nu(\theta)(\omega)}^{\nu(\omega^{\theta(\omega)} + \mathbf{T}_{\theta(\omega)}(\tilde{\omega}))}(T)(\omega^{\theta(\omega)} + \mathbf{T}_{\theta(\omega)}(\tilde{\omega})) \right) d\mathbb{P}(\omega) d\mathbb{P}(\tilde{\omega}) \\ &= \int \mathbb{E} \left[J \left(\theta(\omega), Z_{t,z}^\nu(\theta)(\omega); \nu(\omega^{\theta(\omega)} + \mathbf{T}_{\theta(\omega)}(\cdot)) \right) \right] d\mathbb{P}(\omega), \end{aligned}$$

where $\tilde{\omega} \in \Omega \mapsto \nu(\omega^{\theta(\omega)} + \mathbf{T}_{\theta(\omega)}(\tilde{\omega}))$ is viewed as a control in $\mathcal{U}_{\theta(\omega), Z_{t,z}^\nu(\theta)(\omega)}$ for fixed $\omega \in \Omega$. Since $J \leq V^*$ and $J(T, \cdot) = f$, it follows that

$$\mathbb{E} [f(Z_{t,z}^\nu(T))] \leq \mathbb{E} [V^*(\theta, Z_{t,z}^\nu(\theta)) \mathbf{1}_{\theta < T} + f(Z_{t,z}^\nu(T)) \mathbf{1}_{\theta=T}],$$

and the result follows from the arbitrariness of $\nu \in \mathcal{U}_{t,z}$.

2. We now turn to the second inequality. It follows from Lemma A.2 in the appendix that the set $\Gamma = \{(t, z, \nu) \in [0, T] \times \mathbb{R}^{d+1} \times \mathcal{U} : \nu \in \mathcal{U}_{t,z}\}$ is an analytic set. Clearly, J is Borel measurable and therefore upper semianalytic. It thus follows from Proposition 7.50 in [3] that, for each $\varepsilon > 0$, we can find an analytically measurable map $\hat{\nu}^\varepsilon$ such that $J(t, z; \hat{\nu}^\varepsilon(t, z)) \geq V(t, z) - \varepsilon$ and $\hat{\nu}^\varepsilon(t, z) \in \mathcal{U}_{t,z}$ on D . Since analytically measurable maps are also universally measurable, it follows from Lemma 7.27 in [3] that, for any probability measure m on $[0, T] \times \mathbb{R}^{d+1}$, we can find a Borel measurable map $\hat{\nu}_m^\varepsilon : (t, z) \in D \mapsto \mathcal{U}_{t,z}$ such that $J(t, z; \hat{\nu}_m^\varepsilon(t, z)) \geq V(t, z) - \varepsilon \geq V_*(t, z) - \varepsilon$ for m -almost every element of D . Let us now fix $\nu_1 \in \mathcal{U}_{t_0, z_0}$ for some $(t_0, z_0) \in \text{int}_p(D)$. Let m be the measure induced by $(\theta, Z_{t_0, z_0}^{\nu_1}(\theta))$ on $[0, T] \times \mathbb{R}^{d+1}$. By Theorem 2.1, $(\theta, Z_{t_0, z_0}^{\nu_1}(\theta)) \in D$ \mathbb{P} -a.s. so that

$$\hat{\nu}_m^\varepsilon(\theta, Z_{t_0, z_0}^{\nu_1}(\theta)) \in \mathcal{U}_{\theta, Z_{t_0, z_0}^{\nu_1}(\theta)},$$

and

$$J(\theta, Z_{t_0, z_0}^{\nu_1}(\theta); \hat{\nu}_m^\varepsilon(\theta, Z_{t_0, z_0}^{\nu_1}(\theta))) \geq V_*(\theta, Z_{t_0, z_0}^{\nu_1}(\theta)) - \varepsilon \text{ } \mathbb{P}\text{-a.s.}$$

Moreover, it follows from Lemma 2.1 of [21] that we can find $\nu_2^\varepsilon \in \mathcal{U}$ such that

$$\nu_2^\varepsilon \mathbf{1}_{[\theta, T]} = \hat{\nu}_m^\varepsilon(\theta, Z_{t_0, z_0}^{\nu_1}(\theta)) \mathbf{1}_{[\theta, T]} \text{ } dt \times d\mathbb{P}\text{-a.e.}$$

This implies that $\nu^\varepsilon := \nu_1 \mathbf{1}_{[t_0, \theta)} + \nu_2^\varepsilon \mathbf{1}_{[\theta, T]} \in \mathcal{U}_{t_0, z_0}$ and

$$(6.2) \quad \mathbb{E} \left[f(Z_{\theta, Z_{t_0, z_0}^{\nu_1}(\theta)}^{\nu_2^\varepsilon}(T)) \mid (\theta, Z_{t_0, z_0}^{\nu_1}(\theta)) \right] = J(\theta, Z_{t_0, z_0}^{\nu_1}(\theta); \hat{\nu}_m^\varepsilon(\theta, Z_{t_0, z_0}^{\nu_1}(\theta))) \\ \geq V_*(\theta, Z_{t_0, z_0}^{\nu_1}(\theta)) - \varepsilon \text{ } \mathbb{P}\text{-a.s.}$$

and therefore

$$V(t_0, z_0) \geq \mathbb{E} \left[\mathbb{E} \left[f(Z_{\theta, Z_{t_0, z_0}^{\nu_1}(\theta)}^{\nu_2^\varepsilon}(T)) \mid (\theta, Z_{t_0, z_0}^{\nu_1}(\theta)) \right] \right] \\ \geq \mathbb{E} [V_*(\theta, Z_{t_0, z_0}^{\nu_1}(\theta)) \mathbf{1}_{\theta < T} + f(Z_{t_0, z_0}^{\nu_1}(T)) \mathbf{1}_{\theta = T}] - \varepsilon .$$

The required result then follows from the arbitrariness of $\nu_1 \in \mathcal{U}_{t_0, z_0}$ and $\varepsilon > 0$. \square

Remark 6.1. Note that the restriction to controls independent of \mathcal{F}_t in the definition of $V(t, z)$ seems to be important in the above proof. Indeed, if we did not restrict to such controls, there could be nontrivial correlations between the quantities $\hat{\nu}_m^\varepsilon(\theta, Z_{t_0, z_0}^{\nu_1}(\theta))$ and $(\theta, Z_{t_0, z_0}^{\nu_1}(\theta))$ above so that the left-hand side equality in (6.2) would not remain valid. Otherwise stated, we could have

$$\mathbb{E} \left[f(Z_{\theta, Z_{t_0, z_0}^{\nu_1}(\theta)}^{\hat{\nu}_m^\varepsilon(\theta, Z_{t_0, z_0}^{\nu_1}(\theta))}(T)) \mid (\theta, Z_{t_0, z_0}^{\nu_1}(\theta)) \right] (\omega) \\ \neq \mathbb{E} \left[f(Z_{\theta(\omega), Z_{t_0, z_0}^{\nu_1}(\theta)(\omega)}^{\hat{\nu}_m^\varepsilon(\theta(\omega), Z_{t_0, z_0}^{\nu_1}(\theta)(\omega))}(T)) \right] \\ =: J(\theta(\omega), Z_{t_0, z_0}^{\nu_1}(\theta)(\omega); \hat{\nu}_m^\varepsilon(\theta(\omega), Z_{t_0, z_0}^{\nu_1}(\theta)(\omega))) .$$

Compare with, e.g., Lemma 3.2 in Chapter 4 in [23]. However, one could actually allow for more general controls without changing the nature of the value function. To see this, let $\tilde{\mathcal{U}}_{t,z}$ and $\tilde{V}(t, z)$ be defined by

$$\tilde{\mathcal{U}}_{t,z} := \{ \nu \in \mathcal{U} : g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P}\text{-a.s.} \}, \quad \tilde{V}(t, z) := \sup_{\nu \in \tilde{\mathcal{U}}_{t,z}} J(t, z; \nu),$$

meaning that we now allow for a dependence with respect to \mathcal{F}_t . Clearly, $\tilde{V}(t, z) \geq V(t, z)$ since $\tilde{\mathcal{U}}_{t,z} \supset \mathcal{U}_{t,z}$. We claim that

$$\tilde{V}(t, z) = V(t, z) .$$

Indeed, if $\tilde{\mathcal{U}}_{t,z}$ is empty, then $\mathcal{U}_{t,z}$ is empty too and $\tilde{V} = V = -\infty$. If not, then any control $\nu \in \tilde{\mathcal{U}}_{t,z}$ can be written as a measurable map $\omega \in \Omega \mapsto \nu(\omega^t, \mathbf{T}_t(\omega))$. For fixed ω^t , the map $\nu_{\omega^t} : \tilde{\omega} \in \Omega \mapsto \nu(\omega^t, \mathbf{T}_t(\tilde{\omega}))$ can then be viewed as control independent of \mathcal{F}_t . Moreover, the fact that $\nu \in \tilde{\mathcal{U}}_{t,z}$, Fubini's lemma, and the independence of the Brownian increments imply that

$$1 = \mathbb{P} [g(Z_{t,z}^\nu(T)) \geq 0] = \int \mathbb{P} [g(Z_{t,z}^{\nu_{\omega^t}}(T)) \geq 0] d\mathbb{P}(\omega) ,$$

which readily implies that $\nu_{\omega^t} \in \mathcal{U}_{t,z}$ for \mathbb{P} -a.e. $\omega \in \Omega$. Using Fubini's lemma and the independence of the Brownian increments again, we then obtain

$$J(t, z; \nu) = \int \mathbb{E} [f(Z_{t,z}^{\nu_{\omega^t}}(T))] d\mathbb{P}(\omega) \leq \int V(t, z) d\mathbb{P}(\omega) ,$$

where the latter equals $V(t, z)$. By arbitrariness of $\nu \in \mathcal{U}_{t,z}$, this implies that $\tilde{V}(t, z) \leq V(t, z)$.

6.2. The Hamilton–Jacobi–Bellman equation in the general case. The proof of Theorem 3.1 follows from rather standard arguments based on Theorem 6.1 except that we have to handle the stochastic target constraint.

6.2.1. Supersolution property for $t < T$.

Proof of (3.7). Let φ be a smooth function such that $(t_0, z_0) \in \text{int}_p(D) \cup \partial_Z D$ achieves a strict minimum (equal to 0) of $V_* - \varphi$ on $\text{cl}(D)$. We also consider $\phi \in \mathcal{W}^*(t_0, x_0)$.

Case 1. We first consider the case where $(t_0, z_0) \in \text{int}_p(D)$. We argue by contradiction and assume that

$$(-\partial_t \varphi + H^* \varphi)(t_0, z_0) < 0 .$$

Since the coefficients μ_Z and σ_Z are continuous, we can find a closed bounded neighborhood $B \subset \text{int}_p(D)$ of (t_0, z_0) and $\hat{u} \in U$ such that

$$(6.3) \quad -\mathcal{L}_Z^{\hat{u}} \varphi \leq 0 \quad \text{on } B .$$

Let $(t_n, z_n)_n$ be a sequence in B such that $V(t_n, z_n) \rightarrow V_*(t_0, z_0)$ and $(t_n, z_n) \rightarrow (t_0, z_0)$. Denote $\hat{Z}^n := Z_{t_n, z_n}^{\hat{u}}$, where \hat{u} is viewed as a constant control in \mathcal{U}^{t_n} , and let θ_n denote the first exit time of $(s, \hat{Z}^n(s))_{s \geq t_n}$ from B . Since $(\theta_n, \hat{Z}^n(\theta_n)) \in \partial B \subset \text{int}_p(D)$, it follows from Theorem 2.1 that there exists a control $\nu^n \in \mathcal{U}_{t_n, z_n}$ such that $\nu^n = u$ on $[t_n, \theta_n)$. We now set $Z^n = (X^n, Y^n) := Z_{t_n, z_n}^{\nu^n}$ and observe that $Z^n = \hat{Z}^n$ on $[t_n, \theta_n]$ by continuity of the paths of both processes. It then follows from Itô's lemma and (6.3) that, for n large enough,

$$\varphi(t_n, z_n) \leq \mathbb{E} [\varphi(\theta_n, Z^n(\theta_n))] \leq \mathbb{E} [V_*(\theta_n, Z^n(\theta_n))] - \zeta ,$$

where

$$\zeta := \min_{(t,z) \in \partial_p B} (V_* - \varphi) > 0 .$$

Since $(\varphi - V)(t_n, z_n) \rightarrow (\varphi - V_*)(t_0, z_0) = 0$ as $n \rightarrow \infty$, this contradicts Theorem 6.1 for n large enough.

Case 2. We now turn to the case where $(t_0, z_0) \in \partial_Z D$. As above, we argue by contradiction and assume that

$$\limsup_{(t,z) \rightarrow (t_0,z_0), \gamma \downarrow 0} \inf_{u \in U_{0,\gamma}(t,z,\phi)} (-\mathcal{L}_Z^u \varphi(t, z)) < 0 ;$$

recall (3.4). Note that this implies that $\mathcal{N}_0(t, z, D\phi(t, x)) \neq \emptyset$ for (t, z) in a neighborhood of (t_0, z_0) . Using Assumption 3.1, we then deduce that there exists an open neighborhood \mathcal{O} of (t_0, z_0) and a Lipschitz continuous map \hat{u} on \mathcal{O} such that for all $(t, z) \in \mathcal{O}$

$$(6.4) \quad \left. \begin{aligned} -\mathcal{L}_Z^{\hat{u}(t,z)} \varphi(t, z) &\leq 0 \\ \mu_Y(z, \hat{u}(t, z)) - \mathcal{L}_X^{\hat{u}(t,z)} \phi(t, x) &> 0 \\ \hat{u}(t, z) &\in \mathcal{N}_0(z, D\phi(t, x)) \end{aligned} \right\} .$$

Let $(t_n, z_n)_n$ be a sequence in $\mathcal{O} \cap \text{int}_p(D)$ such that $V(t_n, z_n) \rightarrow V_*(t_0, z_0)$ and $(t_n, z_n) \rightarrow (t_0, z_0)$. Let $\hat{Z}^n = (\hat{X}^n, \hat{Y}^n)$ denote the solution of (1.3) for the Markovian control $\hat{\nu}^n$ associated with \hat{u} and initial condition z_n at time t_n . Clearly, $\hat{\nu}^n \in \mathcal{U}^{t_n}$. Let θ_n denote the first exit time of $(s, \hat{Z}^n(s))_{s \geq t_n}$ from \mathcal{O} . It follows from Itô's lemma and the two last inequalities in (6.4) that

$$\hat{Y}^n(\theta_n) \geq \phi(\theta_n, \hat{X}^n(\theta_n)) + y_n - \phi(t_n, x_n) .$$

Since $w - \phi$ achieves a strict maximum at (t_0, x_0) and $(w - \phi)(t_0, x_0) = 0$, we can find $\kappa > 0$ such that $w - \phi \leq -\kappa$ on $\partial \mathcal{O}$. Since, by definition of θ_n , $\hat{Z}^n(\theta_n) \in \partial \mathcal{O}$, it follows from the previous inequality that

$$\hat{Y}^n(\theta_n) \geq w(\theta_n, \hat{X}^n(\theta_n)) + \kappa + y_n - \phi(t_n, x_n) .$$

Using the fact that $y_n - \phi(t_n, x_n) \rightarrow y_0 - w(t_0, x_0) = 0$ (recall (2.5)), we thus deduce that, for n large enough,

$$\hat{Y}^n(\theta_n) > w(\theta_n, \hat{X}^n(\theta_n)) .$$

In view of Corollary 2.1, we can then find a control $\nu^n \in \mathcal{U}_{t_n, z_n}$ such that $\nu^n = \hat{\nu}^n$ on $[t_n, \theta_n)$. Let us set $Z^n = (X^n, Y^n) := Z_{t_n, z_n}^{\nu^n}$ and observe that $Z^n = \hat{Z}^n$ on $[t_n, \theta_n)$ by continuity of the paths of both processes. We can now appeal to the first inequality in (6.4) to obtain by the same arguments as in Case 1 that

$$V(t_n, z_n) < \mathbb{E} [V_*(\theta_n, Z^n(\theta_n))] ,$$

for n large enough. Since $\nu^n \in \mathcal{U}_{t_n, z_n}$, this contradicts Theorem 6.1. \square

Remark 6.2. The relaxation in (3.4) is used in the above proof to ensure that (6.4) holds on a neighborhood of (t_0, x_0) . In the case where the function w is smooth and the assumptions of Theorem 3.2 hold, it is no longer required and we can replace F^{ϕ^*} by $F_{0,0}^w$ in the supersolution property of Theorem 3.1. Indeed, in this case,

$$-\partial_t \varphi(t_0, x_0, w(t_0, x_0)) + F_{0,0}^w \varphi(t_0, x_0, w(t_0, x_0)) < 0$$

implies that

$$-\mathcal{L}_Z^{\hat{u}(t_0, x_0)} \varphi(t_0, x_0, w(t_0, x_0)) < 0 ,$$

where $\hat{u}(t, x) := \check{u}(x, w(t, x), Dw(t, x))$ by Assumption 3.3, where, by Theorem 3.2,

$$(6.5) \quad \begin{aligned} \mu_Y(x, w(t, x), \hat{u}(t, x)) - \mathcal{L}_X^{\hat{u}(t, x)} w(t, x) &\geq 0 \quad \text{and} \\ \hat{u}(t, x) &\in \mathcal{N}_0(x, w(t, x), Dw(t, x)) . \end{aligned}$$

Since \check{u} is locally Lipschitz, it follows that

$$(6.6) \quad -\mathcal{L}_Z^{\hat{u}(t, x)} \varphi(t, x, y) < 0$$

on a neighborhood of (t_0, x_0, y_0) . Now observe that (6.5), Itô’s lemma, the fact that $y_n \geq w(t_n, x_n)$, and a standard comparison result for stochastic differential equations shows that

$$Y_{t_n, x_n, y_n}^{\hat{\nu}^n} \geq w(\cdot, X_{t_n, x_n}^{\hat{\nu}^n}) \quad \text{on } [t_n, T] ,$$

where $\hat{\nu}^n$ is the Markovian control associated with \hat{u} and initial conditions (t_n, x_n) , i.e., defined by $\hat{\nu}_t^n = \hat{u}(t, X_{t_n, x_n}^{\hat{\nu}^n}(t))$ for $t \geq t_n$. In view of our standing assumption $g(\cdot, w(T, \cdot)) \geq 0$, this shows that $\hat{\nu}^n \in \mathcal{U}_{t_n, x_n, y_n}$; recall that g is nondecreasing with respect to its second argument. Using (6.6) instead of the first inequality in (6.4) then allows us to show by the same arguments as in the above proof that

$$V(t_n, z_n) < \mathbb{E}[V_*(\theta_n, Z^n(\theta_n))]$$

for some well-chosen stopping time θ_n and n large enough.

6.2.2. Subsolution property for $t < T$.

Proof of (3.9). Let φ be a smooth function such that $(t_0, z_0) \in \text{int}_p(D) \cup \partial_Z D$ achieves a strict maximum (equal to 0) of $V^* - \varphi$ on $\text{cl}(D)$. We argue by contradiction and assume that the subsolution property does not hold at (t_0, z_0) for φ , i.e.,

$$-\partial_t \varphi(t_0, z_0) + H_* \varphi(t_0, z_0) > 0 .$$

Since the coefficients μ_Z and σ_Z are continuous, we can then find an open bounded neighborhood \mathcal{O} of the form $(t_0 - \iota, t_0 + \iota) \times \mathcal{O}$ of (t_0, z_0) such that $\mathcal{O} \cap \partial_T D = \emptyset$ and for all $u \in U$

$$(6.7) \quad -\mathcal{L}_Z^u \varphi(t, z) \geq 0 \quad \text{on } \mathcal{O} .$$

Let $(t_n, z_n)_n$ be a sequence in $\mathcal{O} \cap \text{int}_p(D)$ such that $V(t_n, z_n) \rightarrow V^*(t_0, z_0)$ and $(t_n, z_n) \rightarrow (t_0, z_0)$. For each n , since $(t_n, z_n) \in D$, there exists a control $\nu^n \in \mathcal{U}_{t_n, z_n}$. Hence we set $Z^n := Z_{t_n, z_n}^{\nu^n}$ and we let θ_n denote the first exit time of $(s, Z^n(s))_{s \geq t_n}$ from $\mathcal{O} \cap (\text{int}_p(D) \cup \partial_Z D)$. Note that, by definition of \mathcal{U}_{t_n, z_n} and Theorem 2.1, $(s, Z^n(s)) \in \text{int}_p(D) \cup \partial_Z D$ on $[t_n, T]$ and therefore $(\theta_n, Z^n(\theta_n)) \in \partial_p \mathcal{O}$. Applying Itô’s lemma and using (6.7), we then obtain

$$\varphi(t_n, z_n) \geq \mathbb{E}[\varphi(\theta_n, Z^n(\theta_n))] \geq \mathbb{E}[V^*(\theta_n, Z^n(\theta_n))] + \zeta ,$$

where, since (t_0, z_0) achieves a strict maximum,

$$-\zeta := \max_{(t, z) \in \partial_p \mathcal{O}} (V^* - \varphi) < 0 .$$

Since $(\varphi - V)(t_n, z_n) \rightarrow (\varphi - V^*)(t_0, z_0) = 0$ as $n \rightarrow \infty$, this contradicts Theorem 6.1 for n large enough. \square

6.2.3. Terminal condition. The proof of the boundary condition at $t = T$ follows from the preceding arguments and the usual trick of perturbing the test function by adding a term of the form $\pm\sqrt{T-t}$. We prove only the supersolution property for $z_0 = (x_0, y_0)$ such that $y_0 > w(T, x_0)$. The subsolution property is proved similarly by combining the arguments below with the arguments of section 6.2.2; it turns out that it is easier to handle.

Proof of (3.8). Fix $z_0 := (x_0, y_0)$ satisfying $y_0 > w(T, x_0)$, and let φ be a smooth function such that (T, z_0) achieves a strict minimum (equal to 0) of $V_* - \varphi$ on $\text{cl}(D)$ and such that $H^*\varphi(T, z_0) < \infty$.

We argue by contradiction and assume that $V_*(T, z_0) < f_*(z_0)$. It follows that we can find $r, \eta > 0$ such that

$$(6.8) \quad \varphi \leq f_* - \eta \quad \text{on } (\{T\} \times B_r(z_0)) \cap \text{cl}(D).$$

Let $(t_n, z_n)_n$ be a sequence in $\text{int}_p(D)$ such that $V(t_n, z_n) \rightarrow V_*(T, z_0)$ and $(t_n, z_n) \rightarrow (T, z_0)$. As usual we modify the test function and introduce $\tilde{\varphi} : (t, z) \mapsto \varphi(z) - (T-t)^{\frac{1}{2}}$, so that (T, z_0) is also a strict minimum point of $V_* - \tilde{\varphi}$. Since $H^*\varphi(T, z_0) < \infty$ by assumption and $-\partial_t \tilde{\varphi}(t, z) = -\partial_t \varphi(t, z) - \frac{1}{2}(T-t)^{-\frac{1}{2}}$, where $(T-t)^{-\frac{1}{2}} \rightarrow \infty$ as $t \rightarrow T$, we can choose $r > 0$ small enough and $\hat{u} \in U$ such that

$$(6.9) \quad -\mathcal{L}_Z^{\hat{u}} \tilde{\varphi} \leq 0 \quad \text{on } \mathcal{V}_0 := [T-r, T] \times B_r(z_0).$$

Since w is continuous and $y_0 > w(T, x_0)$, we can choose $r > 0$ small enough so that

$$\text{cl}(\mathcal{V}_0) \subset \{(t, z) \in D : y \geq w(t, x) + r/2\} \subset D.$$

Set $\hat{Z}^n := Z_{t_n, z_n}^{\hat{u}}$, where \hat{u} is viewed as a constant control in \mathcal{U}^{t_n} , and let θ_n denote the first exit time of $(s, \hat{Z}^n(s))_{s \geq t_n}$ from \mathcal{V}_0 . Since $(\theta_n, \hat{Z}^n(\theta_n)) \in \partial \mathcal{V}_0 \subset D$, it follows from Theorem 2.1 that there exists a control $\nu^n \in \mathcal{U}_{t_n, z_n}$ such that $\nu^n = \hat{u}$ on $[t_n, \theta_n)$. Using (6.9), we then deduce that

$$\tilde{\varphi}(t_n, z_n) \leq \mathbb{E}[\tilde{\varphi}(\theta_n, Z^n(\theta_n))] ,$$

where $Z^n := Z_{t_n, z_n}^{\nu^n}$. Since (T, z_0) is a strict minimum point for $V_* - \tilde{\varphi}$ and $(V_* - \tilde{\varphi})(T, z_0) = 0$, we can then find $\zeta > 0$ such that $V_* - \zeta \geq \tilde{\varphi}$ on $\{(t, z) \in \partial \mathcal{V}_0 : t < T\}$. Recalling (6.8) and the previous inequality, it follows that

$$\tilde{\varphi}(t_n, z_n) \leq \mathbb{E}[(f_*(Z^n(\theta_n)) - \eta)\mathbf{1}_{\{\theta_n=T\}} + (V_*(\theta_n, Z^n(\theta_n)) - \zeta)\mathbf{1}_{\{\theta_n < T\}}] .$$

Since $\nu^n \in \mathcal{U}_{t_n, z_n}$ and $\tilde{\varphi}(t_n, z_n) - V(t_n, z_n) \rightarrow 0$ as $n \rightarrow \infty$, the above inequality contradicts Theorem 6.1 for n large enough. \square

6.3. Subsolution property in the continuous boundary case. In this section, we provide the proof of Proposition 3.1. We divide it in several steps. From now on, we let φ be a smooth function and $(t_0, z_0) \in \partial_Z D$ be such that (t_0, z_0) is a strict local maximum point of $v - \varphi$, where v is as in Proposition 3.1. We also fix $\phi \in \mathcal{W}_*(t_0, x_0)$, where $(x_0, y_0) = z_0$.

LEMMA 6.1. *There exists $u \in \mathcal{N}_0(z_0, D\phi(t_0, x_0))$ such that $-\mathcal{L}_Z^u \varphi(t_0, z_0) \leq 0$.*

Proof. Given $\eta > 0$, we define

$$\varphi_\eta(t, z) := \varphi(t, z) + f_\eta(t, z),$$

where, for $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}$,

$$f_\eta(t, z) := \frac{1}{\eta}(y - \phi(t, x)) - \eta(y - \phi(t, x))^2 .$$

Observe that $y_0 = w(t_0, x_0) = \phi(t_0, x_0)$ while $y \geq w(t, x) \geq \phi(t, x)$ on $\text{int}_p(D) \cup \partial_Z D$, since $(t_0, z_0) \in \partial_Z D$ and w is continuous on $[0, T) \times \mathbb{R}^d$; recall (2.4)–(2.5). Observe also that $f_\eta(t, z) \geq 0$ for $0 \leq y - \phi(t, x) \leq \frac{1}{\eta^2}$.

It follows that (t_0, z_0) is a strict local maximum point of $v - \varphi_\eta$ on a neighborhood of (t_0, z_0) on which $0 \leq y - \phi(t, x) \leq \frac{1}{\eta^2}$. We then deduce from the subsolution property of v that

$$-\partial_t \varphi_\eta(t_0, z_0) + H_*(z_0, D\varphi_\eta(t_0, z_0), D^2\varphi_\eta(t_0, z_0)) \leq 0 .$$

For all $\varepsilon > 0$, we can then find $(z_\varepsilon, q_\varepsilon, A_\varepsilon) \in \text{cl}(D) \times \mathbb{R}^{d+1} \times \mathbb{S}^{d+1}$ and $u_\varepsilon^\eta \in U$ such that

$$(6.10) \quad |(z_\varepsilon, q_\varepsilon, A_\varepsilon) - (z_0, D\varphi_\eta(t_0, z_0), D^2\varphi_\eta(t_0, z_0))| \leq \varepsilon$$

and, recalling that $y_0 = \phi(t_0, x_0)$,

$$(6.11) \quad -\partial_t \varphi(t_0, z_0) + \frac{1}{\eta} \partial_t \phi(t_0, x_0) + H^{u_\varepsilon^\eta}(z_\varepsilon, q_\varepsilon, A_\varepsilon) \leq \varepsilon .$$

Using (6.10), the definitions of φ_η , and H^u , we thus obtain

$$(6.12) \quad \left\{ \begin{aligned} & -\partial_t \varphi(t_0, z_0) - \mu_Z(z_\varepsilon, u_\varepsilon^\eta) \cdot D\varphi(t_0, z_0) - \frac{1}{2} \text{Tr} [\sigma_Z \sigma_Z^\top(z_\varepsilon, u_\varepsilon^\eta) D^2\varphi(t_0, z_0)] \\ & + \frac{1}{\eta} \partial_t \phi(t_0, x_0) + H^{u_\varepsilon^\eta}(z_\varepsilon, Df_\eta(t_0, z_0), D^2f_\eta(t_0, z_0)) \\ & \leq \varepsilon(1 + |\mu_Z(z_\varepsilon, u_\varepsilon^\eta)| + |\sigma_Z(z_\varepsilon, u_\varepsilon^\eta)|^2) . \end{aligned} \right\}$$

Now straightforward computations show that

$$(6.13) \quad H^{u_\varepsilon^\eta}(z_\varepsilon, Df_\eta(t_0, z_0), D^2f_\eta(t_0, z_0)) = \eta (\sigma_Y(z_\varepsilon, u_\varepsilon^\eta) - \sigma_X^\top(x_\varepsilon, u_\varepsilon^\eta) D\phi(t_0, x_0))^2 + \frac{1}{\eta} \left(\mu_X(x_\varepsilon, u_\varepsilon^\eta) \cdot D\phi(t_0, x_0) + \frac{1}{2} (\sigma_X \sigma_X^\top)(x_\varepsilon, u_\varepsilon^\eta) D^2\phi(t_0, x_0) - \mu_Y(z_\varepsilon, u_\varepsilon^\eta) \right) ;$$

recall that $y_0 = \phi(t_0, x_0)$. We now use Assumption 3.2 in order to deduce that $(u_\varepsilon^\eta)_{\varepsilon \in (0, \varepsilon_0)}$ is bounded for ε_0 small enough. Indeed, either U is bounded or the right-hand side of (6.12) is controlled by the first term of $H^{u_\varepsilon^\eta}(z_\varepsilon, Df_\eta(t_0, z_0), D^2f_\eta(t_0, z_0))$. We thus get a converging subsequence $u_{\varepsilon_n}^\eta \rightarrow u_0^\eta$ with $\varepsilon_n \rightarrow 0$. We then deduce from (6.12) and (6.13) the following inequality:

$$\begin{aligned} & -\mathcal{L}_Z^{u_0^\eta} \varphi(t_0, z_0) + \frac{1}{\eta} \partial_t \phi(t_0, x_0) + \eta (\sigma_Y(z_0, u_0^\eta) - \sigma_X^\top(x_0, u_0^\eta) D\phi(t_0, x_0))^2 \\ & + \frac{1}{\eta} \left(\mu_X(x_0, u_0^\eta) \cdot D\phi(t_0, x_0) + \frac{1}{2} (\sigma_X \sigma_X^\top)(x_0, u_0^\eta) D^2\phi(t_0, x_0) - \mu_Y(z_0, u_0^\eta) \right) \leq 0 . \end{aligned}$$

We now use Assumption 3.2 and the continuity of the coefficients again to obtain the required result by sending $\eta \rightarrow \infty$. □

LEMMA 6.2. For all $u_i \in \mathcal{N}_0(z_0, D\phi(t_0, x_0))$ and $\gamma > 0$ there exists $u_f \in \mathcal{N}_0(z_0, D\phi(t_0, x_0))$ such that

$$-\mathcal{L}_Z^{u_f} \varphi(t_0, z_0) - \gamma [\mu_Y(z_0, u_i) - \mathcal{L}_X^{u_i} \phi(t_0, x_0)]^- (\mu_Y(z_0, u_f) - \mathcal{L}_X^{u_f} \phi(t_0, x_0)) \leq 0 .$$

Proof. For $\gamma > 0$, define $\varphi_\gamma(t, z) := \varphi(t, z) + \ell_\gamma(t, z)$, where, for $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}$,

$$\ell_\gamma(t, z) := \gamma [\mu_Y(z_0, u_i) - \mathcal{L}_X^{u_i} \phi(t_0, x_0)]^- (y - \phi(t, x)) .$$

Since $\ell_\gamma(t, z) \geq 0$ when $y \geq \phi(t, x)$, we conclude that $v - \varphi_\gamma$ achieves a strict maximum point of $v - \varphi_\gamma$ on $\text{cl}(D)$. We thus can apply Lemma 6.1 to $\varphi = \varphi_\gamma$ and get $u_f \in \mathcal{N}_0(z_0, D\phi(t_0, x_0))$ such that $-\mathcal{L}_Z^{u_f} \varphi_\gamma(t_0, z_0) \leq 0$. Direct computations imply the required result. \square

Proof of Proposition 3.1. Given $\eta > 0$, it follows from Lemmas 6.1 and 6.2 that we can find a sequence $(u_k^\eta)_k$ in $\mathcal{N}_0(z_0, D\phi(t_0, x_0))$ such that for all $k \geq 1$

$$-\mathcal{L}_Z^{u_{k+1}^\eta} \varphi(t_0, z_0) - \eta [\mu_Y(z_0, u_k^\eta) - \mathcal{L}_X^{u_k^\eta} \phi(t_0, x_0)]^- (\mu_Y(z_0, u_{k+1}^\eta) - \mathcal{L}_X^{u_{k+1}^\eta} \phi(t_0, x_0)) \leq 0 .$$

Since $\mathcal{N}_0(z_0, D\phi(t_0, x_0))$ is compact under Assumption 3.2 (see Remark 3.4), it follows from the continuity of the coefficients that, for all $\eta > 0$, we can find $u_\eta \in \mathcal{N}_0(z_0, D\phi(t_0, x_0))$ such that

$$-\mathcal{L}_Z^{u_\eta} \varphi(t_0, z_0) + \eta \left([\mu_Y(z_0, u_\eta) - \mathcal{L}_X^{u_\eta} \phi(t_0, x_0)]^- \right)^2 \leq 0 .$$

Sending $\eta \rightarrow \infty$ and using once again the fact that $\mathcal{N}_0(z_0, D\phi(t_0, x_0))$ is compact leads to the required result. \square

6.4. Boundary condition when w is smooth. In this section, we prove Proposition 3.2.

6.4.1. Supersolution property in the domain. We assume $\hat{u}_0 := \hat{u}(t_0, x_0) \in \text{int}(U)$; recall Theorem 3.2. Let ϕ be a smooth function and $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ be a strict minimum point for $\mathcal{V}_* - \phi$ such that $(\mathcal{V}_* - \phi)(t_0, x_0) = 0$. We can assume without loss of generality that ϕ has polynomial growth. Set $y_0 := w(t_0, x_0)$ and define φ_n by $\varphi_n(t, x, y) := \phi(t, x) - n(y - w(t, x)) - \psi(t, x, y)$ with $\psi(t, x, y) := |x - x_0|^{2p} + |y - y_0|^{2p} + |t - t_0|^2$, $n \geq 1$, for some p to be chosen later on. Since V and w have polynomial growth, for $p \geq 2$ large enough, $V_* - \varphi_n$ admits a global minimum point (t_n, z_n) on $\text{cl}(D)$. z_n denotes (x_n, y_n) . Writing that

$$\begin{aligned} (\mathcal{V}_* - \phi)(t_0, x_0) &= (V_* - \varphi_n)(t_0, x_0, w(t_0, x_0)) \\ &\geq (V_* - \varphi_n)(t_n, z_n) \\ &= (V_* - \phi)(t_n, z_n) + n(y_n - w(t_n, x_n)) + \psi(t_n, x_n, y_n) \end{aligned}$$

we deduce from the growth condition on V that, after possibly passing to a subsequence, $(t_n, x_n, y_n) \rightarrow (t_\infty, x_\infty, y_\infty := w(t_\infty, x_\infty))$ for some $(t_\infty, x_\infty) \in [0, T] \times \mathbb{R}^d$. It

follows that

$$\begin{aligned}
 (\mathcal{V}_* - \phi)(t_0, x_0) &\geq \limsup_{n \rightarrow \infty} ((V_* - \phi)(t_n, z_n) + n(y_n - w(t_n, x_n)) + \psi(t_n, x_n, y_n)) \\
 &\geq \liminf_{n \rightarrow \infty} (V_* - \phi)(t_n, z_n) + \limsup_{n \rightarrow \infty} (n(y_n - w(t_n, x_n)) + \psi(t_n, x_n, y_n)) \\
 &\geq (V_* - \phi)(t_\infty, x_\infty, w(t_\infty, x_\infty)) + \limsup_{n \rightarrow \infty} n(y_n - w(t_n, x_n)) \\
 &\quad + \psi(t_\infty, x_\infty, y_\infty) \\
 &\geq (V_* - \phi)(t_\infty, x_\infty, w(t_\infty, x_\infty)) + \liminf_{n \rightarrow \infty} n(y_n - w(t_n, x_n)) \\
 &\quad + \psi(t_\infty, x_\infty, y_\infty) \\
 &\geq (\mathcal{V}_* - \phi)(t_\infty, x_\infty) \geq (\mathcal{V}_* - \phi)(t_0, x_0),
 \end{aligned}$$

which implies that

$$(6.14) \quad (t_n, x_n, y_n, V_*(t_n, z_n), n(y_n - w(t_n, x_n))) \rightarrow (t_0, x_0, y_0, \phi(t_0, x_0), 0).$$

In particular, $t_n < T$ for n large enough. Moreover, it follows from Theorem 3.2 that $\mathcal{N}_0(x_n, w(t_n, x_n), Dw(t_n, x_n)) \neq \emptyset$. Set $\hat{u}_n := \hat{u}(t_n, x_n)$ and note that $\hat{u}_n \rightarrow \hat{u}_0 \in \text{int}(U)$ by Assumption 3.3. It follows that $\hat{u}_n \in \text{int}(U)$ for n large enough.

We have two cases:

1. If $y_n = w(t_n, x_n)$, then it follows from Remark 6.2 that

$$(6.15) \quad -\mathcal{L}_Z^{\hat{u}_n} \varphi_n(t_n, x_n, y_n) \geq 0.$$

2. If $y_n > w(t_n, x_n)$, then $(t_n, x_n, y_n) \in \text{int}_p(D)$ and it follows Theorem 3.1 that (6.15) holds too.

In both cases, we thus have, by replacing φ_n in (6.15),

$$-\mathcal{L}_X^{\hat{u}_n} \phi(t_n, x_n) + n \left(\mu_Y(x_n, y_n, \hat{u}_n) - \mathcal{L}_X^{\hat{u}_n} w(t_n, x_n) \right) + \mathcal{L}_Z^{\hat{u}_n} \psi(t_n, x_n, y_n) \geq 0.$$

Since μ_Y is Lipschitz continuous, and $n(y_n - w(t_n, x_n)) \rightarrow 0$ and $\mathcal{L}_Z^{\hat{u}_n} \psi(t_n, x_n, y_n) \rightarrow 0$ by (6.14), this implies that

$$-\mathcal{L}_X^{\hat{u}_0} \phi(t_0, x_0) + \liminf_{n \rightarrow \infty} n \left(\mu_Y(x_n, w(t_n, x_n), \hat{u}_n) - \mathcal{L}_X^{\hat{u}_n} w(t_n, x_n) \right) \geq 0.$$

Since $\hat{u}_n \in \text{int}(U)$ for n large enough, Theorem 3.2 implies that $\mu_Y(x_n, w(t_n, x_n), \hat{u}_n) - \mathcal{L}_X^{\hat{u}_n} w(t_n, x_n) = 0$, which proves the required result.

6.4.2. Subsolution property in the domain. Let ϕ be a smooth function, let $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ be a global strict maximum point for $\mathcal{V}^* - \phi$, and let $y_0 > w(t_0, x_0)$. Given $\varepsilon > 0$, define for $n \geq 1$ the function φ_n by

$$\varphi_n(t, x, y) := \phi(t, x) + \psi(t, x, y) + \varepsilon n(y - w(t, x)) + \varepsilon n^2(y - w(t, x))(w(t, x) - y + n^{-1})$$

with $\psi(t, x, y) := |t - t_0|^2 + |x - x_0|^{2p} + |y - y_0|^{2p}$. Let $(t_n, x_n, y_n)_n$ be a sequence such that, for each n , (t_n, x_n, y_n) is a maximizer on $\{(t, x, y) \in \text{cl}(D) : y - w(t, x) \leq n^{-1}\}$ of $\mathcal{V}^* - \varphi_n$. Arguing as in the previous section, one easily checks that, for $p \geq 2$ large enough and after possibly passing to a subsequence,

$$(6.16) \quad (t_n, x_n, n(y_n - w(t_n, x_n))) \rightarrow (t_0, x_0, 0).$$

In particular, we can assume, after possibly passing to a subsequence, that $t_n < T$ and $y_n - w(t_n, x_n) < n^{-1}$ for all $n \geq 1$. It then follows from Corollary 3.1 that, for each $n \geq 1$, we can find $u_n \in U$ such that

$$n^{-1}(1 + |\mu_Z(t_n, x_n, y_n)| + |\sigma_Z(t_n, x_n, y_n)|^2) \geq -\mathcal{L}_Z^{u_n} \varphi_n(t_n, x_n, y_n),$$

which, by direct computations, implies

$$(6.17) \quad \begin{aligned} n^{-1}(1 + |\mu_Z(t_n, x_n, y_n)| + |\sigma_Z(t_n, x_n, y_n)|^2) &\geq -\mathcal{L}_X^{u_n} \phi(t_n, x_n) \\ &\quad - 2\varepsilon n^2 (n^{-1} + w(t_n, x_n) - y_n) (\mu_Y(x_n, y_n, u_n) - \mathcal{L}_X^{u_n} w(t_n, x_n)) \\ &\quad + \varepsilon n^2 |N^{u_n}(x_n, y_n, Dw(t_n, x_n))|^2 - \mathcal{L}_Z^{u_n} \psi(t_n, x_n, y_n). \end{aligned}$$

Note that (6.16) implies that

$$(6.18) \quad n^2 (n^{-1} + w(t_n, x_n) - y_n) = O(n).$$

Using (3.11) and the previous inequality, we then deduce that the sequence $(u_n)_n$ is bounded and therefore converges, after possibly passing to a subsequence, to some $u_0 \in U$. It also follows that $n|N^{u_n}(x_n, y_n, Dw(t_n, x_n))|^2$ is bounded, so that $N^{u_0}(x_0, w(t_0, x_0), Dw(t_0, x_0)) = \lim_n N^{u_n}(x_n, y_n, Dw(t_n, x_n)) = 0$, which by Assumption 3.3 leads to $u_0 = \hat{u}(t_0, x_0)$. Moreover, it follows from (6.16)–(6.18), the Lipschitz continuity of the coefficients, and the convergence $u_n \rightarrow \hat{u}(t_0, x_0)$ that

$$\begin{aligned} \alpha(n) + n^{-1}(|\mu_Z(t_n, x_n, y_n)| + |\sigma_Z(t_n, x_n, y_n)|^2) &\geq -\mathcal{L}_X^{\hat{u}(t_0, x_0)} \phi(t_0, x_0) \\ &\quad + 2\varepsilon n^2 |n^{-1} + w(t_n, x_n) - y_n| (\mathcal{L}_X^{u_n} w(t_n, x_n) - \mu_Y(x_n, w(t_n, x_n), u_n)) \\ &\quad + \varepsilon n^2 |N^{u_n}(x_n, w(t_n, x_n), Dw(t_n, x_n))|^2, \end{aligned}$$

where $\alpha(n) \rightarrow 0$ as $n \rightarrow \infty$. In view of Assumption 3.4 and (6.18), taking the limit as $n \rightarrow \infty$ leads to

$$0 \geq -\mathcal{L}_X^{\hat{u}(t_0, x_0)} \phi(t_0, x_0) - \varepsilon C$$

for some $C > 0$, which proves the required result by arbitrariness of $\varepsilon > 0$.

6.4.3. Subsolution property on the boundary. Fix $x_0 \in \mathbb{R}^d$ and a smooth function φ such that x_0 achieves a strict maximum of $\mathcal{V}^*(T, \cdot) - \varphi$ such that $\mathcal{V}^*(T, x_0) - \varphi(x_0) = 0$. If $\mathcal{W}_*(T, x_0) \neq \emptyset$, we can find a smooth function ϕ such that x_0 reaches a local minimum of $w(T, \cdot) - \phi$ such that $w(T, x_0) - \phi(x_0) = 0$. For $n \geq 1$, set $\tilde{\varphi}_n(z) := \varphi(x) + n(y - \phi(x)) + \psi(x, y) + n^2(y - \phi(x))(\phi(x) - y + n^{-1})$, where $\psi(x, y) := |x - x_0|^{2p} + |y - y_0|^{2p}$ for some $p \geq 2$. Arguing as above, we obtain that, for $p \geq 2$ large enough, there exists a sequence of local maximizers $(\tilde{x}_n, \tilde{y}_n)_n$ of $V^*(T, \cdot) - \tilde{\varphi}_n$ such that

$$(6.19) \quad (\tilde{x}_n, n(\tilde{y}_n - \phi(\tilde{x}_n)), V^*(T, \tilde{x}_n, \tilde{y}_n)) \rightarrow (x_0, 0, \mathcal{V}^*(T, x_0)).$$

For all $u \in U$ and $(x, y) \in \mathbb{R}^d \times \mathbb{R}$, we now compute that

$$\begin{aligned} -\mathcal{L}_Z^u \tilde{\varphi}_n(x, y) &= -\mathcal{L}_X^u \varphi(x) - 2n^2 (n^{-1} + \phi(x) - y) (\mu_Y(x, y, u) - \mathcal{L}_X^u \phi(x)) \\ &\quad + n^2 |N^u(x, y, D\phi(x))|^2 - \mathcal{L}_Z^u \psi(x, y). \end{aligned}$$

In view of (3.11), this implies that $H_* \tilde{\varphi}_n(\tilde{x}_n, \tilde{y}_n) > -\infty$. It then follows from Theorem 3.1 that $V^*(T, \tilde{x}_n, \tilde{y}_n) \leq f^*(\tilde{x}_n, \tilde{y}_n)$ for all n . Sending $n \rightarrow \infty$ and using (6.19) shows that $\mathcal{V}^*(T, x_0) \leq f^*(x_0, \phi(x_0)) = f^*(x_0, w(T, x_0))$, which concludes the proof.

6.4.4. Supersolution property on the boundary. In order to get the supersolution property of \mathcal{V} at $t = T$, we first need the following lemma about V when $t = T$ and $y_0 = w(t_0, x_0)$ (i.e., in the “corner” of the domain).

LEMMA 6.3. *For any smooth function φ and (T, z_0) with $z_0 = (x_0, y_0) \in \mathbb{R}^d \times \mathbb{R}$, such that $y_0 = w(t_0, x_0)$, which is a strict minimum point for $V_* - \varphi$ such that $(V_* - \varphi)(T, z_0) = 0$, we have*

$$(6.20) \quad \varphi(T, z_0) \geq f_*(z_0) \quad \text{if} \quad \limsup_{(t', x') \rightarrow (T, x_0), t' < T} |\hat{u}(t', x')| < \infty ,$$

where $\hat{u} := \check{u}(\cdot, w, Dw)$.

Proof. We argue by contradiction and assume that $\varphi(T, z_0) < f_*(z_0)$. Then, there exist $\eta, r > 0$ small enough so that

$$(6.21) \quad \varphi \leq f_* - \eta \quad \text{on} \quad \{T\} \times B_r(z_0) .$$

Let $(t_n, z_n)_n$ be a sequence in $\text{int}_p(D)$ such that $V(t_n, z_n) \rightarrow V_*(T, z_0)$ and $(t_n, z_n) \rightarrow (T, z_0)$. As in section 6.2.3, we modify the test function and introduce $\tilde{\varphi} : (t, z) \mapsto \varphi(z) - (T - t)^{\frac{1}{2}}$, so that (T, z_0) is also a strict minimum point of $V_* - \tilde{\varphi}$. Recalling the right-hand side of (6.20) and observing that $-\partial_t \tilde{\varphi}(t, z) = -\partial_t \varphi(t, z) - (T - t)^{-\frac{1}{2}}$, where $(T - t)^{-\frac{1}{2}} \rightarrow \infty$ as $t \rightarrow T$, we deduce that we can choose $r > 0$ small enough such that

$$(6.22) \quad -\mathcal{L}_Z^{\hat{u}} \tilde{\varphi} < 0 \quad \text{on} \quad \mathcal{V}_0 := [T - r, T] \times B_r(z_0) .$$

In view of Theorem 3.2, w satisfies

$$(6.23) \quad \mu_Y(\cdot, w, \hat{u}) - \mathcal{L}_X^{\hat{u}} w \geq 0 \quad \text{on} \quad [T - r, T] \times \mathbb{R}^d .$$

Set $\hat{Z}^n := Z_{t_n, z_n}^{\hat{u}^n}$, where \hat{u}^n is the Markovian control associated with \hat{u} . Since $y_n \geq w(t_n, x_n)$, a standard comparison argument combined with (6.23) implies that $Y^n(T) \geq w(T, X^n(T))$. Hence, $\hat{u}^n \in \mathcal{U}_{t_n, z_n}$. Moreover, (6.22) implies that

$$\tilde{\varphi}(t_n, z_n) \leq \mathbb{E} \left[\tilde{\varphi}(\theta_n, \hat{Z}^n(\theta_n)) \right] ,$$

where θ_n is the first exit time of $(s, \hat{Z}^n(s))_{s \geq t_n}$ from \mathcal{V}_0 . Since (T, z_0) is a strict minimum point for $V_* - \tilde{\varphi}$ and $(V_* - \tilde{\varphi})(T, z_0) = 0$, we can then find $\zeta > 0$ such that $V_* - \zeta \geq \tilde{\varphi}$ on $\{(t, z) \in \partial \mathcal{V}_0 : t < T\}$. Recalling (6.21) and the previous inequality, it follows that

$$\tilde{\varphi}(t_n, x_n, y_n) \leq \mathbb{E} \left[(f_*(\hat{Z}^n(\theta_n)) - \eta) \mathbf{1}_{\{\theta_n = T\}} + (V_*(\theta_n, \hat{Z}^n(\theta_n)) - \zeta) \mathbf{1}_{\{\theta_n < T\}} \right] .$$

Since $\nu^n \in \mathcal{U}_{t_n, x_n, y_n}$ and $\tilde{\varphi}(t_n, z_n) - V(t_n, z_n) \rightarrow 0$ as $n \rightarrow \infty$, the above inequality contradicts Theorem 6.1 for n large enough. \square

The proof of the supersolution property at $t = T$ easily follows from this lemma. Indeed, it is obtained by arguing as in section 6.4.3 and adapting the test function as in section 6.4.1.

Appendix: Dynamic programming for stochastic targets. This appendix is devoted to the proof of Theorem 2.1, which is essentially known from [21]. The only difference comes from the definition of the set of controls. Since the above arguments are almost the same as the one used in [21], we insist only on the differences and

refer the reader to this paper for some auxiliary results. We start with two important lemmas.

LEMMA A.1. *The set $\{(t, z, \nu) \in [0, T] \times \mathbb{R}^{d+1} \times \mathcal{U} : \nu \in \mathcal{U}^t\}$ is closed in $[0, T] \times \mathbb{R}^{d+1} \times L^2([0, T] \times \Omega)$ and the set $\Gamma := \{(t, z, \nu) \in [0, T] \times \mathbb{R}^d \times \mathcal{U} : \nu \in \mathcal{U}_{t,z}\}$ is a Borel set in $[0, T] \times \mathbb{R}^{d+1} \times L^2([0, T] \times \Omega)$.*

Proof. Let $(t_n, \nu^n)_{n \geq 1}$ be a sequence in $[0, T] \times \mathcal{U}$ such that $\nu^n \in \mathcal{U}^{t_n}$ for each n , and let $(t_n, \nu^n) \rightarrow (t, \nu) \in [0, T] \times L^2([0, T] \times \Omega)$ strongly. Let ξ be a continuous adapted bounded process. For $n \geq 1$, we have

$$\mathbb{E} \left[\int_0^T \nu_s^n \xi_{s \wedge t_n} ds \right] = \int_0^T \mathbb{E} [\nu_s^n] \mathbb{E} [\xi_{s \wedge t_n}] ds$$

since ν^n is independent of \mathcal{F}_{t_n} and $\xi_{\cdot \wedge t_n}$ is \mathcal{F}_{t_n} -measurable. Sending $n \rightarrow \infty$ and using the continuity of ξ , we then obtain

$$\mathbb{E} \left[\int_0^T \nu_s \xi_{s \wedge t} ds \right] = \int_0^T \mathbb{E} [\nu_s] \mathbb{E} [\xi_{s \wedge t}] ds .$$

By arbitrariness of ξ , this shows that ν is independent of \mathcal{F}_t . On the other hand, it follows exactly from the same arguments as in the proof of Lemma 3.1 of [21] that $\{(t, z, \nu) \in [0, T] \times \mathbb{R}^{d+1} \times \mathcal{U} : g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P}\text{-a.s.}\}$ is a Borel set. In view of the previous result, this implies that Γ is the intersection of two Borel sets. \square

LEMMA A.2. *For any probability measure m on (D, \mathcal{B}_D) , there exists a Borel measurable map $\phi_m : (D, \mathcal{B}_D) \rightarrow (\mathcal{U}, \mathcal{B}_\mathcal{U})$ such that $\phi_m(t, z) \in \mathcal{U}_{t,z}$ for m -a.e. $(t, z) \in D$.*

Proof. Appealing to Lemma A.1, the proof follows the same arguments as in the proof of Lemma 3.1 in [21]. \square

Proof of Theorem 2.1. The first assertion follows from the same argument as in the proof of Theorem 3.1 in [21] by appealing to Lemma A.2 instead of Lemma 3.1 in [21]. We now prove the second one. Fix $(t, z) \in [0, T] \times \mathbb{R}^{d+1}$, θ a $[t, T]$ -valued stopping time, $\nu \in \mathcal{U}^t$, and $\tilde{\nu} \in \mathcal{U}_{t,z}$ such that $\nu = \tilde{\nu}$ on $[t, \theta]$. Then, the flow property implies that

$$\begin{aligned} 1 &= \mathbb{P} \left[g(Z_{\theta, Z_{t,z}^\nu}^{\tilde{\nu}}(T)) \geq 0 \right] \\ &= \int \int \mathbf{1} \left\{ g \left(Z_{\theta(\omega), Z_{t,z}^{\nu(\omega)}}^{\tilde{\nu}(\omega^\theta(\omega) + \mathbf{T}_{\theta(\omega)}(\tilde{\omega}))} (T)_{(\omega^\theta(\omega) + \mathbf{T}_{\theta(\omega)}(\tilde{\omega}))} \right) \geq 0 \right\} d\mathbb{P}(\omega) d\mathbb{P}(\tilde{\omega}) , \end{aligned}$$

where we use the notations $\omega^r := (\omega_{t \wedge r})_{t \leq T}$ and $\mathbf{T}_r(\omega) := (\omega_{s+r} - \omega_r)_{s \leq T-r}$ for $r \in [0, T]$. Viewing $\tilde{\omega} \in \Omega \mapsto \tilde{\nu}(\omega^\theta(\omega) + \mathbf{T}_{\theta(\omega)}(\tilde{\omega}))$ as a control independent of $\mathcal{F}_{\theta(\omega)}$ for fixed $\omega \in \Omega$, this shows that, for \mathbb{P} -a.e. $\omega \in \Omega$, $(\theta, Z_{t,z}^\nu(\theta))(\omega) \in D$. \square

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