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Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations

Abstract. This paper is concerned with Hölder regularity of viscosity solutions of second-order, fully non-linear elliptic integro-differential equations. Our results rely on two key ingredients: first we assume that, at each point of the domain, either the equation is strictly elliptic in the classical fully non-linear sense, or (and this is the most original part of our work) the equation is strictly elliptic in a non-local non-linear sense we make precise. Next we impose some regularity and growth conditions on the equation. These results concern a large class of integro-differential operators whose singular measures depend on x and also a large class of equations, including Bellman–Isaacs equations.

1. Introduction

The aim of this paper is to show that viscosity solutions of fully non-linear elliptic integro-differential equations are Hölder continuous under general suitable strict ellipticity and regularity/growth conditions on the equations. We also obtain explicit $C^{0,\alpha}$ estimates in terms of the space dimension, the non-linearity, the singular measure and the (local) L^∞-bound of the solution.

To be more specific, we describe the general framework we consider. We are interested in equations of the type

$$F(x, u, Du, D^2u, \mathcal{I}[x, u]) = 0 \quad \text{in } \Omega \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^N$ (not necessarily bounded) and $\mathcal{I}[x, u]$ is an integro-differential operator. The function u is real-valued and Du, D^2u stand respectively for its gradient and Hessian matrix. The non-linearity F is a (continuous) function which is degenerate elliptic: in this context, this means that F is non-increasing with respect to

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its last two variables (see below for a precise statement and [4] for further details). The integro-differential operators we will consider in the present paper are defined as follows:

$$\mathcal{I}[x, u] = \int_{\mathbb{R}^N} (u(x + z) - u(x) - Du(x) \cdot z \mathbf{1}_B(z)) \, \mu_x(dz) \qquad (1.2)$$

where $\mathbf{1}_B$ denotes the indicator function of the unit ball B and $\{\mu_x\}_{x \in \mathbb{R}^N}$ is a family of *Lévy measures*, i.e. non-negative, possibly singular, Borel measures on $\mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} \min(|z|^2, 1) \, \mu_x(dz) < +\infty. \qquad (1.3)$$

We point out that the solution u has to be given in the whole space $\mathbb{R}^N$ even if (1.1) is satisfied on a domain Ω; see Section 2 for further details.

Important "special cases" of operators of the form (1.2) are *Lévy–Itô operators*, namely

$$\mathcal{I}_{LI}[x, u] = \int_{\mathbb{R}^N} (u(x + j(x, z)) - u(x) - Du(x) \cdot j(x, z) \mathbf{1}_B(z)) \, \mu(dz) \qquad (1.4)$$

where μ is a Lévy measure and $j(x, z)$ is the size of the jumps at x. For the operator to be well-defined, one usually assumes

$$|j(x, z)| \leq C_0 |z| \quad \text{for some constant } C_0 \text{ and for any } x \in \Omega, z \in \mathbb{R}^N.$$

Model examples for a Lévy measure μ and jump function j are the ones associated with the fractional Laplacian: $\mu(dz) = dz/|z|^{N+\alpha}$ (with $0 < \alpha < 2$) and $j(x, z) = z$. This class of operators appears in the context of stochastic control of jump processes and it is also worth pointing out that, at least to the best of our knowledge, general comparison results (for viscosity solutions) are only available for operators in the form (1.4).

Many papers deal with Hölder estimates for fully non-linear elliptic equations. There are two kinds of approaches: for uniformly elliptic equations, one can use the powerful approach by Harnack inequalities which leads also to further regularity results; we refer the reader to Cabré and Caffarelli [7] or Trudinger [16, 17] and references therein for results in this direction. A simpler method, more closely related to classical viscosity solutions theory, was introduced by Ishii and Lions in [13]. Besides its simplicity, it has the further advantage of providing results under weaker ellipticity assumptions and even for some degenerate equations; it was used in [1], [10] where further regularity results are also proved and later in [5, 3]. As far as integro-differential elliptic equations are concerned, many papers were published in potential theory, with equations being linear in most of those papers; moreover they rely on probabilistic techniques. See for instance [6]. One of the first papers about Hölder estimates for integro-differential equations with PDE techniques is probably [15]. It mainly deals with linear equations where singular measures μ_x have a very general x-dependence, improving on the previous literature; it also deals with quite particular non-linear equations, more precisely, $F(\mathcal{I}[x, u], \mathcal{J}[x, u]) = 0$ where $\mathcal{I}$ and $\mathcal{J}$ are two different non-local terms, and for "strictly elliptic" functions F (in a suitable sense).

In the present paper, we deal with fully non-linear elliptic equations and we obtain local $C^{0,\alpha}$ regularity and estimates for a quite general class of integro-differential operators

of the form (1.2) satisfying proper assumptions (see (3.2)–(3.4)). Even if important operators (1.4) can be seen as special cases of the general one, we will give specific results with specific assumptions. In other words, our second main theorem is not a corollary of the first one.

Let us mention that, on the one hand, we do not cover all the examples given in [15] (in particular those in Section 3.4 of [15]), but on the other hand, we can treat examples outside the scope of [15]; indeed, we only assume that the measure μ_x is bounded at infinity (uniformly with respect to x), while condition (2.2) of [15] requires a (small) power of $|z|$ to be integrable at infinity. Moreover, we can handle much more general non-linear equations, including the important Bellman–Isaacs equations, and we can also identify the critical Hölder exponent α of the solution. To be more precise, we are able to prove that the solution is α-Hölder continuous for any $\alpha < \beta$ (and even $\alpha = \beta$ under stronger assumptions in the case $\beta < 1$) where β characterizes the singularity of the measure associated with the integral operator.

In order to treat a large class of non-linear elliptic equations, we decided to present the main results by assuming that the non-linearity F in (1.1) satisfies a proper ellipticity-growth condition (see (H) in Section 3). Loosely speaking, this structure condition ensures that either the equation is strictly elliptic in the classical fully non-linear sense or it is strictly elliptic in a non-linear and non-local sense. Since this condition is rather technical, a whole section is devoted to comments and examples (see Section 4).

The techniques we use in the present paper do not seem to yield Lipschitz regularity when $\beta \geq 1$ and we intend to investigate this question in a future work. Let us mention that we proved [2] in quite a general framework that there exists a viscosity solution of the Dirichlet problem without loss of the boundary condition. The local Hölder estimate we obtain in the present paper applies to the Dirichlet problem too and we will address naturally in a future work the question of boundary estimates. We also point out that the techniques we develop here can be readily applied to parabolic integro-differential equations. Finally, another possible interesting application of these results is the study of the ergodicity of non-local equations.

We conclude this introduction by mentioning that after this work was completed, we found out that Caffarelli and Silvestre [8] obtained regularity results for a large class of non-linear integro-differential equations that are invariant under x-translations and uniformly elliptic in a non-local way. In particular, they are able to get $C^{1,\alpha}$ regularity of solutions.

The paper is organized as follows. In Section 2, we recall the definition of a viscosity solution of (1.1). In Section 3, we state two main results: the first one deals with a general x-dependent Lévy measure and the second with integro-differential operators in the Lévy–Itô form. In Section 4, we comment on the main structure assumption on the non-linearity F and we give examples of direct applications of our results. Section 5 is devoted to the proofs of the main results.

Notation. The scalar product in $\mathbb{R}^N$ is denoted by $x \cdot y$. A ball centered at x of radius r is denoted by $B(x, r)$. If $x = 0$, we simply write B_r, and if $r = 1$, we even write B. Moreover, $\mathcal{S}^{N-1}$ denotes the unit sphere of $\mathbb{R}^N$.

The transpose of a matrix A is denoted A^* and $\|A\|$ stands for the usual norm of A, namely $\|A\| := \max_{|z|=1} |Az|$. $\mathbb{S}^N$ is the space of $N \times N$ real symmetric matrices. We recall that $X \geq Y$ if all the eigenvalues of $X - Y$ are non-negative. If $X \in \mathbb{S}^N$ and $\varepsilon \in (0, 1)$ is such that all the eigenvalues of X are strictly less than 1 (resp. strictly greater than -1), we set $X^\varepsilon = (I - \varepsilon X)^{-1} X$ (resp. $X_\varepsilon = (I + \varepsilon X)^{-1} X$). These matrices are obtained from X by applying a sup-convolution procedure (resp. an inf-convolution procedure), namely, for any $\xi \in \mathbb{R}^N$,

$$X^\varepsilon \xi \cdot \xi = \sup_{\zeta \in \mathbb{R}^N} \left\{ X\zeta \cdot \zeta - \frac{|\xi - \zeta|^2}{\varepsilon} \right\}, \quad X_\varepsilon \xi \cdot \xi = \inf_{\zeta \in \mathbb{R}^N} \left\{ X\zeta \cdot \zeta + \frac{|\xi - \zeta|^2}{\varepsilon} \right\}.$$

2. Viscosity solutions for PIDE

In this section, we recall the notion of degenerate ellipticity for non-linear non-local equations and the definition of viscosity solutions for such equations.

2.1. Degenerate ellipticity

Throughout the paper, the domain Ω is an open subset of $\mathbb{R}^N$ and the non-linearity F is a continuous function. We also assume that (1.1) is degenerate elliptic. In this framework, this means that we make the following

Assumption (E). For any $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $X, Y \in \mathbb{S}^N$, $l_1, l_2 \in \mathbb{R}$,

$$F(x, u, p, X, l_1) \leq F(x, u, p, Y, l_2) \quad \text{if } X \geq Y, \ l_1 \geq l_2.$$

2.2. Non-local operators

In order to define viscosity solutions for (1.1), we introduce two associated operators $\mathcal{I}^{1,\delta}$ and $\mathcal{I}^{2,\delta}$ by

$$\mathcal{I}^{1,\delta}[x, p, u] = \int_{|z|<\delta} [u(x + z) - u(x) - p \cdot z \mathbf{1}_B(z)] \, \mu_x(dz),$$

$$\mathcal{I}^{2,\delta}[x, p, u] = \int_{|z|\geq\delta} [u(x + z) - u(x) - p \cdot z \mathbf{1}_B(z)] \, \mu_x(dz).$$

In the case of Lévy–Itô operators (1.4), $\mathcal{I}^{1,\delta}$ and $\mathcal{I}^{2,\delta}$ are defined as follows:

$$\mathcal{I}^{1,\delta}[x, p, u] = \int_{|z|<\delta} [u(x + j(x, z)) - u(x) - p \cdot j(x, z) \mathbf{1}_B(z)] \, \mu(dz),$$

$$\mathcal{I}^{2,\delta}[x, p, u] = \int_{|z|\geq\delta} [u(x + j(x, z)) - u(x) - p \cdot j(x, z) \mathbf{1}_B(z)] \, \mu(dz).$$

2.3. Definition

We now recall the definition of a viscosity solution for (1.1). We assume that we are given a function u defined on the whole space $\mathbb{R}^N$.

Definition 2.1 (Viscosity solutions). An upper semi-continuous (usc for short) function $u : \mathbb{R}^N \to \mathbb{R}$ is a *subsolution* of (1.1) if for any test function $\phi \in C^2(B(x, \delta))$ such that $u - \phi$ attains a maximum on $B(x, \delta)$ at $x \in \Omega$,

$$F(x, u(x), D\phi(x), D^2\phi(x), \mathcal{I}^{1,\delta}[x, p, \phi] + \mathcal{I}^{2,\delta}[x, p, u]) \geq 0$$

where $p = D\phi(x)$.

A lower semi-continuous (lsc for short) function $u : \mathbb{R}^N \to \mathbb{R}$ is a *supersolution* of (1.1) if for any test function $\phi \in C^2(\mathbb{R}^N)$ such that $u - \phi$ attains a maximum on $B(x, r)$ at $x \in \Omega$ for some $r > 0$,

$$F(x, u(x), D\phi(x), D^2\phi(x), \mathcal{I}^{1,r}[x, p, \phi] + \mathcal{I}^{2,r}[x, u]) \leq 0,$$

where $p = D\phi(x)$.

A continuous function $u : \mathbb{R}^N \to \mathbb{R}$ is a *solution* of (1.1) if it is both a subsolution and a supersolution.

Remark 2.2. It is possible to construct solutions of (1.1) in the case where $\Omega = \mathbb{R}^N$. If $\Omega \neq \mathbb{R}^N$, boundary conditions must be imposed. For instance, as far as the Dirichlet problem is concerned, the function u can be prescribed outside of Ω. See [2] for further details.

Remark 2.3. As remarked in [4], one can choose $r = 0$ in the previous definition, at least in the case of Lévy–Itô operators. See [4] for further details.

3. Main results

In this section, we state the two main results of this paper: the first one is concerned with non-local operators of the form (1.2) (Theorem 3.1) and the second one provides other results for Lévy–Itô operators (1.4) (Theorem 3.2).

The two results rely on a structure condition imposed on the non-linearity F. In order to formulate it, we consider two functions $\Lambda_1, \Lambda_2 : \overline{\Omega} \to [0, +\infty)$ such that $\Lambda_1(x) + \Lambda_2(x) \geq \Lambda_0 > 0$ on $\overline{\Omega}$.

(H) (*Ellipticity-growth condition*) For any $R > 0$, there exist constants $k \geq 0, \tau, \theta \in (0, 1]$, a locally bounded function $\chi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, a modulus of continuity $\omega_F : (0, +\infty) \to (0, +\infty)$ with $\omega_F(0+) = 0$ and two constants $\eta, \bar{\varepsilon}_0 > 0$ such that

for any $x, y \in \Omega$ with $|x - y| \leq \eta$, $u, v \in \mathbb{R}$ with $|u|, |v| \leq R$, $p, q \in \mathbb{R}^N$, $|q| \leq R$, $l_1, l_2 \in \mathbb{R}$ with $l_1 \leq l_2$, $\varpi \in (0, 1/3)$, $L > 0$, $\bar{\varepsilon} \in (0, \bar{\varepsilon}_0)$ and $\hat{a} \in \mathcal{S}^{N-1}$, we have

$$F(y, u, p, Y, l_2) - F(x, v, p + q, X, l_1)$$

$$\leq \Lambda_1(x) \left(\mathrm{Tr}(X - Y) + \frac{\omega_F(|x - y|)}{\bar{\varepsilon}} + |x - y|^\tau |p|^{2+\tau} + |p|^2 + \chi(L, \eta) \right)$$

$$+ \Lambda_2(x) \left((l_1 - l_2) + \frac{|x - y|^{2\theta}}{\bar{\varepsilon}} + |x - y|^\tau |p|^{k+\tau} + C_F |p|^k + \chi(L, \eta) \right)$$

if the matrices X, Y satisfy

$$-\frac{4}{\bar{\varepsilon}} I \leq \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \frac{2}{\bar{\varepsilon}} \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix} + L \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.1}$$

where I denotes the identity matrix and $Z = I - (1 + \varpi)\hat{a} \otimes \hat{a}$.

In the next subsection, we comment on this structure condition and give several examples. The general results are the following.

Theorem 3.1 (Hölder continuity for general non-local operators). *Assume that the measures μ_x satisfy the following: there exist $\beta \in (0, 2)$, a constant $\tilde{C}_\mu > 0$, a modulus of continuity $\omega_\mu : (0, +\infty) \to (0, +\infty)$ with $\omega_\mu(0+) = 0$ and, for $\eta \in (0, 1)$, a constant $C_\mu(\eta) > 0$ such that for any $x, y \in \Omega$, $d \in \mathcal{S}^{N-1}$ and $\eta, \delta \in (0, 1)$,*

$$\int_B |z|^2 \mu_x(dz) + \int_{\mathbb{R}^N \setminus B} \mu_x(dz) \leq \tilde{C}_\mu,$$

$$\int_{\{z : |z| \leq \delta, |d \cdot z| \geq (1 - \eta)|z|\}} |z|^2 \mu_x(dz) \geq C_\mu(\eta)\delta^{2-\beta}, \tag{3.2}$$

$$\int_{B_\delta} |z|^2 |\mu_x - \mu_y|(dz) \leq \omega_\mu(|x - y|)\delta^{2-\beta}, \tag{3.3}$$

$$\int_{B \setminus B_\delta} |z| |\mu_x - \mu_y|(dz) \leq \begin{cases} \omega_\mu(|x - y|)\delta^{1-\beta} & \text{if } \beta \neq 1, \\ \omega_\mu(|x - y|)|\ln \delta| & \text{if } \beta = 1, \end{cases} \tag{3.4}$$

with, if $\beta = 1$, $\omega_\mu(\cdot)$ such that $\omega_\mu(r)|\ln r| \to 0$ as $r \to 0$. Suppose also that the non-linearity F satisfies (H) *for some parameters k, τ, θ.*

(i) *If*

$$\theta > \frac{1}{2}(2 - \beta) \quad \text{and} \quad \begin{cases} k = \beta & \text{if } \beta > 1, \\ k < \beta & \text{if } \beta \leq 1, \end{cases}$$

then any bounded continuous viscosity solution u of (1.1) *is locally α-Hölder continuous for α small enough. More precisely, α must satisfy: $\alpha < 1$ if $\beta \geq 1$ and $\alpha < (\beta - k)/(1 - k)$ if $\beta < 1$.*

(ii) *If $\beta < 1$ and if we assume moreover that $C_F = 0$ in* (H) *and $\tau > k(\beta^{-1} - 1)$, then u is β-Hölder continuous.*

Moreover, in both cases (i) *and* (ii), *the* $C^{0,\alpha}$ *and* $C^{0,\beta}$ *estimates depend on* $\|u\|_\infty$, *N* *(dimension), the constants* $\tilde{C}_\mu$, $C_\mu(\eta)$ *and the function* ω_μ *appearing in* (3.2)–(3.4), *and on the constants and functions appearing in* (H).

We now turn to Lévy–Itô operators.

Theorem 3.2 (Hölder continuity for Lévy–Itô operators). *Assume that the function j appearing in the definition of* $\mathcal{I}_{LI}$ *satisfies: there exist* $c_0, C_0 >$ *such that, for any* $x \in \Omega$ *and* $z \in \mathbb{R}^N$,

$$c_0|z| \le |j(x,z)| \le C_0|z|, \quad |j(x,z) - j(y,z)| \le C_0|z| |x - y|^{\tilde{\theta}} \quad (3.5)$$

with some $\tilde{\theta} \in (0,1)$. *Assume, in addition, that the measure* μ *satisfies: there exist* $\beta \in (0,2)$, *a constant* $\tilde{C}_\mu > 0$ *and, for any* $\eta \in (0,1)$, *a constant* $C_\mu(\eta) > 0$ *such that for any* $x \in \Omega$, $d \in \mathcal{S}^{N-1}$, *and* $\eta, \delta \in (0,1)$,

$$\int_B |j(x,z)|^2\, \mu(dz) + \int_{\mathbb{R}^N \setminus B} \mu(dz) \le \tilde{C}_\mu, \quad \int_{\mathcal{C}_{\delta,\eta}(x)} |j(x,z)|^2\, \mu(dz) \ge C_\mu(\eta)\delta^{2-\beta} \quad (3.6)$$

where $\mathcal{C}_{\delta,\eta}(x) := \{z : |j(x,z)| \le \delta, \ |d \cdot j(x,z)| \ge (1-\eta)|j(x,z)|\}$, *and that, moreover, for* δ *small enough,*

$$\int_{B \setminus B_\delta} |z|\, \mu(dz) \le \begin{cases} \tilde{C}_\mu \delta^{1-\beta} & \text{if } \beta \ne 1, \\ \tilde{C}_\mu |\ln \delta| & \text{if } \beta = 1. \end{cases} \quad (3.7)$$

Assume, finally, that the non-linearity F satisfies (H) *with parameters* k, τ, θ. *If*

$$\theta, \tilde{\theta} > \frac{1}{2}(2-\beta) \quad and \quad \begin{cases} k = \beta & \text{if } \beta > 1, \\ k < \beta & \text{if } \beta \le 1, \end{cases}$$

then any bounded continuous viscosity solution u of (1.1) *with* $\mathcal{I}[x,u]$ *replaced with* (1.4) *is locally* α-*Hölder continuous for any* $\alpha < \min(1,\beta)$.

If, in addition, $C_F = 0$ *in* (H) *and* $\tau > k(1-\beta)\beta^{-1}$, *then u is* β-*Hölder continuous when* $\beta < 1$.

Moreover, the $C^{0,\alpha}$ *estimate depends on* $\|u\|_\infty$, *N and the constants and functions appearing in* (H) *and* (3.5)–(3.7).

Remark 3.3. It is worth pointing out that (3.5)–(3.7) are analogues of (3.2)–(3.4) but they do not imply them. It is easy to see that the first condition of (3.5) together with (3.6) implies (3.2) but (3.3) and (3.4) do not derive from the second condition of (3.5) and (3.7).

Typically we have in mind the measures μ which satisfy, for some $C_\mu^\pm > 0$ and $\beta \in (0,2)$,

$$\frac{C_\mu^-}{|z|^{N+\beta}} dz \le \mu(dz) \le \frac{C_\mu^+}{|z|^{N+\beta}} dz. \quad (3.8)$$

and functions $j(x, z)$ such that $z \mapsto j(x, z)$ has an inverse function $J(x, Z)$ and there exist $c_0, C_0 > 0$ such that

$$\begin{cases} \forall (x, z) \in B_r(x_0, 0), & c_0|z| \leq |j(x, z)| \leq C_0|z|, \\ \forall (x, Z) \in B_R(x_0, 0), & c_0|Z| \leq |j(x, Z)| \leq C_0|Z|, \\ \forall Z \in \mathbb{R}^N, & c_0 \leq |\det D_z J(x_0, Z)|, \\ \forall z \in \mathbb{R}^N, & |j(x, z) - j(y, z)| \leq C_0|z| \, |x - y|^{\tilde{\theta}}, \end{cases} \tag{3.9}$$

and these are the properties we will use. We are in such a case if, for instance, for any x, $D_z j(x, z)$ exists for $|z|$ small enough, is continuous in (x, z) and non-singular for $z = 0$. Such a condition appears in [6].

4. Comments and examples

In this section, we comment on the assumptions of the main theorems and give examples of applications. More precisely, we illustrate the different terms appearing in the structure condition (H); we give examples of non-local operators of type (1.2) and (1.4); eventually, we give a regularity result that applies to the Bellman–Isaacs equation.

4.1. Non-linearities

In this subsection, we illustrate the structure condition (H) we used in Theorems 3.1 and 3.2 and, to do so, we consider the model equation

$$- \operatorname{Tr}(A(x)D^2 u) - c(x)\mathcal{I}[x, u] + H(x, u, Du) = 0 \quad \text{in } \Omega, \tag{4.1}$$

where $A : \Omega \to \mathbb{S}^N$, $c : \Omega \to \mathbb{R}$ and $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are continuous functions and $\mathcal{I}[x, u]$ is a non-local term of type (1.2) or (1.4).

First, the equation (4.1) has to be degenerate elliptic and therefore we assume that for all $x \in \Omega$, $A(x) \geq 0$ and $c(x) \geq 0$. For A, we are even going to use the more restrictive assumption (but natural in the probabilistic framework)

$$\text{for all } x \in \Omega, \quad A(x) = \sigma(x)\sigma^*(x),$$

where σ is a continuous function which maps Ω into the space of $N \times p$ matrices for some $p \leq N$.

We go back to the structure condition (H). It combines two different terms. The first one permits handling equations that are strictly elliptic in the usual sense. The second permits one to handle non-local equations that are strictly elliptic in a generalized (non-local) sense. Notice that imposing $\Lambda_1(x) + \Lambda_2(x) \geq \Lambda_0 > 0$ means that, at each point $x \in \Omega$, the non-linearity is either strictly elliptic in the classical (non-linear) sense or strictly elliptic in the generalized (non-local) sense.

A typical situation is the following: we are given two open subsets $\mathcal{O}_1, \mathcal{O}_2$ of Ω such that $\mathcal{O}_1 \cup \mathcal{O}_2 = \Omega$ and the closure of $(\mathcal{O}_1)^c$ is included in $\mathcal{O}_2$; moreover we know that F satisfies (H) in $\mathcal{O}_1$ with $\Lambda_1(x) \equiv 1$, $\Lambda_2(x) \equiv 0$, while F satisfies (H) in $\mathcal{O}_2$ with

$\Lambda_1(x) \equiv 0$, $\Lambda_2(x) \equiv 1$. Then, if Λ is a continuous function in Ω which equals 1 on the closure of $(\mathcal{O}_1)^c$ and has support in $\mathcal{O}_2$, it is easy to check that F satisfies (H) with $\Lambda_1(x) \equiv 1 - \Lambda(x)$, $\Lambda_2(x) \equiv \Lambda(x)$.

For equation (4.1), the structure condition (H) means that we assume

$$A(x) \geq \Lambda_1(x)I \quad \text{and} \quad c(x) \geq \Lambda_2(x) \quad \text{in } \Omega.$$

Typically this means that the second-order operator $-\operatorname{Tr}(A(x)D^2u)$ is uniformly elliptic in $\mathcal{O}_1$, while there is no degeneracy in the non-local variable l in $\mathcal{O}_2$. Of course, in the conditions "$\Lambda_1(x) \equiv 1$ in $\mathcal{O}_1$" or "$\Lambda_2(x) \equiv 1$ in $\mathcal{O}_2$", the "$\equiv 1$" may be replaced by "$\equiv \Lambda$" with $\Lambda > 0$.

Besides this ellipticity assumption, we have to assume that A (or more precisely σ) satisfies suitable continuity assumptions in $\mathcal{O}_1$ and $\mathcal{O}_2$: this appears in (H) in the second subterms of the Λ_1, Λ_2-terms. To describe these assumptions, we recall a standard computation which appeared for the first time in Ishii [12]: we assume that σ is bounded and uniformly continuous in Ω and we denote by ω_σ its modulus of continuity. We apply (3.1) to the vector $z = (z_1, z_2)$ with $z_1 = \sigma(\bar{x})e$, $z_2 = \sigma(\bar{y})e$ and an arbitrary $e \in \mathcal{S}^{N-1}$, and get

$$\sigma^T(\bar{x})X\sigma(\bar{x})e \cdot e - \sigma^T(\bar{y})Y\sigma(\bar{y})e \cdot e \leq \frac{1}{\varepsilon}\omega_\sigma^2(|\bar{x} - \bar{y}|) + L\|\sigma\|_\infty^2$$

(we used that $Z \leq I$). Therefore

$$\operatorname{Tr}(A(\bar{x})X) - \operatorname{Tr}(A(\bar{y})Y) \leq \frac{1}{\varepsilon}d\omega_\sigma^2(|\bar{x} - \bar{y}|) + Ld\|\sigma\|_\infty^2.$$

Hence, choose $\omega_F(r) = d\omega_\sigma^2(r)$ and $\chi(L, \eta) = Ld\|\sigma\|_\infty^2$. It is worth pointing out that, in the uniformly elliptic case (i.e. in $\mathcal{O}_1$), A or σ is only required to be continuous while, in $\mathcal{O}_2$, σ has to be Hölder continuous (the same computations as above provide the $|x - y|^{2\theta}$-term if $\omega_\sigma(r) = Cr^\theta$) in order to take advantage of the ellipticity of the non-local term.

We now turn to the non-local term. Our main remark is the following: in our formulation, the non-local term l is in fact $c(x)\mathcal{I}[x, u]$ and not only $\mathcal{I}[x, u]$. In that way, assumption (H) is obviously satisfied by (4.1). On the other hand, in order to verify the assumptions of Theorem 3.1, one has to replace the measure $\mu_x(dz)$ with $\tilde{\mu}_x(dz) := c(x)\mu_x(dz)$ and check if the conditions are satisfied by this new measure. In other words, in the case of (1.2), we rewrite $c(x)\mathcal{I}[x, u]$ as

$$\int_{\mathbb{R}^N} (u(x + z) - u(x) - Du(x) \cdot z\mathbf{1}_B(z)) \, \tilde{\mu}_x(dz).$$

Therefore, the continuity assumptions on μ_x in Theorem 3.1 are indeed continuity assumptions on $\tilde{\mu}_x$ and, for this reason, they contain continuity assumptions on both μ_x and c.

A priori a similar approach could be used for (1.4) but this really means that we use Theorem 3.1 instead of Theorem 3.2 in this case.

It remains to consider the first order term $H(x, u, Du)$ in (4.1): this appears in (H) in the third, fourth and (part of the) fifth subterms of Λ_1, Λ_2-terms. As is classical for (local)

equations, we have to require growth conditions with respect to Du: quadratic growth ($|p|^2$) when the equation is uniformly elliptic, and a $|p|^k$ growth with k depending on the measure when the strong ellipticity comes from the non-local term.

For example, in the case of the fractional Laplacian $(-\Delta)^{\beta/2}$, the natural growth turns out to be $k = \beta$, even if Theorem 3.1 shows that the case $\beta \leq 1$ is a little more particular. We refer the reader to Subsection 4.3 where an example of the equation involving a fractional Laplacian is treated in detail.

These growth conditions on the gradient have to be combined with the regularity of coefficients: we are able to treat gradient terms of the form $c(x)|Du(x)|^m$ with $m = 2$ if c is merely bounded, and $m = 2 + \tau$ if c is locally τ-Hölder continuous. We leave details to the reader.

We say more about these assumptions in the next subsections. In particular, we treat equations that are not exactly of the form (1.1) but can be handled with the same techniques (see Subsection 4.4).

4.2. Singular measures

The model singular measure is the Lévy measure associated with the fractional Laplacian $(-\Delta)^{\beta/2}$. In this case, $d\mu_x(z) = d\mu(z) = dz/|z|^{N+\beta}$ with $0 < \beta < 2$.

A second simple example of a measure μ_x is $c(x, z)\mu(dz)$ with a Lévy measure μ satisfying (3.6)–(3.7) with $j(x, z) = z$ and $c(x, z)$ satisfying, for any $x, y \in \Omega$ and $z \in B$,

$$|c(x, z) - c(y, z)| \leq \omega(|x - y|) \quad \text{where } \omega(t) \to 0 \text{ when } t \downarrow 0,$$

and, for any $x \in \Omega$, $z \in \mathbb{R}^N$, $0 < \underline{c} \leq c(x, z) \leq \overline{c}$ for some constants $\underline{c}, \overline{c}$. One can thus easily check (3.2)–(3.4) and Theorem 3.1 applies for suitable non-linearities F. Lévy measures associated with tempered stable Lévy processes satisfy (3.6)–(3.7). Indeed, in this case

$$\mu(dz) = \mathbf{1}_{(0,+\infty)}(z)\left(G^+ e^{-\lambda^+|z|}\frac{dz}{|z|^{N+\alpha}}\right) + \mathbf{1}_{(-\infty,0)}(z)\left(G^- e^{-\lambda^-|z|}\frac{dz}{|z|^{N+\alpha}}\right).$$

These measures appear in financial modeling (see for instance [9]).

4.3. A non-local equation involving the fractional Laplacian

In order to further illustrate our results, we next consider the model equation

$$(-\Delta)^{\beta/2}u + b(x)|Du|^{k+\tau} + |Du|^r = 0$$

where $b \in C^{0,\tau}$, $0 < \tau < 1$, $0 < k, r < 2$. In this case, condition (H) is satisfied with $\Lambda_1 = 0$, $\Lambda_2(x) = 1 > 0$, $\theta > 0$ arbitrary and τ, k, r as in the equation. It is easy to

check that (3.2) is satisfied with exponent β: first, it is a Lévy measure and if $\mathcal{C}_{\delta,\eta}$ denotes $\{|z| < |\delta|, d \cdot z \geq (1 - \eta)|z|\}$, then by homogeneity and symmetry of μ,

$$\int_{\mathcal{C}_{\delta,\eta}} |z|^2 \, \mu(dz) = \frac{|\mathcal{C}_{\delta,\eta}|}{|B_\delta|} \int_{B_\delta} |z|^2 \, \mu(dz) = \frac{|\mathcal{C}_{1,\eta}|}{|B_1|} \int_{B_\delta} |z|^2 \, \mu(dz)$$

$$= \frac{|\mathcal{C}_{1,\eta}|}{|B_1|} \int_{B_\delta} |z|^{2-\beta-N} \, dz = C_\mu(\eta)\delta^{2-\beta}.$$

Moreover, the other hypotheses we make on μ (namely (3.3) and (3.4)) are automatically satisfied since μ is independent of x (in other words, one can choose $\omega_\mu = 0$). Since $\beta < 2$, we cannot allow here a quadratic growth for the gradient term; indeed, Theorems 3.1 and 3.2 work only for $k, r \leq \beta$ in the absence of local ellipticity. It is also worth pointing out that, if $r \leq 1$, we have $|p|^r - |p + q|^r \leq |q|^r \leq R^r$ and therefore, even if $\beta < 1$, any such r works since the R^r can be absorbed in the $\chi(L, \eta)$-term.

4.4. The Bellman–Isaacs equation

Let us illustrate Theorem 3.2 by considering an important second-order non-linear elliptic integro-differential equations appearing in the study of stochastic control of processes with jumps, namely the Bellman–Isaacs equation. Let us mention the work of Jakobsen and Karlsen [14] in the evolution case where the authors use completely different techniques.

Corollary 4.1. *Consider the following Bellman–Isaacs equation in $\mathbb{R}^N$:*

$$cu + \sup_{\lambda \in \Lambda} \inf_{\gamma \in \Gamma} \left\{ -\frac{1}{2}\mathrm{Tr}(\sigma_{\lambda,\gamma}(x)\sigma_{\lambda,\gamma}^*(x)D^2u) - b_{\lambda,\gamma}(x) \cdot Du - \mathcal{I}_{LI}^{\lambda,\gamma}[x, u] - f_{\lambda,\gamma}(x) \right\} = 0$$
$$(4.2)$$

with $c \geq 0$ and where $\mathcal{I}_{LI}^{\lambda,\gamma}[x, u]$ is a family of Lévy–Itô operators associated with a common Lévy measure μ and a family of functions $j_{\lambda,\gamma}(x, z)$. Assume that

1. μ *satisfies* (3.6)–(3.7) *with constants independent of λ, μ,*
2. *there exist $c_0, C_0 > 0$ and $\tilde{\theta} \in (0, 1)$ such that for any $(\lambda, \gamma) \in \Lambda \times \Gamma$, $j_{\lambda,\gamma}$ satisfies* (3.5),
3. $\sigma_{\lambda,\gamma}$, $b_{\lambda,\gamma}$ *and $f_{\lambda,\gamma}$ satisfy, for some $\theta \in (0, 1)$ (and some constant $C_F > 0$),*

$$\forall \alpha, \beta, \quad \|\sigma_{\lambda,\gamma}\|_{0,\theta} + \|b_{\lambda,\gamma}\|_{0,\theta} + \|f_{\lambda,\gamma}\|_{0,\theta} \leq C_F.$$

If $\theta, \tilde{\theta} > \frac{1}{2}(2 - \beta)$, then any bounded viscosity solution u of (4.2) is α-Hölder continuous for any $\alpha < 1$ if $\beta \geq 1$, and for $\alpha < (\beta - k)/(1 - k)$ if $\beta < 1$.

Proof. Note that (4.2) is not exactly of the form (1.1). Nevertheless, the proof of Theorem 3.2 we present below can be adapted to this framework. It is enough to check that the structure condition (H) is satisfied by the linear equations

$$F_{\lambda,\gamma}(x, u, p, A, l) = cu - \frac{1}{2}\mathrm{Tr}(\sigma_{\lambda,\gamma}(x)\sigma_{\lambda,\gamma}^*(x)A) - b_{\lambda,\gamma}(x) \cdot p - l_{\lambda,\gamma} - f_{\lambda,\gamma}(x),$$

with constants and functions appearing in (H) independent of λ, γ.

Indeed, if this is the case, then we just have to use the standard inequality

$$\sup_{\lambda} \inf_{\gamma} (\cdots) - \sup_{\lambda} \inf_{\gamma} (\cdots) \leq \sup_{\lambda, \gamma} (\cdots - \cdots),$$

and we can conclude in this more general case too. □

5. Proofs of Theorems 3.1 and 3.2

We prove successively Theorems 3.1 and 3.2. On one hand, the proofs are very similar and we will skip details in the proof of the second theorem when adapting arguments used in the proof of the first one. On the other hand, we need to use every parameter very precisely. This is the reason why constants are computed from line to line and explicit formulae are given for each of them in order to use them later.

5.1. Proof of Theorem 3.1

We first prove (i). Without loss of generality, we assume that $C_F = 1$ in the ellipticity-growth condition (H). In order to prove the local Hölder continuity of u, we are going to show that, for any $x_0 \in \Omega$, there exists $L_2 = L_2(x_0)$ such that, for some well chosen $\alpha \in (0, 1)$ and for $L_1 = L_1(x_0) > 0$ large enough, we have

$$M = \sup_{x, y \in \mathbb{R}^N} \{u(x) - u(y) - \phi(x - y) - \Gamma(x)\} \leq 0$$

where $\phi(z) = L_1 |z|^{\alpha}$ and $\Gamma(x) = L_2 |x - x_0|^2$. We point out that the role of the term $\Gamma(x)$ is to localize around x_0, while the term $\phi(x - y)$ is concerned with the Hölder continuity. Proving such a result with a suitable control on α, L_1, L_2 clearly implies the desired property.

If $\Omega \neq \mathbb{R}^N$, we first choose L_2 so that $u(x) - u(y) - \phi(x - y) - \Gamma(x) \leq 0$ if $x \notin \Omega$: to do so, we first choose

$$L_2 \geq \frac{8\|u\|_{\infty}}{[d(x_0, \partial\Omega)]^2}.$$

If $\Omega = \mathbb{R}^N$, L_2 is arbitrary. Then we argue by contradiction: we assume that $M > 0$ and we are going to get a contradiction for L_1 large enough and for a suitable choice of α.

If the supremum defining M is attained at $(\bar{x}, \bar{y})$, we deduce from $M > 0$ that $\bar{x} \neq \bar{y}$ and

$$\begin{aligned}
&|\bar{x} - \bar{y}| \leq (2\|u\|_{\infty}/L_1)^{1/\alpha} =: A, \\
&|\bar{x} - x_0| < \sqrt{2\|u\|_{\infty}/L_2} =: R_2 \leq d(x_0, \partial\Omega)/2, \\
&u(\bar{x}) > u(\bar{y}).
\end{aligned} \tag{5.1}$$

If L_1 is so large that $A < d(x_0, \partial\Omega)/2$, then we have $\bar{x}, \bar{y} \in \Omega$.

Next, we pick some $\nu_0 \in (0, 1)$ and we define

$$a = \bar{x} - \bar{y}, \quad \varepsilon = |a|, \quad \hat{a} = a/|a|, \quad \delta = \nu_0 \varepsilon < \varepsilon.$$

First, ν_0 will be chosen small enough but fixed (independent of L_1 and ε).

From the study of the maximum point property for $(\bar{x}, \bar{y})$, we also get

$$L_1 \varepsilon^\alpha \leq u(\bar{x}) - u(\bar{y}) \leq \omega_u(\varepsilon) \tag{5.2}$$

where ω_u denotes the modulus of continuity of u on $B(x_0, d(x_0, \partial\Omega)/2)$. We will use this piece of information below (see Step 4). Notice also that if $\chi(x, y)$ denotes $\phi(x - y) + \Gamma(x)$, then $x \mapsto u(x) - \chi(x, \bar{y})$ (resp. $y \mapsto u(y) + \chi(\bar{x}, y)$) attains a global maximum (resp. minimum) at $\bar{x}$ (resp. $\bar{y}$) with $\chi(\cdot, \bar{y})$ (resp. $-\chi(\bar{x}, y)$) of class C^2 on $B(\bar{x}, \delta)$. In particular, we can use $\chi(\bar{x}, \cdot)$ and $-\chi(\cdot, \bar{y})$ as test functions in Definition 2.1 with any $\delta' < \delta$.

The remainder of the proof is divided into four steps. We write down the viscosity inequalities and combine them (Step 1), we get suitable matrix inequalities from non-local Jensen–Ishii's lemma (Step 2), we estimate from above the difference of the non-local terms (Step 3) and we conclude (Step 4).

Step 1: Writing down viscosity inequalities. Let p denote $D\phi(a)$ and q denote $D\Gamma(\bar{x})$. We use Corollary 1 of [4] which, for $\iota > 0$ small enough, provides us with two matrices $X_\iota, Y_\iota \in \mathbb{S}^N$ such that, for any $\delta' \ll 1$,

$$F(\bar{x}, u(\bar{x}), p + q, X_\iota, \mathcal{I}^{1,\delta'}[\bar{x}, p + q, \chi_\iota(\cdot, \bar{y})] + \mathcal{I}^{2,\delta'}[\bar{x}, p + q, u] + o_\iota(1)) \leq 0,$$
$$F(\bar{y}, u(\bar{y}), p, Y_\iota, \mathcal{I}^{1,\delta'}[\bar{y}, p, -\chi_\iota(\bar{x}, \cdot)] + \mathcal{I}^{2,\delta'}[\bar{y}, p, u] + o_\iota(1)) \geq 0$$

(where χ_ι is an approximation of χ by a localized inf-convolution, see [4]), together with the following matrix inequality:

$$-\frac{1}{\iota} I \leq \begin{bmatrix} X_\iota & 0 \\ 0 & -Y_\iota \end{bmatrix} \leq \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix} + 2L_2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + o_\iota(1), \tag{5.3}$$

where $Z = D^2\phi(a)$.

Our aim is to first let ι tend to 0 in order to get rid of all the artificial ι dependences in these inequalities; in order to do so, but also in order to apply (H) which requires a two-side bound on the matrices, we are first going to build matrices X, Y such that the above viscosity inequalities still hold if we replace X_ι, Y_ι by X, Y and such that the matrices X, Y satisfy the required inequality in (H).

Then, if we set

$$l_1 := \mathcal{I}^{1,\delta'}[\bar{x}, p + q, \chi_\iota(\cdot, \bar{y})] + \mathcal{I}^{2,\delta'}[\bar{x}, p + q, u],$$
$$l_2 := \mathcal{I}^{1,\delta'}[\bar{y}, p, -\chi_\iota(\bar{x}, \cdot)] + \mathcal{I}^{2,\delta'}[\bar{y}, p, u],$$

and if we subtract the viscosity inequalities, dropping all the ι dependences, we will have

$$0 \leq F(\bar{y}, u(\bar{y}), p, Y, l_2) - F(\bar{x}, u(\bar{x}), p + q, X, l_1). \tag{5.4}$$

In order to get the desired contradiction, in the rest of the proof we obtain various estimates, in particular on the differences $X - Y$ and $\mathcal{I}^{2,\delta'}[\bar{x}, p + q, u] - \mathcal{I}^{2,\delta'}[\bar{y}, p, u]$, in order to apply the ellipticity-growth condition (H) to show that the right-hand side of this

inequality is strictly negative. We point out that, since we are going to let first δ' tend to 0, the terms $\mathcal{I}^{1,\delta'}[\bar{x},\, p+q,\, \chi_\iota(\cdot,\bar{y})]$, $\mathcal{I}^{1,\delta'}[\bar{y},\, p,\, -\chi_\iota(\bar{x},\cdot)]$ create no difficulty because they tend to 0 with δ'.

Step 2: Building and estimating the matrices X, Y. We follow here ideas introduced by Crandall and Ishii [11] to obtain these matrices, by using only the upper bounds on X_ι, Y_ι. We rewrite (5.3) as: for any $z_1, z_2 \in \mathbb{R}^N$,

$$X_\iota z_1 \cdot z_1 - Y_\iota z_2 \cdot z_2 \leq Z(z_1 - z_2) \cdot (z_1 - z_2) + 2L_2|z_1|^2.$$

We have dropped the $o_\iota(1)$-term on the right-hand side for the sake of simplicity since it plays no role. In fact, we use the previous matrix inequality in the form

$$(X_\iota - 2L_2 I)z_1 \cdot z_1 - Y_\iota z_2 \cdot z_2 \leq Z(z_1 - z_2) \cdot (z_1 - z_2).$$

Next we have to compute Z, as well as, for the rest of the proof, the derivatives of ϕ. It will be convenient to do the proof for $\phi(x) = \varphi(|x|)$ for a general smooth function $\varphi : \mathbb{R}^+ \to \mathbb{R}$. We thus get, for any $b \in \mathbb{R}^N$,

$$D\phi(b) = \varphi'(|b|)\hat{b}, \quad D^2\phi(b) = \frac{\varphi'(|b|)}{|b|}P_{b\perp} + \varphi''(|b|)\hat{b} \otimes \hat{b} \tag{5.5}$$

where $\hat{b} = b/|b|$ and $P_{b\perp} = I - \hat{b} \otimes \hat{b}$ is the projection on the orthogonal complement of b. Hence, if $\varphi(r) = L_1 r^\alpha$, we get

$$\begin{aligned}
D\phi(b) &= L_1\alpha|b|^{\alpha-2}b, \\
D^2\phi(b) &= L_1(\alpha|b|^{\alpha-2}I + \alpha(\alpha-2)|b|^{\alpha-4}b \otimes b) \\
&= L_1\alpha|b|^{\alpha-4}(|b|^2 I - (2-\alpha)b \otimes b), \tag{5.6} \\
D^2\phi(b) &\leq L_1\alpha|b|^{\alpha-2}I. \tag{5.7}
\end{aligned}$$

We have

$$Z = \frac{1}{\bar{\varepsilon}}(I - (2-\alpha)\hat{a} \otimes \hat{a}) \quad \text{with} \quad \bar{\varepsilon} = (L_1\alpha\varepsilon^{\alpha-2})^{-1}.$$

Now we go back to the X_ι, Y_ι inequality; we apply to it a sup-convolution in both variables z_1 and z_2 with parameter $\frac{1}{4}\bar{\varepsilon}$. Noticing that this corresponds to an inf-convolution on the $(Y_\iota z_2 \cdot z_2)$-term, we easily get, with the notation introduced at the end of the Introduction,

$$(X_\iota - 2L_2 I)^{\bar{\varepsilon}/4}z_1 \cdot z_1 - (Y_\iota)_{\bar{\varepsilon}/4}z_2 \cdot z_2 \leq Z^{\bar{\varepsilon}/2}(z_1 - z_2) \cdot (z_1 - z_2).$$

On the other hand, tedious but easy, explicit computations (which are provided in the Appendix) yield

$$Z^{\bar{\varepsilon}/2} = 2\bar{\varepsilon}^{-1}(I - (1+\varpi)\hat{a} \otimes \hat{a}) \quad \text{with} \quad \varpi := \frac{1-\alpha}{3-\alpha} > 0.$$

Notice that $0 < \varpi < 1/3$.

If we set $X = (X_\iota - 2L_2 I)^{\bar{\varepsilon}/4} + 2L_2 I$, $Y = (Y_\iota)_{\bar{\varepsilon}/4}$, then X, Y satisfy (3.1) with $L = 2L_2$, and since $X_\iota \leq X$ and $Y \leq Y_\iota$, the viscosity inequalities still hold for X and Y because F is degenerate elliptic.

From this new form of inequality (5.3), we can obtain several types of estimates on X and Y. First, choosing $z_2 = -z_1 = \hat{a}$, we get

$$X\hat{a} \cdot \hat{a} - Y\hat{a} \cdot \hat{a} \leq -8\varpi/\bar{\varepsilon} + O(L_2) = -8L_1\alpha\varpi\varepsilon^{\alpha-2} + O(L_2).$$

Next choosing $z_2 = z_1 = z$ with z being orthogonal to $\hat{a}$, we have

$$Xz \cdot z - Yz \cdot z \leq O(L_2).$$

In particular, this yields

$$\mathrm{Tr}(X - Y) \leq -8L_1\alpha\varpi\varepsilon^{\alpha-2} + O(L_2). \tag{5.8}$$

Step 3: Estimates of the non-local terms. The difference of the non-local terms, denoted by T_{nl}, can be rewritten as follows (we recall that $B = B(0, 1)$):

$$T_{nl} = \int_{|z|\geq\delta'} [u(\bar{x} + z) - u(\bar{x}) - (p + q) \cdot z \mathbf{1}_B(z)] \, \mu_{\bar{x}}(dz)$$

$$- \int_{|z|\geq\delta'} [u(\bar{y} + z) - u(\bar{y}) - p \cdot z \mathbf{1}_B(z)] \, \mu_{\bar{y}}(dz).$$

In order to estimate it, we split the domain of integration $\{|z| \geq \delta'\}$ into three pieces: $\mathbb{R}^N \setminus B$ which leads to the T_1 term below; $\mathcal{C} = \{z \in B_\delta : |z \cdot \hat{a}| \geq (1 - \eta)|z|\} \subset B$ which leads to the T_2 term below; and $B \setminus \mathcal{C}$ which leads to the T_3 term below.

In order not to add further technicalities, we assume from now on that $\delta' = 0$; the reader can check that if $\delta' > 0$, the estimates we present below remain valid up to $o_{\delta'}(1)$.

Therefore we have to estimate from above $T_{nl} = T_1 + T_2 + T_3 + o_{\delta'}(1)$ with

$$T_1 = \int_{|z|\geq 1} [u(\bar{x} + z) - u(\bar{x})] \, \mu_{\bar{x}}(dz) - \int_{|z|\geq 1} [u(\bar{y} + z) - u(\bar{y})] \, \mu_{\bar{y}}(dz)$$

$$T_2 = \int_{\mathcal{C}} [u(\bar{x} + z) - u(\bar{x}) - (p + q) \cdot z] \, \mu_{\bar{x}}(dz) - \int_{\mathcal{C}} [u(\bar{y} + z) - u(\bar{y}) - p \cdot z] \, \mu_{\bar{y}}(dz)$$

$$T_3 = \int_{B\setminus\mathcal{C}} [u(\bar{x} + z) - u(\bar{x}) - (p + q) \cdot z] \, \mu_{\bar{x}}(dz)$$

$$- \int_{B\setminus\mathcal{C}} [u(\bar{y} + z) - u(\bar{y}) - p \cdot z] \, \mu_{\bar{y}}(dz).$$

For the reader's convenience, we recall that $p = D\phi(a)$ and $q = D\Gamma(\bar{x})$.

Estimate of T_1. Since u is bounded and so are the measures μ_x away from the origin, we conclude that T_1 is bounded, uniformly with respect to all the parameters we introduced. More precisely,

$$T_1 \leq C_1 \tag{5.9}$$

where

$$C_1 = 4\|u\|_\infty \sup_{x \in B(x_0, d(x_0, \partial\Omega)/2)} \mu_x(\mathbb{R}^N \setminus B). \tag{5.10}$$

Estimate of T_2. We estimate T_2 from above by using the definition of M. Indeed, its definition provides the following key inequality:

$$
\begin{aligned}
u(\bar{x} + d) - u(\bar{x}) - (p + q) \cdot d \leq{} & u(\bar{y} + d') - u(\bar{y}) - p \cdot d' \\
& + \{\phi(a + d - d') - \phi(a) - D\phi(a) \cdot (d - d')\} \\
& + \{\Gamma(\bar{x} + d) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot d\}. \tag{5.11}
\end{aligned}
$$

We then use (5.11) with $d = z$ and $d' = 0$ (resp. with $d = 0$ and $d' = z$). We obtain

$$
\begin{aligned}
T_2 \leq{} & \int_{\mathcal{C}} [\phi(a + z) - \phi(a) - D\phi(a) \cdot z] \, \mu_{\bar{x}}(dz) \\
& + \int_{\mathcal{C}} [\phi(a - z) - \phi(a) + D\phi(a) \cdot z] \, \mu_{\bar{y}}(dz) \\
& + \int_{\mathcal{C}} [\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z] \, \mu_{\bar{x}}(dz)
\end{aligned}
$$

We now use a second-order Taylor expansion in each integral. First, according to the form of Γ, we have

$$\int_{\mathcal{C}} [\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z] \, \mu_{\bar{x}}(dz) = L_2 \int_{\mathcal{C}} |z|^2 \, \mu_{\bar{x}}(dz) = O(L_2).$$

Next, for the other two terms and for a general $\phi(x) = \varphi(|x|)$, we recall (5.5) and we finally get

$$
\begin{aligned}
T_2 \leq{} & \frac{1}{2} \int_{\mathcal{C}} \sup_{t \in (-1,1)} \left(\frac{\varphi'(|a + tz|)}{|a + tz|} P_{(a+tz)^\perp} z \cdot z + \varphi''(|a+tz|)(\widehat{a + tz} \cdot z)^2 \right) (\mu_{\bar{x}} + \mu_{\bar{y}})(dz) \\
& + O(L_2).
\end{aligned}
$$

and using next the equality $\varphi(r) = L_1 r^\alpha$, we obtain

$$
\begin{aligned}
T_2 \leq{} & \frac{L_1 \alpha}{2} \int_{\mathcal{C}} \sup_{t \in (-1,1)} |a + tz|^{\alpha-4} \big(|a + tz|^2 |z|^2 - (2 - \alpha)((a + tz) \cdot z)^2\big)(\mu_{\bar{x}} + \mu_{\bar{y}})(dz) \\
& + O(L_2).
\end{aligned}
$$

We use the notation b for $a + tz$ and we estimate $|b|$ and $b \cdot z$ for $z \in \mathcal{C}$ as follows:

$$
\begin{aligned}
|b| &\leq \varepsilon + t|z| \leq \varepsilon + \delta \leq (1 + \nu_0)\varepsilon, \\
|b \cdot z| &= |a \cdot z + t|z|^2| \geq (1 - \eta)\varepsilon|z| - \delta|z| \geq (1 - \eta - \nu_0)\varepsilon|z|, \\
|b|^2 |z|^2 - (2 - \alpha)(b \cdot z)^2 &\leq (1 + \nu_0)^2 \varepsilon^2 |z|^2 - (2 - \alpha)(1 - \eta - \nu_0)^2 \varepsilon^2 |z|^2 \\
&\leq ((1 + \nu_0)^2 - (2 - \alpha)(1 - \eta - \nu_0)^2)\varepsilon^2 |z|^2,
\end{aligned}
$$

where we choose η, ν_0 small enough so that $1 - \eta - \nu_0 > 0$ and

$$(2 - \alpha)(1 - \eta - \nu_0)^2 - (1 + \nu_0)^2 > 0. \tag{5.12}$$

Hence

$$T_2 \leq -L_1 C_4 \varepsilon^{\alpha - 2} \int_{\mathcal{C}} |z|^2 \, (\mu_{\bar{x}} + \mu_{\bar{y}})(dz) + O(L_2)$$

with

$$C_4 = \frac{\alpha}{2}((2 - \alpha)(1 - \eta - \nu_0)^2 - (1 + \nu_0)^2)(1 + \nu_0)^{\alpha - 4}. \tag{5.13}$$

Using (3.2), we thus obtain

$$T_2 \leq -L_1 C_4 C_\mu(\eta) \varepsilon^{\alpha - 2} \delta^{2 - \beta} + O(L_2), \tag{5.14}$$

and finally,

$$T_2 \leq -L_1 C_5 \varepsilon^{\alpha - \beta} + O(L_2) \tag{5.15}$$

with

$$C_5 = C_4 C_\mu(\eta) \nu_0^{2 - \beta}. \tag{5.16}$$

Estimate of T_3. In order to estimate T_3 from above, it is convenient to introduce

$$\mathcal{U}_1(z) = u(\bar{x} + z) - u(\bar{x}) - (p + q) \cdot z, \quad \mathcal{U}_2(z) = u(\bar{y} + z) - u(\bar{y}) - p \cdot z,$$

and write

$$T_3 = \int_{B \setminus \mathcal{C}} \mathcal{U}_1(z) \, \mu_{\bar{x}}(dz) - \int_{B \setminus \mathcal{C}} \mathcal{U}_2(z) \, \mu_{\bar{y}}(dz).$$

We first remark that (5.11) with, successively, $(d, d') = (z, z)$, $(d, d') = (z, 0)$ and $(d, d') = (0, z) = 0$ yields

$$\begin{cases} \mathcal{U}_1(z) - \mathcal{U}_2(z) \leq \Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z, \\ \mathcal{U}_1(z) \leq (\phi(a + z) - \phi(a) - D\phi(a) \cdot z) + (\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z), \\ \mathcal{U}_2(z) \geq -(\phi(a - z) - \phi(a) + D\phi(a) \cdot z). \end{cases} \tag{5.17}$$

We next consider the signed measure $\mu(dz) = \mu_{\bar{x}}(dz) - \mu_{\bar{y}}(dz)$. It can be represented by two non-negative measures $\mu^{\pm}$ using the Hahn–Jordan decomposition: we write $\mu = \mu^+ - \mu^-$ where μ^+, μ^- are respectively the positive and negative parts of μ. We would like next to introduce a measure $\min(\mu_{\bar{x}}, \mu_{\bar{y}})$. To make it precise, we use the Hahn decomposition of $\mathbb{R}^N$ with respect to μ: if K denotes the support of μ^+, we define $\bar{\mu} = \mathbf{1}_K \mu_{\bar{y}} + (1 - \mathbf{1}_K)\mu_{\bar{x}}$. We now rewrite T_3 with these measures. We use

$$\mu_{\bar{x}} = \mu^+ + \bar{\mu} \quad \text{and} \quad \mu_{\bar{y}} = \mu^- + \bar{\mu}$$

together with (5.17) to get

$$
\begin{aligned}
T_3 &= \int_{B \setminus \mathcal{C}} (\mathcal{U}_1(z) - \mathcal{U}_2(z))\, \overline{\mu}(dz) + \int_{B \setminus \mathcal{C}} \mathcal{U}_1(z)\, \mu^+(dz) - \int_{B \setminus \mathcal{C}} \mathcal{U}_2(z)\, \mu^-(dz) \\
&\leq \int_{B \setminus \mathcal{C}} [\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z]\, \overline{\mu}(dz) \\
&\quad + \int_{B \setminus \mathcal{C}} [\phi(a + z) - \phi(a) - D\phi(a) \cdot z]\, \mu^+(dz) \\
&\quad + \int_{B \setminus \mathcal{C}} [\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z]\, \mu^+(dz) \\
&\quad + \int_{B \setminus \mathcal{C}} [\phi(a - z) - \phi(a) + D\phi(a) \cdot z]\, \mu^-(dz).
\end{aligned}
$$

In order to estimate the right hand side, we first remark that

$$
\int_{B \setminus \mathcal{C}} [\Gamma(\bar{x} + z) - \Gamma(\bar{x}) - D\Gamma(\bar{x}) \cdot z]\, (\overline{\mu} + \mu^+)(dz) \leq L_2 \int_B |z|^2\, \mu_{\bar{x}}(dz) \leq \tilde{C}_\mu L_2.
$$

Next, for the other two terms, we split the integration domain into $B \setminus B_\delta$ and $B_\delta \setminus \mathcal{C}$. On $B_\delta \setminus \mathcal{C}$, we use once again a second-order Taylor expansion for ϕ while, on $B \setminus B_\delta$, we use the concavity of the function $t \mapsto L_1 t^\alpha$ on $(0, +\infty)$ to obtain

$$
\begin{aligned}
\phi(a + z') - \phi(a) - D\phi(a) \cdot z' &\leq L_1(|a| + |z'|)^\alpha - L_1 |a|^\alpha - D\phi(a) \cdot z' \\
&\leq \alpha L_1 |a|^{\alpha-1} |z'| + |D\phi(a) \cdot z'| \leq 2\alpha L_1 |a|^{\alpha-1} |z'|. \quad (5.18)
\end{aligned}
$$

Using (5.18) for $z' = z$ and $-z$ and (3.2), we are led to

$$
\begin{aligned}
T_3 &\leq \tilde{C}_\mu L_2 + L_1 \int_{B_\delta \setminus \mathcal{C}} (\varepsilon - \delta)^{\alpha-2} |z|^2\, |\mu|(dz) + 2\alpha L_1 \varepsilon^{\alpha-1} \int_{B \setminus B_\delta} |z|\, |\mu|(dz) \\
&\leq \tilde{C}_\mu L_2 + (1 - v_0)^{\alpha-2} L_1 \varepsilon^{\alpha-2} \int_{B_\delta \setminus \mathcal{C}} |z|^2\, |\mu_{\bar{x}} - \mu_{\bar{y}}|(dz) \\
&\quad + 2\alpha L_1 \varepsilon^{\alpha-1} \int_{B \setminus B_\delta} |z|\, |\mu_{\bar{x}} - \mu_{\bar{y}}|(dz),
\end{aligned}
$$

where $|\mu| = \mu^+ + \mu^- = |\mu_{\bar{x}} - \mu_{\bar{y}}|$.

We now use (3.3) and (3.4). It is convenient to introduce

$$
\psi_\beta(r) = \begin{cases} r^{1-\beta} & \text{if } \beta \neq 1, \\ |\ln r| & \text{if } \beta = 1. \end{cases}
$$

We finally get

$$
T_3 \leq \tilde{C}_\mu L_2 + v_0^{\alpha-\beta} L_1 \omega_\mu(\varepsilon) \varepsilon^{\alpha-\beta} + 2\alpha L_1 \omega_\mu(\varepsilon) \varepsilon^{\alpha-1} \psi_\beta(v_0 \varepsilon) \quad (5.19)
$$

for v_0 small enough. We use here the fact that $v_0^{\alpha-\beta}$ controls $(1 - v_0)^{\alpha-2} v_0^{2-\beta}$ from above.

Final estimate for T_{nl}. Combining (5.9), (5.15), (5.19), the final estimates are: for $\beta \neq 1$,

$$T_{nl} \leq -L_1 C_5 \varepsilon^{\alpha-\beta} + o(\varepsilon^{\alpha-\beta}) + O(L_2), \tag{5.20}$$

and for $\beta = 1$,

$$T_{nl} \leq -L_1 C_5 \varepsilon^{\alpha-1} + \nu_0^{\alpha-1} L_1 \omega_\mu(\varepsilon) \varepsilon^{\alpha-1} + 2\alpha L_1 \omega_\mu(\varepsilon) \varepsilon^{\alpha-1} |\ln(\nu_0 \varepsilon)| + O(L_2).$$

We see that if $\omega_\mu(r)$ satisfies $\omega_\mu(r)|\ln r| \to 0$ as $r \to 0$ (which is the case when $\beta = 1$), (5.20) still holds true in this case.

Step 4: Conclusion. For all $\alpha < 1$, we deduce from (5.20) that $T_{nl} \leq 0$ if L_1 is large enough. Using inequality (5.4) together with (H) with $L = R = O(L_2)$ and estimates (5.8) & (5.20), and recalling that $\bar{\varepsilon} := (L_1 \alpha \varepsilon^{\alpha-2})^{-1}$, we are thus led to

$$0 \leq \Lambda_1(\bar{x}) A_1 + \Lambda_2(\bar{x}) A_2 \tag{5.21}$$

with

$$A_1 = L_1 \alpha \varepsilon^{\alpha-2}[-8\varpi + \omega_F(\varepsilon)] + \varepsilon^\tau (\alpha L_1 \varepsilon^{\alpha-1})^{2+\tau} + (\alpha L_1 \varepsilon^{\alpha-1})^2 + \tilde{\chi}(L_2),$$

$$A_2 = [-L_1 C_5 \varepsilon^{\alpha-\beta} + o(\varepsilon^{\alpha-\beta})] + \frac{\varepsilon^{2\theta}}{\bar{\varepsilon}} + \varepsilon^\tau (\alpha L_1 \varepsilon^{\alpha-1})^{k+\tau} + (\alpha L_1 \varepsilon^{\alpha-1})^k + \tilde{\chi}(L_2),$$

where we have gathered in $\tilde{\chi}(L_2)$ the terms that either depend on L_2 or are bounded. We use the assumption $\Lambda_1 + \Lambda_2 \geq \Lambda_0 > 0$ by rewriting (5.21) as follows:

$$0 \leq \Lambda_0 \max(A_1, A_2).$$

To get the desired contradiction and to obtain the $C^{0,\alpha}$-estimate, it is enough to prove that $A_i < 0$ for $i = 1, 2$ and L_1 large enough, and to control the size of such L_1.

As far as A_1 is concerned and as soon as $\alpha < 1$, we can ensure that $-8\varpi + \omega_F(\varepsilon) \leq -4\varpi$ if L_1 is large enough; using (5.1), it is clear that, in order to do it, the size of L_1 depends only on $\|u\|_\infty$, α and $d(x_0, \partial\Omega)$. This yields an estimate of the type

$$A_1 \leq L_1 \varepsilon^{\alpha-2}[-4\varpi\alpha + \alpha^{2+\tau}(L_1 \varepsilon^\alpha)^{1+\tau} + \alpha^2 L_1 \varepsilon^\alpha] + \tilde{\chi}(L_2).$$

Now there are two ways to estimate $L_1 \varepsilon^\alpha$: either to use the first part of inequality (5.2) which yields the estimate $L_1 \varepsilon^\alpha \leq 2\|u\|_\infty$, or to use the second part and the estimate of $L_1 \varepsilon^\alpha$ through the modulus of continuity of u. The strategy of the proof consists in proving the result for α small enough by using the first estimate of $L_1 \varepsilon^\alpha$ and then to use this first step (which provides a modulus of continuity of u) to prove it for all α by using the second estimate of $L_1 \varepsilon^\alpha$.

Using $L_1 \varepsilon^\alpha \leq 2\|u\|_\infty$ in the above inequality yields

$$A_1 \leq L_1 \varepsilon^{\alpha-2}[-4\varpi\alpha + \alpha^{2+\tau}(2\|u\|_\infty)^{1+\tau} + 2\alpha^2\|u\|_\infty] + \tilde{\chi}(L_2).$$

For α small enough (depending only on $\|u\|_\infty$ and τ), the bracket is less than $-2\varpi\alpha < 0$ and, recalling (5.1), it is clear that the right-hand side is (strictly) negative if L_1 is large enough (depending on α, ϖ, $\tilde{\chi}(L_2)$). Hence, we get the desired inequality: $A_1 < 0$.

For the A_2-term, we first write

$$
\begin{aligned}
A_2 = & -L_1 C_5 \varepsilon^{\alpha-\beta} + o(\varepsilon^{\alpha-\beta}) + \varepsilon^{2\theta} L_1 \varepsilon^{\alpha-2} + \varepsilon^{\tau}(\alpha L_1 \varepsilon^{\alpha-1})^{k+\tau} + (L_1 \alpha \varepsilon^{\alpha-1})^k \\
& + \tilde{\chi}(L_2) \\
= & L_1 \varepsilon^{\alpha-\beta}[-C_5 + o_\varepsilon(1) + \varepsilon^{2\theta-2+\beta} + \alpha^{k+\tau} \varepsilon^{\beta-k}(L_1 \varepsilon^\alpha)^{k+\tau-1} \\
& + \alpha^k \varepsilon^{\beta-k}(L_1 \varepsilon^\alpha)^{k-1}] + \tilde{\chi}(L_2) \\
= & L_1 \varepsilon^{\alpha-\beta}[-C_5 + o_\varepsilon(1) + \alpha^{k+\tau} \varepsilon^{\beta-k}(L_1 \varepsilon^\alpha)^{k+\tau-1} + \alpha^k \varepsilon^{\beta-k}(L_1 \varepsilon^\alpha)^{k-1}] + \tilde{\chi}(L_2)
\end{aligned}
$$

where C_5 is given by (5.16); we also used $2\theta + \beta - 2 > 0$.

The key difference from A_1 is the fact that the exponents of the term $L_1 \varepsilon^\alpha$ can be non-positive and we have to argue differently if it is indeed the case.

We have the following cases.

- If $\beta > 1$, since $C_5 \geq \alpha C_5'$ for some constant C_5' independent of α (at least for $\alpha \leq 1/2$), then one can argue as for A_1 with $k = \beta$ since $k + \tau > 1$ and $k = \beta > 1$ and obtain the $C^{0,\alpha}$ regularity and estimates for α small enough.
- If $\beta \leq 1$, then we cannot use this argument anymore since k must satisfy (at least) $k \leq 1$. In order to conclude, it is enough to ensure

$$
\varepsilon^{\beta-k}(L_1 \varepsilon^\alpha)^{k+\tau-1} = o_\varepsilon(1) \quad \text{and} \quad \varepsilon^{\beta-k}(L_1 \varepsilon^\alpha)^{k-1} = o_\varepsilon(1). \tag{5.22}
$$

Writing $\varepsilon^{\beta-k}(L_1 \varepsilon^\alpha)^{k-1} = L_1^{k-1} \varepsilon^{\beta-k+\alpha(k-1)}$, we see that this term is $o_\varepsilon(1)$ if $k < 1$ and $\beta - k + \alpha(k-1) \geq 0$; notice that we do not know how to compare, in full generality, L_1-terms and ε-terms. The second condition implies that $k < \beta$. In the same way, for the other term, either $k + \tau - 1 \geq 0$ and the condition $\beta > k$ is sufficient, or $k + \tau - 1 < 0$ and we are led to $\alpha \leq (\beta - k)/(1 - k - \tau)$. Gathering all this information yields the conditions

$$
1 \geq \beta > k \quad \text{and} \quad \alpha \leq \frac{\beta - k}{1 - k}.
$$

At this point, putting together the information on A_1 and A_2, we have shown that u is locally in $C^{0,\bar{\alpha}}$ for $\bar{\alpha}$ small enough (depending only on the data and the L^∞-norm of u) and we have an estimate of the local $C^{0,\bar{\alpha}}$-norm of u. In order to conclude the proof, we need to go back to the estimate on A_1 and A_2 (in the case when $\beta > 1$) and to estimate the terms $L_1 \varepsilon^\alpha$ using (5.2) and the local $C^{0,\bar{\alpha}}$-modulus of continuity. This easily yields the full result and the proof of (i) is complete.

In order to prove the second part of the theorem, we now choose $\alpha = \beta$ and we only need to adapt the final step (Step 4) in the previous proof. We proceed as before by writing (5.21) with A_1 unchanged and, since $C_F = 0$, with A_2 given by

$$
A_2 = L_1[-C_5 + o(1)] + L_1 \varepsilon^{\beta-2+2\theta} + \varepsilon^{\tau}(\beta L_1 \varepsilon^{\beta-1})^{k+\tau} + \tilde{\chi}(L_2),
$$

which we can rewrite as

$$
A_2 = L_1[-C_5 + o(1)] + L_1 \varepsilon^{\beta-2+2\theta} + \beta L_1 \varepsilon^{\tau+(\beta-1)(k+\tau)} + \tilde{\chi}(L_2).
$$

At this stage of the proof, L_2 is fixed and L_1 can be chosen large enough in order to control the term $\tilde{\chi}(L_2)$. Next we notice that $\beta - 2 + 2\theta > 0$. In order to conclude, it is enough to have $\tau + (\beta - 1)(k + \tau) > 0$, i.e. $\tau > k(\beta^{-1} - 1)$.

The proof of Theorem 3.1 is now complete. □

5.2. Proof of Theorem 3.2

This proof follows the lines of the previous one, the only difference is the way of getting estimate (5.20) for the non-local term in the new framework and under the new assumptions. Let us explain this point.

In order to shed light on the fact that j has to be Hölder continuous with respect to x, we let $\omega(r)$ denote $C_0 r^{\tilde{\theta}}$ and we will see at the end of the present proof that ω has to be chosen as a power law. More precisely, in this case,

$$T_{nl} = \int_{|z| \geq \delta'} [u(\bar{x} + j(\bar{x}, z)) - u(\bar{x}) - (p + q) \cdot j(\bar{x}, z)\mathbf{1}_B(z)]\,\mu(dz)$$

$$- \int_{|z| \geq \delta'} [u(\bar{y} + j(\bar{y}, z)) - u(\bar{y}) - p \cdot j(\bar{y}, z)\mathbf{1}_B(z)]\,\mu(dz).$$

We then write $T_{nl} = T_1 + T_2 + T_3$ with

$$T_1 = \int_{|z| \geq 1} [u(\bar{x} + j(\bar{x}, z)) - u(\bar{x})]\mu(dz) - \int_{|z| \geq 1} [u(\bar{y} + j(\bar{y}, z)) - u(\bar{y})]\,\mu(dz),$$

$$T_2 = \int_{z \in \mathcal{C}} [u(\bar{x} + j(\bar{x}, z)) - u(\bar{x}) - (p + q) \cdot j(\bar{x}, z)]\,\mu(dz)$$

$$- \int_{z \in \mathcal{C}} [u(\bar{y} + j(\bar{y}, z)) - u(\bar{y}) - p \cdot j(\bar{y}, z)]\,\mu(dz),$$

$$T_3 = \int_{z \in B,\, z \notin \mathcal{C}} [\ldots]\,\mu_{\bar{x}}(dz) - \int_{z \in B,\, z \notin \mathcal{C}} [\ldots]\,\mu_{\bar{y}}(dz),$$

where $\mathcal{C}$ is defined in the following way:

$$\mathcal{C} := \left\{ z : \left| j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \right| \leq \frac{\delta}{2} \text{ and } \left| j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \cdot \hat{a} \right| \geq \left(1 - \frac{\eta}{2}\right) \left| j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \right| \right\}$$

$$= \mathcal{C}_{\delta/2, \eta/2}\left(\frac{\bar{x} + \bar{y}}{2}\right),$$

where the notation $\mathcal{C}_{\delta, \eta}(x)$ is defined in the statement of Theorem 3.2. Roughly speaking, $\mathcal{C}$ is the analogue of the cone used in the proof of Theorem 3.1 where we have replaced z by $j((\bar{x} + \bar{y})/2, z)$. Notice that, because of (3.5), $\mathcal{C} \subset B$ if δ is small enough.

We have chosen such a set $\mathcal{C}$ for the following reason: if L_1 is large enough (or equivalently ε or δ is small enough), then

$$\mathcal{C} \subset \mathcal{C}_{\delta, \eta}(\bar{x}) \cap \mathcal{C}_{\delta, \eta}(\bar{y}), \tag{5.23}$$

which means that both $j(\bar{x}, z)$ and $j(\bar{y}, z)$ are in the "good" cones.

To check these properties, we write

$$
\begin{aligned}
|j(\bar{x}, z) \cdot \hat{a}| &\geq \left| j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \cdot \hat{a} \right| - \left| j(\bar{x}, z) - j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \right| \\
&\geq (1 - \eta/2) \left| j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \right| - |z|\omega\left(\left|\bar{x} - \frac{\bar{x} + \bar{y}}{2}\right|\right) \\
&\geq (1 - \eta/2)|j(\bar{x}, z)| - (2 - \eta/2)|z|\omega(\varepsilon/2) \\
&\geq (1 - \eta/2 - (2 - \eta)C_0^{-1}\omega(\varepsilon/2))|j(\bar{x}, z)| \geq (1 - \eta)|j(\bar{x}, z)|
\end{aligned}
$$

and

$$
|j(\bar{x}, z)| \leq \left| j\left(\frac{\bar{x} + \bar{y}}{2}, z\right) \right| + \omega(\varepsilon/2)|z| \leq \frac{\delta}{2} + \frac{\delta}{2c_0}\omega(\varepsilon/2) \leq \delta
$$

for L_1 such that

$$
\omega(\varepsilon/2) \leq \min\left(\frac{\eta c_0}{4 - \eta}, c_0\right) = \min(C_0, c_0) = c_0
$$

for $\eta < 1$.

Estimate of T_1. We remark that, thanks to the properties of j and μ, (5.9) still holds true.

Estimate of T_2. One can check that (5.14) and (5.15) still hold true. Indeed, we use (3.6)–(3.7) in order to get (3.2). More precisely, using (3.6)–(3.7) and recalling the computation we made in Subsection 4.3, we obtain

$$
\forall \hat{a}, \quad \int_{\mathcal{C}} |z|^2 \, \mu(dz) \geq C_\mu \delta^{2 - \beta}
$$

where $C_\mu = C(c_0, C_0, C_\mu^-, \eta, d, \beta)$. Notice that (5.12) is slightly modified and so is C_4 (and consequently C_5).

Estimate of T_3. In the case of Lévy–Itô operators, we estimate T_3 as follows. By using (5.11) with $d = j(\bar{x}, z)$ and $d' = j(\bar{y}, z)$, we get

$$
\begin{aligned}
T_3 = \int_{z \in B, \, z \notin \mathcal{C}} &[u(\bar{x} + j(\bar{x}, z)) - u(\bar{x}) - (p + q) \cdot j(\bar{x}, z) \\
&- u(\bar{y} + j(\bar{y}, z)) + u(\bar{y}) + p \cdot j(\bar{y}, z)] \, \mu(dz) \leq T_3^1 + T_3^2
\end{aligned}
$$

with

$$
\begin{aligned}
T_3^1 &= \int_{z \in B, \, z \notin \mathcal{C}} [\Gamma(\bar{x} + j(\bar{x}, z)) - \Gamma(\bar{x}) - q \cdot j(\bar{x}, z)] \, \mu(dz), \\
T_3^2 &= \int_{z \in B, \, z \notin \mathcal{C}} [\phi(a + \Delta(z)) - \phi(a) - p \cdot \Delta(z)] \, \mu(dz),
\end{aligned}
$$

where $\Delta(z) = j(\bar{x}, z) - j(\bar{y}, z)$. Let us first estimate T_3^1 as follows:

$$
T_3^1 \leq \frac{1}{2} \int_B \sup_{t \in (0,1)} [D^2\Gamma(\bar{x} + tj(\bar{x}, z))j(\bar{x}, z) \cdot j(\bar{x}, z)] \, \mu(dz)
$$

and we deduce that

$$T_3^1 \leq C_2 L_2 \qquad (5.24)$$

where

$$C_2 = C_0^2 \int_{|z| \leq 1} |z|^2 \, \mu(dz). \qquad (5.25)$$

We now turn to T_3^2 and we write $T_3^2 = T_3^{2,1} + T_3^{2,2}$ with

$$T_3^{2,1} = \int_{z \in B, \, z \notin \mathcal{C}, \, |\Delta(z)| \geq \delta} [\phi(a + \Delta(z)) - \phi(a) - D\phi(a) \cdot \Delta(z)] \, \mu(dz),$$

$$T_3^{2,2} = \int_{z \in B, \, z \notin \mathcal{C}, \, |\Delta(z)| \leq \delta} [\phi(a + \Delta(z)) - \phi(a) - D\phi(a) \cdot \Delta(z)] \, \mu(dz).$$

We can now estimate $T_3^{2,1}$. In order to clarify computations, we write ω for $\omega(\varepsilon)$ in the following.

We use (5.18); remarking that $\delta \leq |\Delta(z)| \leq \omega|z|$, we deduce

$$T_3^{2,1} \leq \int_{\delta\omega^{-1} \leq |z| \leq 1} 2\alpha L_1 \varepsilon^{\alpha-1} |\Delta(z)| \, \mu(dz) \leq \int_{\delta\omega^{-1} \leq |z| \leq 1} 2\alpha L_1 \varepsilon^{\alpha-1} \omega|z| \, \mu(dz)$$

$$\leq 2\alpha L_1 \varepsilon^{\alpha-1} \omega \psi_\beta(\delta\omega^{-1})$$

where, by (3.7),

$$\psi_\beta(r) = \int_{r \leq |z| \leq 1} |z| \, \mu(dz) \leq \begin{cases} \tilde{C}_\mu r^{1-\beta} & \text{if } \beta \neq 1, \\ \tilde{C}_\mu |\ln r| & \text{if } \beta = 1. \end{cases} \qquad (5.26)$$

In order to estimate $T_3^{2,2}$, we use a Taylor expansion together with (5.7) and the fact that $|a + t\Delta(z)| \geq \varepsilon - \delta > 0$. It follows that

$$T_3^{2,2} \leq \frac{L_1\alpha}{2} \int_{|z| \leq 1} \sup_{t \in (0,1)} |a + t\Delta(z)|^{\alpha-2} |\Delta(z)|^2 \, \mu(dz) \leq \frac{L_1}{2} \int_B |z|^2 \mu(dz) \, (\varepsilon - \delta)^{\alpha-2} \omega^2$$

$$\leq \frac{L_1}{2} \omega^2 \varepsilon^{\alpha-2} (1 - \nu_0)^{\alpha-2} \int_B |z|^2 \, \mu(dz).$$

Gathering the estimates on $T_3^{2,1}$ and $T_3^{2,2}$, we obtain

$$T_3^2 \leq 2\alpha L_1 \varepsilon^{\alpha-1} \omega \psi_\beta(\delta\omega^{-1}) + L_1 C_3 \omega^2 \varepsilon^{\alpha-2} \qquad (5.27)$$

with

$$C_3 = \frac{1}{2} \int_B |z|^2 \, \mu(dz) \, (1 - \nu_0)^{\alpha-2} \qquad (5.28)$$

Final estimate of T_{nl}. Gathering (5.9), (5.15), (5.24) and (5.27), we finally obtain

$$T_{nl} \leq 2\alpha L_1 \omega \psi_\beta(\nu_0 \omega^{-1}\varepsilon)\varepsilon^{\alpha-1} + L_1 C_3 \omega^2 \varepsilon^{\alpha-2} - L_1 C_5 \varepsilon^{\alpha-\beta} + O(L_2). \qquad (5.29)$$

Looking at the $\omega^2\varepsilon^{\alpha-2}$-term, we see that we need to assume that j is Hölder continuous with respect to x and we thus replace $\omega(\varepsilon)$ with $C\varepsilon^{\tilde{\theta}}$. Denoting $C_3' = CC_3$, we get

$$T_{nl} \leq 2\alpha L_1\varepsilon^{\alpha+\tilde{\theta}-1}\psi_\beta(\nu_0\varepsilon^{1-\tilde{\theta}}) + L_1 C_3'\varepsilon^{\alpha-2+2\tilde{\theta}} - L_1 C_5\varepsilon^{\alpha-\beta} + O(L_2).$$

We claim that as in the case of Theorem 3.1, (5.20) holds true under the assumptions of Theorem 3.2. To see this, we write

$$2\alpha L_1\varepsilon^{\alpha+\tilde{\theta}-1}\psi_\beta(\nu_0\varepsilon^{1-\tilde{\theta}}) + L_1\varepsilon^{\alpha-2+2\tilde{\theta}} = L_1\varepsilon^{\alpha-\beta}[\varepsilon^{\beta+\tilde{\theta}-1}\psi_\beta(\nu_0\varepsilon^{1-\tilde{\theta}}) + C_3'\varepsilon^{\beta-2+2\tilde{\theta}}]$$
$$= L_1\varepsilon^{\alpha-\beta}[\varepsilon^{\beta+\tilde{\theta}-1}\psi_\beta(\nu_0\varepsilon^{1-\tilde{\theta}}) + o_\varepsilon(1)]$$

since $\tilde{\theta} > \frac{1}{2}(2-\beta)$ and $\alpha < \beta$. We next distinguish two cases.

- If $\beta \neq 1$, then by (5.26), we get

$$\varepsilon^{\beta+\tilde{\theta}-1}\psi_1(\nu_0\varepsilon^{1-\tilde{\theta}}) = O(\varepsilon^{\tilde{\theta}\beta})$$

  and we conclude in this case.
- If $\beta = 1$,
$$\varepsilon^{\beta+\tilde{\theta}-1}\psi_1(\nu_0\varepsilon^{1-\tilde{\theta}}) = O(\varepsilon^{\tilde{\theta}}|\ln(\varepsilon^{1-\tilde{\theta}})|) = o_\varepsilon(1)$$

  and we can conclude in this case too.

The proof of Theorem 3.2 is now complete. □

Appendix

In this Appendix we provide the explicit computation of $Z^{\bar{\varepsilon}/2}$ used in the proof of Theorem 3.1.

We have $Z = (1/\bar{\varepsilon})(I - (2-\alpha)\hat{a} \otimes \hat{a})$ and

$$Z^{\bar{\varepsilon}/2}q \cdot q = \sup_{p\in\mathbb{R}^N}\left\{Zp \cdot p - \frac{2}{\bar{\varepsilon}}|p-q|^2\right\} = \frac{2}{\bar{\varepsilon}}\sup_{p\in\mathbb{R}^N}\left\{\frac{\bar{\varepsilon}}{2}Zp \cdot p - |p-q|^2\right\}.$$

One checks easily that the (last) sup is achieved at a point p such that $\bar{\varepsilon}Zp = 2(p-q)$, i.e.

$$p - (2-\alpha)(\hat{a} \cdot p)\hat{a} = 2(p-q). \tag{5.30}$$

Taking a scalar product with $\hat{a}$, we deduce

$$\hat{a} \cdot p = \frac{2}{3-\alpha}\hat{a} \cdot q,$$

and inserting in the previous equality yields

$$p - q = q - \frac{2(2-\alpha)}{3-\alpha}(\hat{a} \cdot q)\hat{a}. \tag{5.31}$$

Going back to the value of the supremum and using (5.30), we have

$$Z^{\bar{\varepsilon}/2}q \cdot q = \frac{2}{\bar{\varepsilon}}\left\{(p-q)\cdot p - |p-q|^2\right\} = \frac{2}{\bar{\varepsilon}}(p-q)\cdot q,$$

and now using (5.31), we finally obtain

$$Z^{\bar{\varepsilon}/2}q \cdot q = \frac{2}{\bar{\varepsilon}}\left\{|q|^2 - \frac{2(2-\alpha)}{3-\alpha}(\hat{a}\cdot q)^2\right\}.$$

Therefore $1 + \varpi = 2(2-\alpha)/(3-\alpha)$ and $\varpi = (1-\alpha)/(3-\alpha)$.

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