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Homogenization of First-Order Equations with (u/ε)-Periodic Hamiltonians. Part I: Local Equations

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Abstract

In this paper, we present a result of homogenization of first-order Hamilton–Jacobi equations with (u/ε)-periodic Hamiltonians. On the one hand, under a coercivity assumption on the Hamiltonian (and some natural regularity assumptions), we prove an ergodicity property of this equation and the existence of nonperiodic approximate correctors. On the other hand, the proof of the convergence of the solution, usually based on the introduction of a perturbed test function in the spirit of Evans's work, uses here a twisted perturbed test function for a higher-dimensional problem.

1. Introduction

Setting of the problem. In this paper, we study the limit as $\varepsilon \to 0$ of the viscosity solution of the following first order Hamilton–Jacobi equation with $N \in \mathbf{N}$:

$$\begin{cases} u_t^\varepsilon = H\left(\frac{u^\varepsilon}{\varepsilon}, \frac{x}{\varepsilon}, \nabla u^\varepsilon\right) & \text{for } (t,x) \in (0,+\infty) \times \mathbf{R}^N, \\ u^\varepsilon(0,x) = u_0(x) & \text{for } x \in \mathbf{R}^N \end{cases} \tag{1}$$

with $u_0(\cdot) \in W^{1,\infty}(\mathbf{R}^N)$ (bounded and Lipschitz continuous functions on $\mathbf{R}^N$), and where u_t^ε stands for $\frac{\partial u^\varepsilon}{\partial t}$ and ∇u^ε or $\nabla_x u^\varepsilon$ for the gradient vector $\left(\frac{\partial u^\varepsilon}{\partial x_1}, \dots, \frac{\partial u^\varepsilon}{\partial x_N}\right)$. Our motivation comes from a problem of periodic homogenization for a model of dislocations dynamics [24], where the level sets of the solution describe dislocations. We consider the following assumptions on the Hamiltonian H:

– (A1). **Regularity:** the function $H : \mathbf{R}\times\mathbf{R}^N\times\mathbf{R}^N \to \mathbf{R}$ is Lipschitz continuous and satisfies for a constant $\gamma \geqq 0$ and for almost every $(v, y, p) \in \mathbf{R}\times\mathbf{R}^N\times\mathbf{R}^N$:

$$\left|\nabla_y H(v,y,p)\right| \leqq \gamma(1+|p|), \quad |\partial_v H(v,y,p)| \leqq \gamma, \quad \left|\nabla_p H(v,y,p)\right| \leqq \gamma;$$

– (A2). **Periodicity:** for any $(v, y, p) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N$:

$$H(v + l, y + k, p) = H(v, y, p) \quad \text{for any } l \in \mathbf{Z}, k \in \mathbf{Z}^N;$$

– (A3). **Coercivity:**

$$H(v, y, p) \to +\infty \quad \text{as } |p| \to +\infty \text{ uniformly for } (v, y) \in \mathbf{R} \times \mathbf{R}^N.$$

Under Assumptions (A1) and (A2), there exists a unique bounded continuous viscosity solution u^ε of (1); see Section 3 for references on viscosity solutions and below for a discussion about the regularity assumption (A1).

The aim of this paper is to prove a homogenization result, that is, we want to prove that the limit u^0 of u^ε as $\varepsilon \to 0$ exists and is the solution of a homogenized equation of the form:

$$\begin{cases} u^0_t = \overline{H}^0\left(\nabla u^0\right) & \text{for } (t, x) \in (0, +\infty) \times \mathbf{R}^N, \\ u^0(0, x) = u_0(x) & \text{for } x \in \mathbf{R}^N \end{cases} \tag{2}$$

where $\overline{H}^0$ is a continuous function to be determined. There exists a unique bounded continuous viscosity solution for such an equation (see [7] for instance). To get such a result, the coercivity assumption (A3) is natural, see for instance the seminal work of LIONS et al. [28], and Section 2.2. Typically, our model Hamiltonian is

$$H(v, y, p) = c(y)|p| + g(v) \quad \text{with } c \geqq 1$$
$$c, g \quad \text{Lipschitz continuous and periodic functions.}$$

The literature about homogenization of Hamilton–Jacobi equations has developed a lot since [28] and it is difficult to give an exhaustive list of references. See [1, 2, 5, 6, 9, 16, 20–22, 25, 28, 29, 34, 35] and references therein. However, to our best knowledge, there are very few papers concerning periodic homogenization with equations depending (periodically) on u/ε. Let us cite [32, 33] where PICCININI studied the homogenization of ordinary differential equations of the type $u^\varepsilon_t = f\left(\frac{u^\varepsilon}{\varepsilon}\right)$ and also more general ordinary differential equations (ODEs). In [13, 14], BOCCARDO and MURAT studied the homogenization of second-order equations which include $\operatorname{div}\left(A\left(\frac{u^\varepsilon}{\varepsilon}\right) \cdot \nabla u^\varepsilon\right) = f$.

Main results. Let us now describe the main results of this paper. They concern the ergodicity of the Hamiltonian and the convergence of the oscillating solution towards the homogenized one. We next give more details. In order to determine the limit of the solution u^ε of (1), the first task is to determine the limit equation (2). The so-called effective Hamiltonian $\overline{H}^0$ is uniquely defined by the long-time behavior of an evolution equation in $(0, +\infty) \times \mathbf{R}^N$. Let us be more specific. For any $p \in \mathbf{R}^N$, consider the continuous viscosity solution $w = w(\tau, y)$ of

$$\begin{cases} w_\tau = H(p \cdot y + w, y, p + \nabla w) & \text{for } (\tau, y) \in (0, +\infty) \times \mathbf{R}^N, \\ w(0, y) = 0 & \text{for } y \in \mathbf{R}^N. \end{cases} \tag{3}$$

Then we have the following *ergodicity* of the Hamiltonian.

Theorem 1. (Ergodicity) *Under Assumptions* (A1)–(A3)*, for any* $p \in \mathbf{R}^n$*, there exists a unique* $\lambda \in \mathbf{R}$ *such that the continuous viscosity solution* w *of* (3) *satisfies:* $\frac{w(\tau,y)}{\tau}$ *converges towards* λ *as* $\tau \to +\infty$*, locally uniformly in* y*. The real number* λ *is denoted by* $\overline{H}^0(p)$ *and this defines a continuous function on* $\mathbf{R}^n$*. Moreover we have the following coercivity of* $\overline{H}^0$*:*

$$\overline{H}^0(p) \to +\infty \quad as\ |p| \to +\infty.$$

We can now state the main result of this article.

Theorem 2. (Convergence result) *Under Assumptions* (A1)–(A3)*, the solution* u^ε *of* (1) *converges towards the solution* u^0 *of* (2) *locally uniformly in* (t,x)*, where* $\overline{H}^0$ *is defined in Theorem* 1.

In order to prove the convergence of u^ε towards u^0, we try to construct a so-called *corrector*, that is a solution of the *cell problem* which, in our case, has the following form:

$$\lambda + v_\tau = H(\lambda\tau + p\cdot y + v, y, p + \nabla_y v) \quad \text{for } (\tau, y) \in \mathbf{R}\times\mathbf{R}^N. \tag{4}$$

It turns out that we are only able to construct sub- and supercorrectors, that is, sub- and supersolution of the cell problem—see Theorem 3 in Appendix. However, it is enough to prove our convergence result and Theorem 1 appears as a corollary of this result. Let us also point out that, in contrast to the classical case, that is, Hamiltonians of the form $H(y,p)$, the (sub- and super-) correctors here are not periodic with respect to y in general and may be discontinuous. Moreover true correctors are necessarily time dependent in general.

A specific technical difficulty of our problem is to deal with the case $\lambda = p = 0$. For this reason, the "discontinuous" sub- and supercorrectors of Theorem 3 are not directly used in our proof of convergence. Instead, the proof of Theorem 2 is based on the existence of smoother correctors that are exact correctors for approximate Hamiltonians in higher dimension.

Our approach to construct these nonperiodic correctors in a periodic framework is somehow related to several different works. Let us cite the work of Müller [30], where the periodic homogenization of nonconvex functionals involves some nonperiodic correctors. See Caffarelli [15] for the construction of planar-like nonperiodic minimal surfaces in periodic media. See also Berestycki and Hamel [12], where pulsating fronts are studied in periodic media, these pulsating fronts being very similar to our corrector solutions. Finally, let us mention that the construction of correctors is related to the long-time behavior of solutions to Hamilton–Jacobi equations; such a problem has been studied by many authors, with seminal works by Fathi [22] and Roquejoffre [34]—see also Barles and Roquejoffre [8] and Barles and Souganidis [6] and references therein.

For general references on homogenization, see the book of Bensoussan et al. [10], and the pioneering work of Murat and Tartar [31].

Extensions. Let us first mention that we choose to present our results by assuming that the initial condition and the Hamiltonian are Lipschitz continuous. However, it is well known that the uniformly continuous framework is a natural extension of the Lipschitz framework and, on one hand, we could handle uniformly continuous initial condition and, on the other hand, (A1) can be adapted to deal with more-general Hamiltonians. Let us make a further comment on Assumption (A1). Usually, one only needs to bound from above $\partial_v H$; but in order to ensure a strong maximum principle when perturbing the equation by a nonlocal operator, we strengthen the usual assumption.

The reader can also think of more general Hamiltonians, depending for instance on slow variables. Let us consider two integers $0 \leqq m \leqq N$ and denote $x' = (x_1, \dots, x_m)$ and $x = (x_1, \dots, x_N)$. Let us consider a viscosity solution of

$$\begin{cases} u^\varepsilon_t = H\left(\frac{u^\varepsilon}{\varepsilon}, \frac{x'}{\varepsilon}, u^\varepsilon, t, \nabla u^\varepsilon\right) & \text{for } (t,x) \in (0,+\infty) \times \mathbf{R}^N, \\ u^\varepsilon(0,x) = u_0(x) & \text{for } x \in \mathbf{R}^N \end{cases} \tag{5}$$

where the Hamiltonian $H = H(v, y, u, t, p)$ satisfies classical assumptions ensuring comparison principles. Assuming moreover some coercive condition analogous to (A3), namely,

- (A3′). **Coercivity:** For $p' = (p_1, \dots, p_m)$, $p'' = (p_{m+1}, \dots, p_N)$ and $p = (p', p'')$ we have

$$H(v, y', u, t, p', p'') \to +\infty \quad \text{as } |p'| \to +\infty$$

  uniformly for $(v, y', u, t, p'') \in \mathbf{R} \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{N-m}$

and that $(v, y) \mapsto H(v, y, u, t, p)$ is periodic, we consider the limit problem:

$$\begin{cases} u^0_t = \overline{H}^0\left(u^0, t, \nabla u^0\right) & \text{for } (t,x) \in (0,+\infty) \times \mathbf{R}^N, \\ u^0(0,x) = u_0(x) & \text{for } x \in \mathbf{R}^N \end{cases} \tag{6}$$

where $\overline{H}^0(v, t, p)$ is a continuous function which is defined as the limit of $\frac{w(\tau,y)}{\tau}$ as $\tau \to +\infty$ for the solution of

$$\begin{cases} w_\tau = H(p \cdot y + w, y', v, t, p + \nabla_y w) & \text{for } (\tau, y) \in (0,+\infty) \times \mathbf{R}^N, \\ w(0,y) = 0 & \text{for } y \in \mathbf{R}^N. \end{cases}$$

The homogenized Hamiltonian $\overline{H}^0$ satisfies the following coercivity condition

$$\overline{H}^0(v, t, p', p'') \to +\infty \quad \text{as} \quad |p'| \to +\infty \text{ uniformly for } (v, t, p'').$$

Then a straightforward adaptation of our proofs permits to prove that, for an initial condition $u_0 \in W^{1,\infty}(\mathbf{R}^N)$, the solution u^ε of (5) converges towards the solution u^0 of (6) locally uniformly in (t, x). In a forthcoming work, we will study extensions of these results to some general ODE or nonlocal partial differential equation (PDE) problems.

After this work was completed, BARLES [4] provided much simpler proofs of our results in a special case. Moreover, the extensive use of these ideas permitted him to obtain homogenization results for noncoercive Hamiltonians. In particular, he introduced a different higher-dimensional problem (instead of the original problem in dimension N) by considering a geometric equation in dimension $N+1$.
Organization of the paper. The paper is organized as follows. In Section 2, we present the main difficulties encountered in the present problem of homogenization and the main new ideas we introduce to solve it. In Section 3, we state various comparison principles and gradient estimates for first-order Hamilton–Jacobi equations perturbed by a nonlocal operator. In Section 4, we prove the convergence result (Theorem 2) by using the existence of approximate sub- and supercorrectors (Proposition 7). In Section 5, we state a result on correctors for approximate Hamiltonians (Proposition 8) that encompasses Proposition 7 and the ergodicity theorem (Theorem 1). In Section 6, we prove the cornerstone of our construction, namely Proposition 8. Finally, in the Appendix, we state an independent result, Theorem 3, about the existence of bounded sub- and supercorrectors; we also give a second proof of Theorem 1.

Notations. $B_r(x)$ is the open ball of radius r centered at the point x. $\lceil x \rceil$ is the integer such that $\lceil x \rceil - x \in [0, 1)$ for any real x.

2. Strategies of proofs

The proofs of the main results are quite technical. To maintain easy understanding without technicalities, we propose in this section to indicate to the reader the main arguments used in the proofs, as simply as possible, but with formal arguments.

2.1. Main ideas for the Ansatz used in the proof of convergence

The goal of this subsection is to give some heuristic explanations of the difficulties arising in the homogenization of (1), and the main arguments that we have introduced in our proof of convergence.

First try: the usual Ansatz. Let us first start with a naive approach to the problem. The first Ansatz that we can try is the following [for (t, x) close to $(0, 0)$]:

$$u^\varepsilon(t,x) \simeq u^0(t,x) + \varepsilon\, v\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \tag{7}$$

where v is a corrector to determine. If we plug this expression of u^ε into (1), we find formally with $(\tau, y) = \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$:

$$u_t^0(t,x) + v_\tau(\tau, y) \simeq H\left(\frac{u^0(t,x)}{\varepsilon} + v(\tau,y), y, \nabla_x u^0(t,x) + \nabla_y v(\tau,y)\right).$$

Using the Taylor expansion $\frac{u^0(t,x)}{\varepsilon} \simeq \frac{u^0(0,0)}{\varepsilon} + u_t^0(0,0)\tau + \nabla_x u^0(0,0) \cdot y$, the notation $\lambda = u_t^0(0,0)$, $p = \nabla_x u^0(0,0)$, and decoupling the slow variables (t,x) from the fast variables (τ, y), we get:

$$\lambda + v_\tau \simeq H\left(\frac{u^0(0,0)}{\varepsilon} + \lambda\tau + p \cdot y + v, y, p + \nabla_y v\right).$$

From this easy computation, we can learn two things. The first is that a good candidate for the corrector v seems to be a solution of the equation where we put the term $\frac{u^0(0,0)}{\varepsilon}$ to zero, that is, Equation (4). The second is that Ansatz (7) is not the right ansatz, because the final result depends on the term $\frac{u^0(0,0)}{\varepsilon}$, which is not acceptable. This is why we will have to consider a twisted corrector.

Second try: the twisted corrector. We use the previous notations, and let p denote (p', p_N) with $p' = (p_1, \dots, p_{N-1})$. We also denote $x = (x', x_N)$. In the case where $p_N \neq 0$, we introduce a new ansatz with an x_N-twisted corrector that takes into account the distortion created by the term $\frac{u^\varepsilon}{\varepsilon}$ in the Hamiltonian:

$$\begin{aligned} u^\varepsilon(t,x',x_N) &\simeq u^0(t,x',x_N) + \varepsilon\, v\left(\frac{t}{\varepsilon}, \frac{x'}{\varepsilon}, \frac{u^0(t,x',x_N) - \lambda t - p' \cdot x'}{\varepsilon p_N}\right) \\ &=: A^\varepsilon(t,x',x_N) \end{aligned} \tag{8}$$

where a Taylor expansion immediately shows that $y_N = \frac{u^0(t,x',x_N) - \lambda t - p'\cdot x'}{\varepsilon p_N}$ is a good approximation of $\frac{x_N}{\varepsilon}$. If we denote $y' = \left(\frac{x_1}{\varepsilon}, \dots, \frac{x_{N-1}}{\varepsilon}\right)$ and $y = (y', y_N)$, we get:

$$\frac{A^\varepsilon(t,x',x_N)}{\varepsilon} = \lambda\tau + p' \cdot y' + p_N \cdot y_N + v.$$

Inserting this ansatz into Equation (1), we get with $v_N = \frac{\partial v}{\partial y_N}$ and $u_N = \frac{\partial u}{\partial x_N}$:

$$\begin{aligned} &\lambda + v_\tau(\tau,y) + v_N(\tau,y) \cdot \left(\tfrac{u_t^0(t,x) - \lambda}{p_N}\right) \\ &\simeq H\Bigg(\lambda\tau + p \cdot y + v, y, p' + \nabla_{y'} v + v_N(\tau,y) \cdot \left(\tfrac{\nabla_{x'} u^0(t,x) - p'}{p_N}\right), \\ &\qquad p_N + v_N(\tau,y) \cdot \left(\tfrac{u_N^0(t,x)}{p_N}\right)\Bigg). \end{aligned}$$

From this computation, we now learn two more things. First, for (t,x) close enough to $(0,0)$ and a smooth function u^0, we see that we can replace $\frac{u_t^0(t,x)-\lambda}{p_N}$ and $\frac{\nabla_{x'} u^0(t,x) - p'}{p_N}$ by 0, and $\frac{u_N^0(t,x)}{p_N}$ by 1, creating an error term that is bounded by $||v_N||_\infty$. However, the difficulty is that the corrector is only bounded, and may even be noncontinuous, which makes the term $||v_N||_\infty = \left|\left|\frac{\partial v}{\partial y_N}\right|\right|_\infty$ difficult to control. This difficulty will be overcome using a truncated Hamiltonian H_K whose corresponding correctors are Lipschitz continuous in all variables. Secondly, the previous computation only works for $p_N \neq 0$. In the case $p_N = 0$, we could still consider a x_i-twisted corrector if $p_i \neq 0$, or even a t-twisted corrector if $\lambda \neq 0$. However, in the case where $\lambda = p = 0$, we still have a difficulty. This case, and by the way the general case, will be solved by imbedding the problem in a higher-dimensional problem where by construction $p_{N+1} \neq 0$.

Third try (our definitive choice of the ansatz): a twisted corrector in higher dimension. We first consider a modification H_K of the Hamiltonian H such that H_K is bounded and converges locally uniformly on compact sets to H as $K \to +\infty$. Let us fix $p_{N+1} \in \mathbf{R}\setminus\{0\}$, and consider correctors $V_K(\tau, y, y_{N+1})$ of:

$$\lambda_K + \frac{\partial V_K}{\partial \tau} = H_K\Bigg(\lambda_K \tau + p\cdot y + p_{N+1}\cdot y_{N+1} + V_K, y, p + \nabla_y V_K, p_{N+1} + \frac{\partial V_K}{\partial y_{N+1}}\Bigg) \quad \text{in } \mathbf{R}\times\mathbf{R}^N\times\mathbf{R}. \tag{9}$$

From the boundedness of H_K, we can find a corrector V_K which is Lipschitz continuous in all variables. We now consider the solution $U^\varepsilon(t, x, x_{N+1})$ to the following equation:

$$\begin{cases} U^\varepsilon_t = H\left(\frac{U^\varepsilon}{\varepsilon}, \frac{x}{\varepsilon}, \nabla U^\varepsilon\right) & \text{in } (0, +\infty)\times\mathbf{R}^{N+1}, \\ U^\varepsilon(0, x, x_{N+1}) = u_0(x) + p_{N+1}\cdot x_{N+1} & \text{in } \mathbf{R}^{N+1}. \end{cases} \tag{10}$$

In the following, $U_0(x, x_{N+1}) := u_0(x) + p_{N+1}\cdot x_{N+1}$. In particular, we have $u^\varepsilon(t, x) = U^\varepsilon(t, x, 0)$. We now consider the following ansatz:

$$U^\varepsilon(t, X) \simeq U^0(t, X) + \varepsilon\, V_K\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \frac{U^0(t, X) - \lambda_K t - p\cdot x}{\varepsilon\, p_{N+1}}\right) \tag{11}$$

where $X = (x, x_{N+1})$ and $U^0(t, X) = u^0(t, x) + p_{N+1}\cdot x_{N+1}$. We can then easily check that Ansatz (11) is a good ansatz for which we can control the error terms in the equation, using in particular the fact that λ_K is close enough to λ for K large enough, and V_K is Lipschitz continuous with respect to y_{N+1}. Finally this construction works for any $p_{N+1} \neq 0$ and to simplify the presentation we take $p_{N+1} = 1$.

2.2. Main ideas for the construction of correctors

Boundedness of the corrector and the coercivity condition (A3). In the very first result of homogenization for Hamilton–Jacobi equations, namely in [28], a coercivity condition is used to ensure the existence of correctors. Let us comment this assumption. An example of a simple Hamiltonian without coercivity condition is $H(v, y, p) = \sin(2\pi y)$ in dimension $N = 1$. For this Hamiltonian, the solution of (1) with zero initial condition is

$$u^\varepsilon(t, x) = t\ \sin\left(\frac{2\pi x}{\varepsilon}\right).$$

Because the limit as $t \to +\infty$ of $\frac{u^\varepsilon(t,x)}{t} = \sin\left(\frac{2\pi x}{\varepsilon}\right)$ depends on x, this shows that this Hamiltonian is not ergodic in the sense of Theorem 1. There is no homogenization (with strong convergence) for this example, or more generally for some hyperbolic equations (see [36–38]). We also remark a fundamental property of this solution: the space oscillation of the solution u^ε is proportional to the time t, and is thus unbounded in time.

On the contrary, let us now consider a Hamiltonian $H(v, y, p)$ for which we can bound the space oscillation of the solution by a constant $C > 0$ for all time. In the very particular case where the solution is constant in space, the ergodicity of the equation is reduced to the study of the long-time behavior of the solution to an ODE. In the general case, the bound on the space oscillation of the solution allows us to compare easily the solution at two different times, which is enough to prove the ergodicity of the Hamiltonian.

We now explain how the coercivity condition (A3) can be used here to bound the corrector on the whole space for all time. Even if, in fact, we are not able to prove the existence of such a corrector, let us assume that we have a bounded corrector v which is a solution of (4), that is,

$$\lambda + v_\tau = H(\lambda\tau + p \cdot y + v, y, p + \nabla v) \quad \text{for } (\tau, y) \in \mathbf{R} \times \mathbf{R}^N$$

which is T-periodic in time where $T = 1/|\lambda|$ with $\lambda = \overline{H}^0(p) \neq 0$. Considering the maximum of v, we first remark that:

$$\lambda \leqq \sup_{(v,y)\in\mathbf{R}\times\mathbf{R}^N} H(v, y, p). \tag{12}$$

Let us now define

$$\overline{v}(y) = \inf_{\tau\in\mathbf{R}} v(\tau, y) = v(\overline{\tau}(y), y) \quad \text{for some } \overline{\tau}(y) \in [0, T).$$

Then $\overline{v}$ satisfies

$$\lambda \geqq H(\lambda\overline{\tau}(y) + p \cdot y + \overline{v}, y, p + \nabla\overline{v}).$$

Using (A3), this gives (at least formally) a bound on $p + \nabla\overline{v}$, that is,

$$|\nabla\underline{v}| \leqq C_1(|p|).$$

This implies that the space oscillation of $\overline{v}$ is bounded at short distance. The space oscillation of $\overline{v}$ at large distance is also bounded if we choose a corrector v satisfying moreover:

$$|v(\tau, y + k) - v(\tau, y)| \leqq 1 \quad \forall k \in \mathbf{Z}^N, \quad \forall(\tau, y) \in \mathbf{R} \times \mathbf{R}^N,$$

which follows from the periodicity of the Hamiltonian [see Assumption (A2)].

Under certain conditions, we can formally show that the corrector v can be chosen such that $w(\tau, y) := \lambda\tau + v(\tau, y)$ satisfies $\lambda w_\tau \geqq 0$. Together with the time periodicity of v, we deduce that $0 \leqq w(\tau+s, y)-w(\tau, y) \leqq 1$ for $0 \leqq s \leqq T$, in the case $\lambda > 0$. In general, we obtain that

$$|v(\tau + s, y) - v(\tau, y)| \leqq 1.$$

Up to the subtraction of an integer from v, this is enough to derive the following L^∞ bound on the corrector:

$$|v(\tau, y)| \leqq C_2(|p|).$$

Construction of the correctors. The basic idea behind the construction of correctors is to consider solutions w to (3), that is,

$$\begin{cases} w_\tau = H(p \cdot y + w, y, p + \nabla_y w) & \text{for } (\tau, y) \in (0, +\infty) \times \mathbf{R}^N, \\ w(0, y) = 0 & \text{for } y \in \mathbf{R}^N. \end{cases}$$

If we are able first to bound the space oscillation of w uniformly in time, we can show that $\frac{w(\tau,y)}{\tau} \to \lambda$ as $\tau \to +\infty$, uniformly in $y \in \mathbf{R}^N$, and then consider a limit $w_\infty(\tau, y)$ of $w(\tau + k, y)$ as $k \to +\infty$. Roughly speaking, we then show that $v(\tau, y) = w_\infty(\tau, y) - \lambda\tau$ (or at least a limit of $w_\infty(\tau, y) - \lambda\tau$) is $1/|\lambda|$-periodic in time. This last property can be proven using a strong maximum principle on a perturbed problem.

The perturbed problem consists first of considering a truncated Hamiltonian H_K in place of H and to adding a nonlocal term $\varepsilon\, \mathcal{I}(w)$ on the right-hand side of the equation to ensure the strong maximum principle for $\varepsilon > 0$. At the end, we take the limit as $\varepsilon \to 0$ and, if necessary, as $K \to +\infty$.

3. Comparison principles and gradient estimates for nonlocal equations

In this Section, we state various comparison principles and obtain gradient estimates for viscosity solutions of Hamilton–Jacobi equations perturbed by a zeroth-order nonlocal operator, under appropriate assumptions on the Hamiltonian. For a definition of viscosity solutions and their properties, see in particular the User's Guide [17] for viscosity solutions, the book of Barles [7], the book of Bardi and Capuzzo-Dolcetta [3] or the book of Lions [27].

We first introduce the zeroth-order nonlocal operator:

$$\mathcal{I}(v)(x) = \int_{\mathbf{R}^N} \mathrm{d}z\ J(z)\ \left(v(x-z) - v(x)\right)$$

where the function J satisfies

$$\begin{cases} J \quad \text{is continuous,} \\ J(-z) = J(z) > 0 \quad \text{for every } z \in \mathbf{R}^N, \\ \int_{\mathbf{R}^N} \mathrm{d}z\ J(z) = 1 \quad \text{and } \mathcal{I}_i := \int_{\mathbf{R}^N} \mathrm{d}z\ |z|^i\ J(z) < +\infty, \quad i = 1, 2. \end{cases} \tag{13}$$

These conditions are for instance satisfied for the choice of the following function: $J(z) = c\, e^{-|z|}$ with $c = \left(\int_{\mathbf{R}^N} \mathrm{d}z\ e^{-|z|}\right)^{-1}$.

Let $F : \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ and $w_0 : \mathbf{R}^N \to \mathbf{R}$ be continuous functions, the latter satisfying for some constant $C > 0$:

$$|w_0(x)| \leqq C(1 + |x|) \quad \text{for every } x \in \mathbf{R}^N.$$

For $T > 0$, we consider the equation for $\varepsilon \geqq 0$:

$$w_t = \varepsilon\, \mathcal{I}(w(t, \cdot)) + F(w, x, \nabla w) \tag{14}$$

for $(t, x) \in (0, T) \times \mathbf{R}^N$, with the following initial condition:

$$w(0, x) = w_0(x) \quad \text{for } x \in \mathbf{R}^N. \tag{15}$$

We say that a function $w : [0, T) \times \mathbf{R}^N \to \mathbf{R}$ has *at most a linear growth* if there exists a constant $C > 0$ such that:

$$|w(t, x)| \leqq C(1 + |x|) \quad \text{for every } (t, x) \in [0, T) \times \mathbf{R}^N.$$

Given such a function w, the function w^* designates its upper-semicontinuous (usc) envelope (that is, the smallest usc function above w) and the function w_* its lower-semicontinuous (lsc) envelope.

Throughout the paper, we use the following convention (from the theory of discontinuous solutions developed by Ishii): we say that a locally bounded function w is a subsolution (resp. supersolution) of an equation if its usc envelope (resp. lsc envelope) is a subsolution (resp. supersolution).

Next, we successively give a comparison principle for unbounded viscosity solutions of (14), a comparison principle on bounded domains and a strong maximum principle. The first result is adapted from [18]. Let us mention that the nonlocal term can be easily handled because $\mathcal{I}_i < +\infty$ for $i = 1, 2$. See the Appendix for details.

Proposition 1. (Comparison principle, $\varepsilon \geqq 0$) *Under Assumption* (A1) *and assuming that u_0 is Lipschitz continuous, if u and v are respectively sub- and supersolutions of* (14)–(15) *on* $(0, T) \times \mathbf{R}^N$*, which have at most a linear growth, then $u \leqq v$ on* $[0, T) \times \mathbf{R}^N$.

The following result is easier than the previous one since we assume boundedness of the domain; it can easily be derived from classical results—see [7] for instance and [23] for integro-PDEs. This is the reason why we skip the proof.

Proposition 2. (Comparison principle on bounded open sets, $\varepsilon = 0$) *Under Assumption* (A1)*, let $Q \subset (0, T) \times \mathbf{R}^N$ be a bounded open set and the two functions $u : \overline{Q} \to \mathbf{R}$ and $v : \overline{Q} \to \mathbf{R}$ be, respectively, sub- and supersolutions of* (14) *on Q. If $u_* \leqq v^*$ on ∂Q, then $u_* \leqq v^*$ on Q.*

We next state a strong maximum principle for (14) when $\varepsilon > 0$. The techniques used in the following proof are classical but not really the result; hence we provide the details.

Proposition 3. (Strong maximum principle for $\varepsilon > 0$) *Under Assumption* (A1) *and assuming that u_0 is Lipschitz continuous, let us consider functions $\overline{w}$ and $\underline{w}$ which are respectively super- and subsolution of* (14)–(15) *on* $(0, T) \times \mathbf{R}^N$*, are at most of linear growth, and satisfy $\underline{w}^* \leqq \overline{w}_*$ on $(0, T) \times \mathbf{R}^N$. If $\underline{w}^*(\bar{t}, \bar{x}) = \overline{w}_*(\bar{t}, \bar{x})$ for some point $(\bar{t}, \bar{x}) \in (0, T) \times \mathbf{R}^N$ and if $\varepsilon > 0$, then we have $\underline{w}^* = \overline{w}_*$ on* $(0, \bar{t}] \times \mathbf{R}^N$.

Proof. Let us first notice that we can assume that $\varepsilon = 1$. Let us next define, for any $\alpha \geqq 0$:

$$w_\alpha(t,x) = \overline{w}_*(t,x) - \underline{w}^*(t,x) + \alpha\,|x-\overline{x}|^2 \quad \text{and} \quad m_\alpha(t) := \inf_{x\in\mathbf{R}^N} w_\alpha(t,x).$$

Note that $m_\alpha(t) \geqq 0$ and $m_\alpha(\overline{t}) = 0$. We claim that this function satisfies, in the viscosity sense,

$$m'_\alpha(t) + \gamma m_\alpha(t) \geqq -C_1\alpha \tag{16}$$

for some constant C_1 to be determined.

In order to justify such a result, we first prove that the infimum is realized. Because $\overline{w}$ and $\underline{w}$ are at most of linear growth, we know that for every $\alpha \in (0,1)$, there exists $x_\alpha(t) \in \mathbf{R}^N$ such that $m_\alpha(t) = w_\alpha(t, x_\alpha(t))$ and $|x_\alpha(t)| \leqq \frac{C'}{\alpha}$ for some constant $C' > 0$. In order to prove that (16) is satisfied, we need to consider the lsc envelope of m_α. In view of the bound on $x_\alpha(t)$, it is clear that m_α is lsc and therefore coincides with its lsc envelope. Next, we assert that m_α is the relaxed lower limit of the family of functions $\{m_{\alpha,\varepsilon}\}_{\varepsilon>0}$:

$$\begin{aligned} m_\alpha &= \liminf{}_*(m_{\alpha,\varepsilon}) \text{ with} \\ m_{\alpha,\varepsilon}(t) & \\ &= \inf_{s\geqq 0, x,y\in\mathbf{R}^N} \left\{ \overline{w}_*(t,x) - \underline{w}^*(s,y) + \frac{(t-s)^2}{2\varepsilon} + \frac{|y-x|^2}{2\varepsilon} + \alpha|x-\overline{x}|^2 \right\}. \end{aligned}$$

Using once again the fact that $\overline{w}_*$ and $\underline{w}^*$ have linear growth, we can assert that the infimum defining $m_{\alpha,\varepsilon}$ is attained.

Let us next consider a test function $\phi(t)$ such that $m_\alpha - \phi$ attains a strict local minimum at time $t_0 > 0$. This implies that there exists $\varepsilon_n \to 0$ and $t_n \to t_0$ such that $m_{\alpha,\varepsilon_n} - \phi$ attains a local minimum at time t_n and $m_{\alpha,\varepsilon_n}(t_n) \to m_\alpha(t_0)$ as $n \to +\infty$. As explained above, the infimum defining $m_{\alpha,\varepsilon_n}(t_n)$ is attained and

$$m_{\alpha,\varepsilon_n}(t_n) = \overline{w}_*(t_n,x_n) - \underline{w}^*(s_n,y_n) + \frac{(t_n-s_n)^2}{2\varepsilon_n} + \frac{|y_n-x_n|^2}{2\varepsilon_n} + \alpha|x_n-\overline{x}|^2.$$

Classical penalization results assert that the following properties hold true:

$$\begin{cases} \dfrac{(t_n-s_n)^2}{2\varepsilon_n} \to 0 \quad \text{and in particular } s_n > 0 \text{ for } n \text{ large enough,} \\ \dfrac{|y_n-x_n|^2}{2\varepsilon_n} \to 0, \\ x_n \to x_0 \quad \text{such that } m_\alpha(t_0) = \overline{w}_*(t_0,x_0) - \underline{w}^*(t_0,x_0) + \alpha|x_0-\overline{x}|^2, \\ \overline{w}_*(t_n,x_n) - \underline{w}^*(s_n,y_n) \to \overline{w}_*(t_0,x_0) - \underline{w}^*(t_0,x_0). \end{cases} \tag{17}$$

In the following, $x_0 = x_\alpha(t_0)$ in accordance with the notations introduced previously. Consequently,

$$\begin{aligned} &\overline{w}_*(t,x) - \underline{w}^*(s,y) + \frac{(t-s)^2}{2\varepsilon} + \frac{|y-x|^2}{2\varepsilon} + \alpha|x-\overline{x}|^2 - \phi(t) \\ &\geqq \overline{w}_*(t_n,x_n) - \underline{w}^*(s_n,y_n) + \frac{(t_n-s_n)^2}{2\varepsilon} + \frac{|y_n-x_n|^2}{2\varepsilon} + \alpha|x_n-\overline{x}|^2 - \phi(t_n) \end{aligned}$$

for any $(s,x,y) \in (0;+\infty) \times \mathbf{R}^{2N}$ and t close to t_n. Using the fact that $\overline{w}$ (resp. $\underline{w}$) is a supersolution (resp. subsolution) of (14)–(15), we conclude that:

$$\frac{s_n - t_n}{\varepsilon_n} + \phi'(t_n) \geqq \mathcal{I}(\overline{w}_*(t_n,\cdot))(x_n) + F(\overline{w}_*(t_n,x_n), x_n, p_n + 2\alpha(x_n - \overline{x}))$$
$$\frac{s_n - t_n}{\varepsilon_n} \leqq \mathcal{I}(\underline{w}^*(s_n,\cdot))(y_n) + F(\underline{w}^*(s_n,y_n), y_n, p_n)$$

with $p_n = (y_n - x_n)/\varepsilon_n$. Subtracting both inequalities yields:

$$\begin{aligned}
\phi'(t_n) &\geqq \mathcal{I}(\overline{w}_*(t_n,\cdot))(x_n) - \mathcal{I}(\underline{w}^*(s_n,\cdot))(y_n) - 2\gamma\alpha|x_n - \overline{x}| \\
&\quad -\gamma\left| m_{\alpha,\varepsilon_n}(t_n) - \frac{(t_n - s_n)^2}{2\varepsilon_n} - \frac{|x_n - y_n|^2}{2\varepsilon_n} - \alpha|x_n - \overline{x}|^2 \right| \\
&\quad -\gamma\left(|x_n - y_n| + \frac{|x_n - y_n|^2}{\varepsilon_n}\right) \\
&\geqq \int \mathrm{d}z\, J(z)(\overline{w}_*(t_n, x_n - z) - \underline{w}^*(s_n, y_n - z)) - (\overline{w}_*(t_n,x_n) - \underline{w}^*(s_n,y_n)) \\
&\quad -2\gamma\alpha|x_n - \bar{x}| - \gamma\left| m_{\alpha,\varepsilon_n}(t_n) - \frac{(t_n - s_n)^2}{2\varepsilon_n} - \frac{|x_n - y_n|^2}{2\varepsilon_n} - \alpha|x_n - \overline{x}|^2 \right| \\
&\quad -\gamma\left(|x_n - y_n| + \frac{|x_n - y_n|^2}{\varepsilon_n}\right)
\end{aligned}$$

where we used (A1). Using Fatou's lemma and (17) yields as $n \to +\infty$:

$$\begin{aligned}
\phi'(t_0) &\geqq \mathcal{I}(\overline{w}_*(t_0,\cdot))(x_\alpha(t_0)) - \mathcal{I}(\underline{w}^*(t_0,\cdot))(x_\alpha(t_0)) \\
&\quad -2\gamma\alpha|x_\alpha(t_0) - \overline{x}| - \gamma\left(m_\alpha(t_0) - \alpha|x_\alpha(t_0) - \overline{x}|^2\right) \\
&\geqq \mathcal{I}(w_\alpha(t_0,\cdot))(x_\alpha(t_0)) - \alpha\mathcal{I}(|\cdot - \overline{x}|^2)(x_\alpha(t_0)) \\
&\quad -\gamma m_\alpha(t_0) + \alpha \inf_{r \geqq 0}(\gamma r^2 - 2\gamma r).
\end{aligned}$$

Using the fact that $J(-z) = J(z)$, we get

$$\mathcal{I}(|\cdot - \overline{x}|^2)(x) = \text{constant} = \mathcal{I}_2 := \int_{\mathbf{R}^N} \mathrm{d}z\; J(z)|z|^2$$

and setting $C_1 = -\inf_{r \geqq 0}(\gamma r^2 - 2\gamma r) + \mathcal{I}_2$, we deduce that m_α satisfies, in the viscosity sense,

$$m'_\alpha(t) + \gamma m_\alpha(t) \geqq -\alpha C_1 + \mathcal{I}(w_\alpha(t,\cdot))(x_\alpha(t)) \quad \text{with } \mathcal{I}(w_\alpha(t,\cdot))(x_\alpha(t)) \geqq 0.$$

In particular, we have:

$$m'_\alpha(t) + \gamma m_\alpha(t) \geqq -\alpha C_1.$$

By integration [using $m_\alpha(\overline{t}) = 0$], we get:

$$m_\alpha(t) \leqq \alpha\frac{C_1}{\gamma}\left(e^{\gamma(\overline{t} - t)} - 1\right) \quad \text{for } t \in (0, \overline{t}]. \tag{18}$$

Using the fact that:

$$m_\alpha(t) = \overline{w}_*(t, x_\alpha(t)) - \underline{w}^*(t, x_\alpha(t)) + \alpha\, |x_\alpha(t) - \overline{x}|^2 \quad \text{and} \quad \overline{w}_* - \underline{w}^* \geqq 0,$$

we deduce that:

$$|x_\alpha(t) - \overline{x}|^2 \leqq \frac{C_1}{\gamma} \left(e^{\gamma(\overline{t}-t)} - 1 \right) \quad \text{for } t \in (0, \overline{t}]. \tag{19}$$

First, we remark that $m_\alpha(t) \to m_0(t)$ as $\alpha \to 0$, and then there exists $x_0(t)$ satisfying (19), such that

$$m_0(t) = w_0(t, x_0(t)) = \overline{w}_*(t, x_0(t)) - \underline{w}^*(t, x_0(t)) \quad \text{for } t \in (0, \overline{t}].$$

We also have $m_0 = \limsup {}^* m_\alpha$ and arguing as previously implies that m_0 satisfies:

$$m_0'(t) + \gamma m_0(t) \geqq \mathcal{I}(w_0(t, \cdot))(x_0(t)) \quad \text{for } t \in (0, \overline{t}).$$

From (18), we also deduce that $m_0(t) = 0$ for every $t \in (0, \overline{t})$. Therefore for any $t_0 \in (0, \overline{t})$, we can take a test function tangent from below to m_0 at t_0, and deduce that:

$$0 \geqq \mathcal{I}(w_0(t_0, \cdot))(x_0(t_0)) = \int_{\mathbf{R}^N} \mathrm{d}z\; J(z)\; w_0(t_0, x_0(t_0) - z)$$

with $w_0(t_0, x_0(t_0)) = 0$ and $w_0(t_0, \cdot) \geqq 0$.

Because J is continuous and satisfies $J > 0$, we deduce that $w_0(t_0, x) = 0$ for almost every $x \in \mathbf{R}^N$. Because w_0 is lower-semicontinuous, we deduce that $w_0(t_0, x) = 0$ for every $x \in \mathbf{R}^N$. Now this result is true for every $t_0 \in (0, \overline{t})$, and then still by lower-semicontinuity (and using $w_0 \geqq 0$) $w_0(t, x) = 0$ for every $(t, x) \in (0, \overline{t}] \times \mathbf{R}^N$, that is, $\overline{w}_*(t, x) = \underline{w}^*(t, x)$ for every $(t, x) \in (0, \overline{t}] \times \mathbf{R}^N$. This ends the proof of Proposition 3. □

We next state and prove the existence of a solution of (14)–(15).

Proposition 4. (Existence and uniqueness of a solution, $\varepsilon \geqq 0$) *If* $u_0 \in W^{1,\infty}$ *and* F *satisfies* (A1)*, there exists a unique viscosity solution* w *of* (14)–(15) *on* $(0, T) \times \mathbf{R}^N$ *satisfying* $|w(t, x) - u_0(x)| \leqq Ct$ *for some* $C > 0$. *Moreover this solution* w *is continuous.*

Proof. The existence is classical via Perron's method if one constructs barriers. Let us consider

$$w^\pm(t, x) = \pm Ct + u_0(x)$$

with $C = \sup_{(x,v,p):|p|\leqq \|u_0\|_{1,\infty}} |F(v, y, p)| + \varepsilon \|u_0\|_{1,\infty} \mathcal{I}_1$. We consider $\mathcal{E}$ the set of all subsolutions u of (14) such that $u \leqq w^+$ on $[0, T) \times \mathbf{R}^N$. This set is nonempty since it contains w^-. Define

$$w(t, x) = \sup\{u(t, x) : u \in \mathcal{E}\}$$

It is a subsolution. It is also classical to check that w is a supersolution, which relies on a "bump construction", described for instance in [18]. The uniqueness and the continuity follow from the comparison principle (Proposition 1). □

When constructing the correctors, we will need some gradient estimates, but in an integral form. Let us give a precise definition.

Definition 1. (Gradient estimate) For a function $w_0 : \mathbf{R}^N \to \mathbf{R}$, we say that

$$\xi \cdot \nabla w_0 \leqq M \quad \text{on } \mathbf{R}^N$$

if and only if

$$w_0(x + h\xi) - w_0(x) \leqq hM \quad \text{for all } h \geqq 0,\ \ x \in \mathbf{R}^N.$$

We next state and prove two results concerning gradient estimates satisfied by the viscosity solution of (14)–(15) under certain conditions on the Hamiltonian F.

Proposition 5. (A priori bound on the gradient, $\varepsilon \geqq 0$) *Under assumptions of Proposition* 4, *let us assume moreover that there exists* $\xi \in \mathbf{R}^N$ *with* $|\xi| = 1$, $M \geqq 0$ *and a function* F_0 *such that*

$$F(u, x, p) = F_0(x - (\xi \cdot x)\,\xi, p) \quad \textit{for all } (u, x, p) \textit{ if } \xi \cdot p \geqq M.$$

If $\xi \cdot \nabla w_0 \leqq M$ *on* $\mathbf{R}^N$ *in the sense of Definition* 1, *then the solution* w *of* (14)–(15) *on* $(0, T) \times \mathbf{R}^N$ *satisfies* $\xi \cdot \nabla w \leqq M$ *on* $[0, T) \times \mathbf{R}^N$.

Before proving this proposition, let us derive a straightforward corollary.

Corollary 1. *Under the assumptions of Proposition* 4, *let us assume moreover that there exists a closed set* Ω *that is star shaped with respect to the origin and a function* F_0 *such that*

$$F(u, x, p) = F_0(p) \quad \textit{for all } (u, x, p) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N \textit{ if } p \notin \Omega.$$

If $\nabla w_0 \in \Omega$ *on* $\mathbf{R}^N$ *then the solution* w *of* (14)–(15) *on* $(0, T) \times \mathbf{R}^N$ *satisfies* $\nabla w \in \Omega$ *on* $[0, T) \times \mathbf{R}^N$.

Let us now turn to the proof of the proposition.

Proof of Proposition 5. The proof proceeds by reduction to a simpler case and by approximation. First, there is no restriction in assuming that $\varepsilon = 1$ and $\xi = (1, 0, \dots, 0)$. Hence, if $x = (x_1, x')$ and $p = (p_1, p')$ with $x', p' \in \mathbf{R}^{N-1}$:

$$\partial_1 w^0 \leqq M,$$

$$F(u, x, p) = F_0(x', p) \quad \text{for all } (u, x, p) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N \quad \text{if} \quad p_1 \geqq M \tag{20}$$

and we want to estimate from above $\partial_1 w$. Next, we consider $w_0^\delta \in C^\infty(\mathbf{R}^N)$, $F^\delta \in C^\infty(\mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N)$ such that $w_0^\delta \to w_0$ and $F^\delta \to F$ locally uniformly as $\delta \to 0$. F^δ satisfies (20) and (A1) with $M + \delta$ and $\gamma + \delta$, respectively, with $\partial_1 w_0^\delta \leqq M + \delta$ on $\mathbf{R}^N$.

Then if w^δ is the viscosity solution of:

$$w_t^\delta = \mathcal{I}(w^\delta(t, \cdot)) + F^\delta(w^\delta, x, \nabla w^\delta) + \delta \Delta w^\delta$$

for $(t, x) \in (0, T) \times \mathbf{R}^N$, with the following initial condition:

$$w^\delta(0, x) = w_0^\delta(x) \quad \text{for } x \in \mathbf{R}^N,$$

we can prove that $w^\delta \in C^\infty([0, T) \times \mathbf{R}^N)$ by adapting the classical theory of parabolic equations [26]—see [19, 23] for such an adaptation but with a different integral term. Let us next write down the equation satisfied by $v(t, x) = \partial_1 w^\delta(t, x)$:

$$\begin{aligned} \partial_t v = &\mathcal{I}(v(t, \cdot)) + \partial_w F^\delta(w^\delta, x, v, \nabla' w^\delta)v + \partial_{x_1} F^\delta(w^\delta, x, v, \nabla' w^\delta) \\ &+ \sum_i \partial_{p_i} F^\delta(w^\delta, x, v, \nabla' w^\delta)\partial_i v + \delta \Delta v \end{aligned}$$

where $\nabla' w = (\partial_2 w, \ldots, \partial_N w) \in \mathbf{R}^{N-1}$. Now remark that $M+\delta$ is a supersolution of such an equation and the comparison principle yields $v \leqq M + \delta$, which implies that:

$$w^\delta(x_1 + h, x') \leqq w^\delta(x_1, x') + h(M + \delta).$$

Passing to the limit as $\delta \to 0$ completes the proof. □

Proposition 6. (Monotonicity of the solution, $\varepsilon \geqq 0$) *Under the assumptions of Proposition* 4*, let us assume moreover that there exists* $\xi \in \mathbf{R}^N$ *with* $|\xi| = 1$*, and a function* F_1 *such that:*

$$F(u, x, p) = F_1(u, x - (\xi \cdot x)\, \xi, p) \quad \textit{for all } (u, x, p) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N.$$

If $\xi \cdot \nabla w_0 \leqq 0$ *on* $\mathbf{R}^N$ *in the sense of Definition 1, then the solution* w *of* (14)–(15) *on* $(0, T) \times \mathbf{R}^N$ *satisfies* $\xi \cdot \nabla w \leqq 0$ *on* $[0, T) \times \mathbf{R}^N$.

Proof. The proof is very similar to the proof of Proposition 5. We do the same reduction and the same approximation and v satisfies the same equation but with $\partial_{x_1} F \equiv 0$. It is therefore clear that $v = 0$ is a supersolution and we conclude the same way. □

4. The proof of the convergence

This section is dedicated to the proof of Theorem 2. Before presenting it, we first imbed the problem in a higher-dimensional one. Precisely, we consider the solution U^ε of

$$\begin{cases} U_t^\varepsilon = H\left(\frac{U^\varepsilon}{\varepsilon}, \frac{x}{\varepsilon}, \nabla_x U^\varepsilon\right) & \text{for } (t, x, x_{N+1}) \in (0, +\infty) \times \mathbf{R}^N \times \mathbf{R}, \\ U^\varepsilon(0, x, x_{N+1}) = u_0(x) + x_{N+1} & \text{for } (x, x_{N+1}) \in \mathbf{R}^N \times \mathbf{R}. \end{cases} \tag{21}$$

There exists a unique viscosity solution of such an equation under assumptions (A1)–(A3). Then we have the following lemma.

Lemma 1. (Link between problems on $\mathbf{R}^N$ and on $\mathbf{R}^{N+1}$) *We have* $u^\varepsilon(t, x) = U^\varepsilon(t, x, 0)$.

Proof. Let $u^\varepsilon[x_{N+1}]$ be the solution of (1) on $\mathbf{R}^N$ with initial condition

$$u^\varepsilon[x_{N+1}](0,x) = u_0(x) + x_{N+1}.$$

We now build the function: $V(t,x,x_{N+1}) = u^\varepsilon[x_{N+1}](t,x)$. Let us first justify that V is a continuous function with respect to x_{N+1}. To see this, consider the function $u^\varepsilon[x_{N+1}](t,x) + e^{\gamma t/\varepsilon}\delta$, where γ is given by (A1), and prove that it is above $u^\varepsilon[x_{N+1}+\delta](t,x)$. It suffices to remark that it is a supersolution of (1) [by using Assumption (A1)] and to use the comparison principle (Proposition 1). Hence, V is upper semicontinuous. We prove analogously that V is lower semicontinuous and we conclude that V is continuous. We now check easily (using test functions) that V is a solution of (21). By the comparison principle applied to (21), we deduce that $V = U^\varepsilon$ and then $u^\varepsilon(t,x) = U^\varepsilon(t,x,0)$. This ends the proof of the Lemma. □

As underlined in the third try of Section 2.1, the proof of convergence will use Lipschitz continuous approximate sub- and supercorrectors on $\mathbf{R}\times\mathbf{R}^{N+1}$. More precisely, we will use the following proposition whose proof is postponed.

Proposition 7. (Lipschitz continuous in y_{N+1} approximate sub- and supercorrectors in dimension $N+1$) *Let $p \in \mathbf{R}^N$. For any $\beta \in \mathbf{R}$, let $\lambda(\beta)$ be the constant defined by Theorem 1 for the Hamiltonian $\beta + H$. Then*

$$\begin{cases} \lambda(\beta) \quad \text{is nondecreasing in} \quad \beta, \\ \forall \lambda_0 \in \mathbf{R}, \quad \exists \beta_0 \in \mathbf{R}, \ \text{such that } \lambda(\beta_0) = \lambda_0. \end{cases} \tag{22}$$

For any fixed $\beta \in \mathbf{R}$, there exist real numbers $\lambda_K^+(\beta)$, $\lambda_K^-(\beta)$, a constant $C = C(p) > 0$ (independent on K and β), and bounded super- and subcorrectors V_K^+, V_K^- depending on β, such that

$$\lambda(\beta) = \lim_{K\to+\infty} \lambda_K^+(\beta) = \lim_{K\to+\infty} \lambda_K^-(\beta)$$

with $\lambda_K^+(\beta)$ and $\lambda_K^-(\beta)$ satisfying (22) and, for $\tau \in \mathbf{R}$, $Y = (y, y_{N+1}) \in \mathbf{R}^N \times \mathbf{R}$ and $P = (p,1) \in \mathbf{R}^N \times \mathbf{R}$:

$$|V_K^+(\tau,Y)| \leqq C, \quad |V_K^-(\tau,Y)| \leqq C$$

$$\lambda_K^+ + \frac{\partial V_K^+}{\partial \tau} \geqq \beta + H\left(\lambda_K^+\tau + P\cdot Y + V_K^+,\ y,\ p + \nabla_y V_K^+\right), \tag{23}$$

$$\lambda_K^- + \frac{\partial V_K^-}{\partial \tau} \leqq \beta + H\left(\lambda_K^-\tau + P\cdot Y + V_K^-,\ y,\ p + \nabla_y V_K^-\right). \tag{24}$$

For any $\lambda_0 \in \mathbf{R}$, there exist real numbers $\beta_0^\pm$ and $\beta_K^\pm$ such that

$$\begin{cases} \lambda_K^\pm(\beta_K^\pm) = \lambda(\beta_0^\pm) = \lambda_0 \\ \beta_K^\pm \to \beta_0^\pm \quad as \quad K \to +\infty \\ \left|\dfrac{\partial V_K^\pm}{\partial y_{N+1}}\right| \leqq C_K \end{cases} \tag{25}$$

for the correctors V_K^+ and V_K^- associated to β_K^+ and β_K^-, respectively, and for some constant $C_K = C(K, p) > 0$.

Proof of Theorem 2. Classically, we prove that

$$U^+ = \limsup{}^* U^\varepsilon$$

is a subsolution of

$$\begin{cases} W_t = \overline{H}^0 (\nabla_x W) & \text{in } (0, +\infty) \times \mathbf{R}^N \times \mathbf{R}, \\ W(0, x, x_{N+1}) = u_0(x) + x_{N+1} & \text{in } \mathbf{R}^N \times \mathbf{R}. \end{cases} \tag{26}$$

Analogously, we can prove that $U^- = \liminf_* U^\varepsilon$ is a supersolution of (26). By the barriers (uniform in ε) given in Proposition 4, we deduce that $U^+(0, x) = U^-(0, x)$. The comparison principle for (2) (see Proposition 1) thus implies that $U^+ \leqq U^-$. Since $U^- \leqq U^+$ always holds true, we conclude that the two functions coincide with U^0, the unique continuous viscosity solution of (26). This last fact is equivalent to the local uniform convergence of U^ε towards U^0.

From invariance under translations we also deduce that $U^0(t, x, x_{N+1} + a) - a$ is also a solution for any $a \in \mathbf{R}$. This proves that $U^0(t, x, x_{N+1}) = U^0(t, x, 0) + x_{N+1}$. Therefore $U^0(t, x, 0)$ is the solution u^0 of (2). By Lemma 1, this proves in particular the local uniform convergence of u^ε to u^0.

Step 1: U^+ is a subsolution. Let us thus prove that U^+ is a subsolution of (26). First, by the comparison principle we easily check that, for any sequence $k_\varepsilon \in \mathbf{Z}$, we have

$$U^\varepsilon(t, x, x_{N+1} + \varepsilon k_\varepsilon) = \varepsilon k_\varepsilon + U^\varepsilon(t, x, x_{N+1})$$

For any $a \in \mathbf{R}$, we then choose the sequence such that $\varepsilon k_\varepsilon \to a$. Passing to the limit as $\varepsilon \to 0$, we deduce by definition of U^+ that $U^+(t, x, x_{N+1} + a) = a + U^+(t, x, x_{N+1})$ and then

$$U^+(t, x, x_{N+1}) = U^+(t, x, 0) + x_{N+1}. \tag{27}$$

Applying Evans's technique of the perturbed test function [20, 21], we argue by contradiction: we consider a test function $\phi \in C^2((0; +\infty) \times \mathbf{R}^{N+1})$ such that $U^+ - \phi$ attains a strict zero local maximum at (t_0, X_0) with $t_0 > 0$, $X_0 \in \mathbf{R}^{N+1}$, and we suppose that there exists $\theta > 0$ such that:

$$\phi_t(t_0, X_0) = \overline{H}(\nabla_x \phi(t_0, X_0)) + \theta.$$

In the following, we set $p = \nabla_x \phi(t_0, X_0)$ and $\lambda_0 = \phi_t(t_0, X_0)$. With the notations of Proposition 7, we see that we have

$$\lambda_0 = \lambda(0) + \theta = \lambda(\beta_0^+) \quad \text{for } \beta_0^+ > 0.$$

We deduce from Proposition 7 that there exists $\beta_K^+ \in \mathbf{R}$ such that

$$\lambda_0 = \lambda_K^+(\beta_K^+) \quad \text{and} \quad \beta_K^+ \geqq \beta_0^+/2,$$

for K large enough.

We next construct a perturbed test function in the spirit of Evans's seminal work. Here, this is a x_{N+1}-twisted perturbed test function:

$$\phi^\varepsilon(t,x,x_{N+1}) = \phi(t,x,x_{N+1}) + \varepsilon V_K^+\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \frac{\phi(t,x,x_{N+1}) - \lambda_0 t - p\cdot x}{\varepsilon}\right)$$

where V_K^+ is a supercorrector given by Proposition 7 for $\beta = \beta_K^+$. From (27), we also know that

$$\frac{\partial \phi}{\partial x_{N+1}}(t_0, X_0) = 1.$$

Let us define, for r small enough: $\mathcal{V}_r := (t_0 - r, t_0 + r) \times B_r(X_0) \subset (0,+\infty) \times \mathbf{R}^{N+1}$. We claim that the following lemma (whose proof is postponed) holds true.

Lemma 2. (ϕ^ε is a supersolution on $\mathcal{V}_{r_K}$) *There exist $r_K > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the function ϕ^ε is a supersolution of*

$$\phi_t^\varepsilon = H\left(\frac{\phi^\varepsilon}{\varepsilon}, \frac{x}{\varepsilon}, \nabla_x \phi^\varepsilon\right) \quad \textit{with} \;\; (t,x,x_{N+1}) \in (0;+\infty)\times \mathbf{R}^{N+1} \tag{28}$$

on $\mathcal{V}_{r_K}$.

Now, since (t_0, X_0) is a strict local zero maximum of $U^+ - \phi$, for $0 < r \leqq r_K$ small enough, we have: $U^+ - \phi \leqq -2\eta$ on $\partial\mathcal{V}_r$ for some $\eta > 0$. Then $U^\varepsilon - \phi \leqq -\eta$ on $\partial\mathcal{V}_r$ for ε small enough. From the bound on the corrector given in Proposition 7, we deduce that $U^\varepsilon \leqq \phi^\varepsilon - \eta + C\varepsilon$ on $\partial\mathcal{V}_r$. By definition, U^ε is a solution of (21) and by Lemma 2, ϕ^ε is a supersolution. Remark that the function $\phi^\varepsilon + \varepsilon\left\lceil \frac{-\eta + C\varepsilon}{\varepsilon}\right\rceil$ is still a supersolution. By the comparison principle (Proposition 2), we conclude that $U^\varepsilon \leqq \phi^\varepsilon + \varepsilon\left\lceil\frac{-\eta+C\varepsilon}{\varepsilon}\right\rceil$ on $\mathcal{V}_r$. Letting $\varepsilon \to 0$, we get in particular at (t_0, X_0): $U^+(t_0, X_0) \leqq \phi(t_0, X_0) - \eta$, which is a contradiction.

Step 2: U^- is a supersolution. Let us thus prove that U^- is a supersolution of (26). We proceed as in Step 1 with $U^+ - \phi$ attaining a strict local zero minimum at (t_0, X_0) with $\theta < 0$ such that:

$$\phi_t(t_0, X_0) = \overline{H}(\nabla_x \phi(t_0, X_0)) + \theta$$

and find $\beta_K^- \in \mathbf{R}$ such that $\lambda_0 = \lambda_K^-(\beta_K^-) = \lambda(\beta_0)$, which satisfies $\beta_K^- \leqq \beta_0/2 < 0$ for K large enough. We define

$$\phi^\varepsilon(t,x,x_{N+1}) = \phi(t,x,x_{N+1}) + \varepsilon V_K^-\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \frac{\phi(t,x,x_{N+1}) - \lambda_0 t - p\cdot x}{\varepsilon}\right),$$

where V_K^- is a corrector given by Proposition 7 for $\beta = \beta_K^-$. We claim that the following lemma holds true (its proof is similar to the proof of Lemma 2):

Lemma 3. (ϕ^ε is a subsolution on $\mathcal{V}_{r_K}$) *There exist* $r_K > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$*, the function* ϕ^ε *is a subsolution on* $\mathcal{V}_{r_K}$*, that is, satisfies*

$$\phi_t^\varepsilon \leqq H\left(\frac{\phi^\varepsilon}{\varepsilon}, \frac{x}{\varepsilon}, \nabla_x \phi^\varepsilon\right) \quad \text{on } \mathcal{V}_{r_K}.$$

By using such a lemma, we can get a contradiction as in Step 1.

The proof of Theorem 2 is now complete. □

Proof of Lemma 2. We want to prove that ϕ^ε is a supersolution of (28). Consider a test function ψ and a point $(\bar{t}, \overline{X}) \in \mathcal{V}_r$ such that $\phi^\varepsilon - \psi$ attains a local minimum at $(\bar{t}, \overline{X})$ with $\overline{X} = (\bar{x}, \bar{x}_{N+1})$:

$$\phi^\varepsilon(\bar{t}, \overline{X}) - \psi(\bar{t}, \overline{X}) \leqq \phi^\varepsilon(t, X) - \psi(t, X),$$

that is,

$$\begin{aligned} V_K^+&\left(\frac{\bar{t}}{\varepsilon}, \frac{\bar{x}}{\varepsilon}, \frac{\phi(\bar{t}, \overline{X}) - \lambda_0 \bar{t} - p \cdot \bar{x}}{\varepsilon}\right) - \frac{1}{\varepsilon}(\psi(\bar{t}, \overline{X}) - \phi(\bar{t}, \overline{X})), \\ &\leqq V_K^+\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \frac{\phi(t, X) - \lambda_0 t - p \cdot x}{\varepsilon}\right) - \frac{1}{\varepsilon}(\psi(t, X) - \phi(t, X)). \end{aligned} \quad (29)$$

Let us define $F(t, X) = \phi(t, X) - \lambda_0 t - p \cdot x$. We have

$$\frac{\partial F}{\partial x_{N+1}}(t_0, X_0) = \frac{\partial \phi}{\partial x_{N+1}}(t_0, X_0) = 1.$$

Consequently, there exists $r_0 > 0$ such that the map

$$\begin{aligned} Id \times F : \mathcal{V}_{r_0} &\to U \subset \mathbf{R} \times \mathbf{R}^N \times \mathbf{R} \\ (t, x, x_{N+1}) &\mapsto (t, x, F(t, x, x_{N+1})) \end{aligned}$$

is a C^1-diffeomorphism from $\mathcal{V}_{r_0}$ onto its range U, and let us call $G : U \to \mathbf{R}$ the map such that

$$\begin{aligned} Id \times G : U &\to \mathcal{V}_{r_0} \\ (t, x, \xi_{N+1}) &\mapsto (t, x, G(t, x, \xi_{N+1})) \end{aligned}$$

is the inverse of $Id \times F$.

Let us consider the variables $\tau = t/\varepsilon$, $Y = (y, y_{N+1})$ with $y = x/\varepsilon$ and $y_{N+1} = F(t, X)/\varepsilon$ and define

$$\Gamma^\varepsilon(\tau, Y) = \frac{1}{\varepsilon}\left(\psi\left(\varepsilon\tau, \varepsilon y, G(\varepsilon\tau, \varepsilon y, \varepsilon y_{N+1})\right) - \phi\left(\varepsilon\tau, \varepsilon y, G(\varepsilon\tau, \varepsilon y, \varepsilon y_{N+1})\right)\right).$$

Let $\bar{\tau} = \frac{\bar{t}}{\varepsilon}$, $\bar{y} = \frac{\bar{x}}{\varepsilon}$, $\bar{y}_{N+1} = \frac{F(\bar{t}, \overline{X})}{\varepsilon}$, $\overline{Y} = (\bar{y}, \bar{y}_{N+1})$. Then (29) implies

$$V_K^+(\bar{\tau}, \overline{Y}) - \Gamma^\varepsilon(\bar{\tau}, \overline{Y}) \quad \leqq \quad V_K^+(\tau, Y) - \Gamma^\varepsilon(\tau, Y)$$

for all (τ, Y) in a neighborhood of $(\overline{\tau}, \overline{Y})$, that is, $V_K^+ - \Gamma^\varepsilon$ reaches a local minimum at $(\overline{\tau}, \overline{Y})$. Estimate (25) implies in particular that

$$\left| \frac{\partial \Gamma^\varepsilon}{\partial y_{N+1}} (\overline{\tau}, \overline{Y}) \right| \leqq C_K. \tag{30}$$

Since V_K^+ is a viscosity solution of (24), we conclude that [with $P = (p, 1) \in \mathbf{R}^{N+1}$, $\lambda_0 = \lambda_K^+(\beta_K^+)$ and $\beta_K^+ \geqq \beta_0/2 > 0$]:

$$\lambda_0 + \Gamma_\tau^\varepsilon(\overline{\tau}, \overline{Y}) \;\geqq\; \frac{\beta_0}{2} + H(\lambda_0 \overline{\tau} + P \cdot \overline{Y} + V_K^+(\overline{\tau}, \overline{Y}), \overline{y}, p + \nabla_y \Gamma^\varepsilon(\overline{\tau}, \overline{Y})). \tag{31}$$

Using the fact that $G(t, x, F(t, x, x_{N+1})) = x_{N+1}$, simple computations yield:

$$\begin{cases} \lambda_0 \overline{\tau} + P \cdot \overline{Y} + V_K^+(\overline{\tau}, \overline{Y}) = \frac{\phi^\varepsilon(\overline{t}, \overline{X})}{\varepsilon} \\ \lambda_0 + \Gamma_\tau^\varepsilon(\overline{\tau}, \overline{Y}) = \psi_t(\overline{t}, \overline{X}) + \big(\lambda_0 - \phi_t(\overline{t}, \overline{X})\big) \Big(1 + \frac{\partial \Gamma^\varepsilon}{\partial y_{N+1}}(\overline{\tau}, \overline{Y})\Big) \\ p + \nabla_y \Gamma^\varepsilon(\overline{\tau}, \overline{Y}) = \nabla_x \psi(\overline{t}, \overline{X}) + \big(p - \nabla_x \phi(\overline{t}, \overline{X})\big) \Big(1 + \frac{\partial \Gamma^\varepsilon}{\partial y_{N+1}}(\overline{\tau}, \overline{Y})\Big). \end{cases} \tag{32}$$

Therefore we get

$$\begin{aligned} &\psi_t(\overline{t}, \overline{X}) + \big(\lambda_0 - \phi_t(\overline{t}, \overline{X})\big) \Big(1 + \tfrac{\partial \Gamma^\varepsilon}{\partial y_{N+1}}(\overline{\tau}, \overline{Y})\Big) \\ &\quad \geqq \tfrac{\beta_0}{2} + H\left(\tfrac{\phi^\varepsilon(\overline{t}, \overline{X})}{\varepsilon}, \tfrac{\overline{x}}{\varepsilon}, \nabla_x \psi(\overline{t}, \overline{X}) + \big(p - \nabla_x \phi(\overline{t}, \overline{X})\big) \Big(1 + \tfrac{\partial \Gamma^\varepsilon}{\partial y_{N+1}}(\overline{\tau}, \overline{Y})\Big)\right). \end{aligned}$$

Using the uniform continuity of H on $\mathbf{R} \times \mathbf{R}^N \times B_{2C_K}(0)$, bounds (30), the C^1 regularity of ϕ and the fact that $\lambda_0 = \phi_t(t_0, X_0)$, $p = \nabla_x \phi(t_0, X_0)$, we deduce that there exists $0 < r_K \leqq r_0$, such that (with $\beta_0 > 0$):

$$\psi_t(\overline{t}, \overline{X}) \;\geqq\; \frac{\beta_0}{4} + H\left(\frac{\phi^\varepsilon(\overline{t}, \overline{X})}{\varepsilon}, \frac{\overline{x}}{\varepsilon}, \nabla_x \psi(\overline{t}, \overline{X})\right) \quad \text{for } (\overline{t}, \overline{X}) \in \mathcal{V}_{r_K}.$$

This proves that ϕ^ε is a supersolution on $\mathcal{V}_{r_K}$ and ends the proof of Lemma 2. □

The proof of Lemma 3 is similar and we skip it.

5. Approximate cell problems

In this section, we explain that approximate correctors of Proposition 7 (used in the proof of convergence) are in fact exact correctors for approximate and bounded Hamiltonians. The construction of these approximate correctors also permits one to prove ergodicity (Theorem 1).

First, we define two nondecreasing functions by using (A3):

$$\begin{cases} h(r) = \sup\big\{H(v, y, p), \quad (v, y) \in \mathbf{R} \times \mathbf{R}^N, \;\; p \in B_r(0)\big\}, \\ r(h) = \inf\big\{r \geqq 0, \quad \big(|p| \geqq r\big) \Longrightarrow H(\cdot, \cdot, p) > h\big\}. \end{cases} \tag{33}$$

Because the Hamiltonian H is continuous, we deduce that the function h is continuous and the function r is nondecreasing, upper-semicontinuous and (therefore) continuous from the right. Moreover these functions satisfy in particular $h(r(h)) \geqq h$ and $r(h(r)) \geqq r$.

Definition of $H_K^{\delta,+}$.
For every $\delta \in \mathbf{R}$ and $P = (p, p_{N+1})$, we define:

$$H^\delta(v, y, P) = H(v, y, p) + \delta\, |p_{N+1}|. \tag{34}$$

For $\delta > 0$ fixed for the sequel, we set $h^\delta(K) = h(K) + \delta K$. Using (A3), for every $K > 0$ large enough we have $h^\delta(K) \geqq h(K) > 0$ and we define for some $\mu^+ \geqq 1$ to be chosen later:

$$H_K^{\delta,+}(v, y, P) := \begin{cases} H^\delta(v, y, P) & \text{if } H^\delta(v, y, P) \leqq h^\delta(K), \\ h^\delta(K) + \mu^+\left(H^\delta(v, y, P) - h^\delta(K)\right) & \text{if } h^\delta(K) \leqq H^\delta(v, y, P) \leqq 2h^\delta(K), \\ (1 + \mu^+)\, h^\delta(K) & \text{if } 2h^\delta(K) \leqq H^\delta(v, y, P). \end{cases} \tag{35}$$

Let us define

$$r_K^\delta(p_{N+1}) := r\left(2h^\delta(K) - \delta\, |p_{N+1}|\right) \tag{36}$$

$$\Omega_K^{\delta,+} = \left\{ P = (p, p_{N+1}) \in \mathbf{R}^{N+1}, |p| \leqq r_K^\delta(p_{N+1}), |p_{N+1}| \leqq \frac{2h^\delta(K) - \inf H}{\delta} \right\} \tag{37}$$

with $\inf H := \inf_{(v,y,q)\in\mathbf{R}\times\mathbf{R}^N\times\mathbf{R}^N} H(v, y, q)$.

We can easily check that

$$\Omega_K^{\delta,+} \supset \left\{ P \in \mathbf{R}^{N+1}, \exists (v, y) \in \mathbf{R} \times \mathbf{R}^N, H^\delta(v, y, P) \leqq 2h^\delta(K) \right\} \supset B_K(0).$$

Then the bounded and uniformly continuous Hamiltonian $H_K^{\delta,+}$ satisfies in particular

$$H_K^{\delta,+}(\cdot, \cdot, P) \begin{cases} = H^\delta(\cdot, \cdot, P) & \text{for } |P| \leqq K \\ \geqq H^\delta(\cdot, \cdot, P) & \text{for } P \in \Omega_K^{\delta,+} \\ = M_K^{\delta,+} & \text{for } P \in \mathbf{R}^{N+1} \backslash \Omega_K^{\delta,+} \\ \in \left[\inf H,\ M_K^{\delta,+}\right] & \text{for every } P \in \mathbf{R}^{N+1} \end{cases} \tag{38}$$

if we choose $\mu^+ \geqq 1$ such that $\mu^+ = \mu_K^{\delta,+}$ with

$$(1 + \mu_K^{\delta,+})\, h^\delta(K) = M_K^{\delta,+} := \sup_{P \in \Omega_K^{\delta,+}} (h(|p|) + \delta |p_{N+1}|) \geqq 2h^\delta(K). \tag{39}$$

Definition of $H_K^{\delta,-}$.
We first define

$$\check{H}(v,y,p) := -\min\left(2h(K),\ H(v,y,p)\right).$$

Then we proceed similarly as in the definition of $H_K^{\delta,+}$, with H replaced by $\check{H}$. We define for $\delta > 0$

$$\begin{cases} \check{H}^\delta(v,y,P) = \delta\,|p_{N+1}| + \check{H}(v,y,p) \\ \check{h}^\delta(K) = -\inf H + \delta K, \end{cases}$$

which satisfies $\check{h}^\delta(K) > 0$ for $\delta K > 0$ large enough. We then define for $\mu^- \geqq 1$

$$\check{H}_K^\delta(v,y,P) := \begin{cases} \check{H}^\delta(v,y,P) & \text{if } \check{H}^\delta(v,y,P) \leqq \check{h}^\delta(K) \\ \check{h}^\delta(K) + \mu^-\left(\check{H}^\delta(v,y,P) - \check{h}^\delta(K)\right) & \text{if } \check{h}^\delta(K) \leqq \check{H}^\delta(v,y,P) \leqq 2\check{h}^\delta(K) \\ (1+\mu^-)\,\check{h}^\delta(K) & \text{if } 2\check{h}^\delta(K) \leqq \check{H}^\delta(v,y,P) \end{cases} \tag{40}$$

and the compact set

$$\check{\Omega}_K^\delta = \left\{ p_{N+1} \in \mathbf{R},\ |p_{N+1}| \leqq \frac{2\check{h}^\delta(K) - \inf \check{H}}{\delta} \right\} \quad \text{with } \inf \check{H} = -2h(K). \tag{41}$$

We can easily check that

$$\check{\Omega}_K^\delta \supset \left\{ p_{N+1} \in \mathbf{R},\ \exists (v,y,p),\ H^\delta(v,y,p,p_{N+1}) \leqq 2h^\delta(K) \right\} \supset B_K(0).$$

Then we have in particular

$$\check{H}_K^\delta(\cdot,\cdot,P) \begin{cases} = \check{H}^\delta(\cdot,\cdot,P) & \text{for } |p_{N+1}| \leqq K \\ \geqq \check{H}^\delta(\cdot,\cdot,P) & \text{for } p_{N+1} \in \check{\Omega}_K^\delta \\ = -m_K^{\delta,-} & \text{for } p_{N+1} \in \mathbf{R}\setminus\check{\Omega}_K^\delta \end{cases} \tag{42}$$

if we choose $\mu^- \geqq 1$ such that $\mu^- = \mu_K^{\delta,-}$ with

$$(1+\mu_K^{\delta,-})\,\check{h}^\delta(K) = -m_K^{\delta,-} := 2\check{h}^\delta(K) + 2h(K) - \inf H \geqq 2\check{h}^\delta(K). \tag{43}$$

Let us define the compact set

$$\Omega_K^{\delta,-} = \left\{ P = (p, p_{N+1}) \in \mathbf{R}^{N+1},\ |p| \leqq r_K,\ |p_{N+1}| \leqq \frac{2\check{h}^\delta(K) + 2h(K)}{\delta} \right\}. \tag{44}$$

We can easily check that $\Omega_K^{\delta,-} \supset B_K(0)$. Let us finally define the bounded and uniformly continuous Hamiltonian

$$H_K^{\delta,-}(v,y,P) = -\check{H}_K^\delta(v,y,P) \tag{45}$$

which satisfies (using the definition (34) of $H^{-\delta}$)

$$H_K^{\delta,-}(\cdot,\cdot,P)\ \begin{cases} = H^{-\delta}(\cdot,\cdot,P) & \text{for } |P| \leqq K \\ \leqq H^{-\delta}(\cdot,\cdot,P) & \text{for } P\in\Omega_K^{\delta,-} \\ = f_K^\delta(p_{N+1}) & \text{for } P\in \mathbf{R}^{N+1}\backslash\Omega_K^{\delta,-} \\ = m_K^{\delta,-} & \text{for } |p_{N+1}| \geqq \frac{2\check{h}^\delta(K)+2h(K)}{\delta} \\ \in \left[m_K^{\delta,-},\ 2h(K)\right] & \text{for every } P\in\mathbf{R}^{N+1} \end{cases} \tag{46}$$

where f_K^δ is a Lipschitz continuous function defined by

$$f_K^\delta(p_{N+1}) \begin{cases} = -\delta|p_{N+1}| + 2h(K) & \text{if } |p_{N+1}| \leqq \frac{\check{h}^\delta(K)+2h(K)}{\delta} \\ = -\left\{\check{h}^\delta(K) + \mu^-\left(\delta|p_{N+1}| - 2h(K) - \check{h}^\delta(K)\right)\right\} & \text{if } \frac{\check{h}^\delta(K)+2h(K)}{\delta} \leqq |p_{N+1}| \\ & \qquad \leqq \frac{2\check{h}^\delta(K)+2h(K)}{\delta} \\ = -(1+\mu^-)\,\check{h}^\delta(K) & \text{if } |p_{N+1}| \geqq \frac{2\check{h}^\delta(K)+2h(K)}{\delta} \end{cases}$$

Finally, we can easily check for later use that, for $K > \sqrt{1+|p|^2}$ large enough and for $\delta K > 0$ large enough, we have for every $(v, y, q) \in \mathbf{R}\times\mathbf{R}^N\times\mathbf{R}^N$:

$$\left(h(|p|) \pm \delta \geqq H_K^{\delta,\pm}(v,y,q,1)\right) \Longrightarrow \left(H_K^{\delta,\pm}(v,y,q,1) = \pm\delta + H(v,y,q)\right), \tag{47}$$

$$H_K^{\delta,+}(\cdot,\cdot,Q) \geqq H(\cdot,\cdot,q) \quad \text{for all } Q=(q,q_{N+1})\in\Omega_K^{\delta,+}, \tag{48}$$

$$H_K^{\delta,-}(\cdot,\cdot,Q) \leqq H(\cdot,\cdot,q) \quad \text{for all } Q=(q,q_{N+1})\in\Omega_K^{\delta,-}. \tag{49}$$

We now state the following fundamental result, which will be used as the cornerstone of our construction and will be proved in Section 6.

Proposition 8. (Lipschitz continuous correctors for an approximate Hamiltonian in higher dimension) *Let* $p\in\mathbf{R}^N$ *and* $P=(p,1)\in\mathbf{R}^N\times\mathbf{R}$. *Let us consider the truncated Hamiltonians* $H_K^{\delta,+}$ *and* $H_K^{\delta,-}$ *defined by* (35) *and* (45) *for* $K>\sqrt{1+|p|^2}$ *large enough and for* $\delta K>0$ *large enough, and* H *satisfying* (A1)–(A3). *For any* $\beta\in\mathbf{R}$, *there exist real numbers* $\lambda_K^{\delta,+}(\beta)$, $\lambda_K^{\delta,-}(\beta)$ *and bounded approximate sub- and supercorrectors* $V_K^{\delta,+}$, $V_K^{\delta,-}$ *depending on* β, *satisfying*

$$\begin{cases} \lambda_K^{\delta,\pm}(\beta) \quad \text{is nondecreasing in} \quad \beta \\ \forall\lambda_0\in\mathbf{R},\quad \exists\beta_{K,0}^{\delta,\pm}\in\mathbf{R},\ \text{such that } \lambda_K^{\delta,\pm}(\beta_{K,0}^{\delta,\pm}) = \lambda_0 \end{cases} \tag{50}$$

and

$$\inf_{(v,y)\in\mathbf{R}\times\mathbf{R}^N} H(v,y,p) \leqq \lambda_K^{\delta,\pm}(\beta) - \beta \mp \delta \leqq h(|p|) \tag{51}$$

(*where the function h is defined by* (33)) *and*

$$\lambda_K^{\delta,\pm} + \frac{\partial V_K^{\delta,\pm}}{\partial \tau} = \beta + H_K^{\delta,\pm}\left(\lambda_K^{\delta,\pm}\tau + P\cdot Y + V_K^{\delta,\pm},\ y,\ P + \nabla V_K^{\delta,\pm}\right) \quad (52)$$

for $(\tau, Y) \in \mathbf{R}^{N+2}$.

$\mathbf{W}^{1,\infty}$ **a priori bounds on the correctors.** *We can construct bounded Lipschitz continuous correctors with*:

$$\begin{aligned} |V_K^{\delta,\pm}(\tau, Y)| &\leqq 4 + \sqrt{N}\,(|p| + r(h(|p|))) \\ \left(P + \nabla V_K^{\delta,\pm}(\tau, Y)\right) &\in \Omega_K^{\delta,\pm} \end{aligned} \quad (53)$$

where r and h are defined by (33) *and*

$$0 \leqq 1 + \frac{\partial V_K^{\delta,+}}{\partial y_{N+1}}(\tau, Y) \leqq \frac{2h^\delta(K) - \inf H}{\delta}$$
$$0 \leqq 1 + \frac{\partial V_K^{\delta,-}}{\partial y_{N+1}}(\tau, Y) \leqq \frac{2\check{h}^\delta(K) + 2h(K)}{\delta}$$

$$\inf H - h(|p|) \leqq \delta + \frac{\partial V_K^{\delta,+}}{\partial \tau}(\tau, Y) \leqq M_K^{\delta,+} - \inf H \quad (54)$$

$$m_K^{\delta,-} - h(|p|) \leqq -\delta + \frac{\partial V_K^{\delta,-}}{\partial \tau}(\tau, Y) \leqq 2h(K) - \inf H \quad (55)$$

where $m_K^{\delta,-}$ *and* $M_K^{\delta,+}$ *are defined respectively by* (43) *and by* (39).

Further properties of the correctors. *The correctors satisfy*

$$V_K^{\delta,\pm}(\tau, y, y_{N+1}) = V_K^{\delta,\pm}\left(0, y, \lambda_K^{\delta,+}\tau + y_{N+1}\right), \quad (56)$$

$$\begin{cases} V_K^{\delta,\pm}(\tau, y, y_{N+1}+1) = V_K^{\delta,\pm}(\tau, y, y_{N+1}), \\ |V_K^{\delta,\pm}(\tau, y+k, y_{N+1}) - V_K^{\delta,\pm}(\tau, y, y_{N+1})| \leqq 1, \quad \textit{for every } k \in \mathbf{Z}^N. \end{cases} \quad (57)$$

If $p = P/Q$ *with* $P \in \mathbf{Z}^N$ *and* $Q \in \mathbf{N}\setminus\{0\}$, *then*

$$V_K^{\delta,\pm}(\tau, y+Qk, y_{N+1}) = V_K^{\delta,\pm}(\tau, y, y_{N+1}) \quad \textit{for every } k \in \mathbf{Z}^N.$$

We next deduce from this proposition Theorem 1 about the ergodicity of the problem and Proposition 7 about the existence of approximate correctors (the proposition we used in the proof of convergence).

Proof of Theorem 1. Let us apply Proposition 8 with $\beta = 0$. We have

$$|\lambda_K^{\delta,\pm}| \leqq C, \quad |V_K^{\delta,\pm}| \leqq C$$

for some constant C independent of δ small enough and K large enough. Moreover $V_K^{\delta,\pm}$ is $1/|\lambda_K^{\delta,\pm}|$-periodic in τ if $\lambda_K^{\delta,\pm} \neq 0$ and is independent on τ if $\lambda_K^{\delta,\pm} = 0$ (by (56)) and satisfies (52). Let us call $\lambda^\pm$ any limit of $\lambda_K^{\delta,\pm}$ for a subsequence of $(\delta, K) \to (0, +\infty)$ and

$$V_+^+ = \limsup{}^* V_K^{\delta,+}, \quad V_-^+ = \liminf{}_* V_K^{\delta,+},$$
$$V_+^- = \limsup{}^* V_K^{\delta,-}, \quad V_-^- = \liminf{}_* V_K^{\delta,-}$$

which still satisfy $|\lambda^\pm| \leqq C, \quad |V_\pm^+| \leqq C, \quad |V_\pm^-| \leqq C$ and passing to the limit in (52) we get, in the viscosity sense,

$$\lambda^\pm + \frac{\partial V_+^\pm}{\partial \tau} \leqq H\left(\lambda^\pm \tau + P \cdot Y + V_+^\pm,\ y,\ p + \nabla_y V_+^\pm\right),$$
$$\lambda^\pm + \frac{\partial V_-^\pm}{\partial \tau} \geqq H\left(\lambda^\pm \tau + P \cdot Y + V_-^\pm,\ y,\ p + \nabla_y V_-^\pm\right)$$

for $(\tau, Y) \in \mathbf{R} \times \mathbf{R}^{N+1}$.

Let us define $W_\alpha^\pm(\tau, y) = \lambda^\pm \tau + p \cdot y + V_\alpha^\pm$ for $\alpha = \pm$. Let us choose $k \in \mathbf{N}$ such that $k \geqq 2C$. Then the comparison principle (Proposition 1) applied to the supersolution $W_-^+ + k$ and the subsolution W_+^- implies that $W_-^+ + k \geqq W_+^-$ and then $\lambda^+ \geqq \lambda^-$. Similarly, comparing $W_-^- + k$ with W_+^+ we get $\lambda^- \geqq \lambda^+$, which proves that $\lambda^+ = \lambda^- =: \lambda$. Let us consider the solution w of

$$\begin{cases} w_\tau = H(p \cdot y + w, y, p + \nabla w) & \text{for } (\tau, y) \in (0, +\infty) \times \mathbf{R}^N, \\ w(0, y) = 0 & \text{for } y \in \mathbf{R}^N \end{cases} \tag{58}$$

and the solution W of

$$\begin{cases} W_\tau = H(P \cdot Y + W, y, p + \nabla_y W) & \text{for } (\tau, Y) \in (0, +\infty) \times \mathbf{R}^{N+1}, \\ W(0, Y) = 0 & \text{for } Y \in \mathbf{R}^{N+1}. \end{cases}$$

By Lemma 1, we have $w(\tau, y) = W(\tau, y, 0)$. The comparison principle implies $W_+^+ - k \leqq W \leqq W_-^+ + k$, which proves that $\frac{W(\tau,y)}{\tau} \to \lambda$ as $\tau \to +\infty$ uniformly for $Y \in \mathbf{R}^{N+1}$, and then $\frac{w(\tau,y)}{\tau} \to \lambda$ as $\tau \to +\infty$ uniformly for $y \in \mathbf{R}^N$. This ends the proof of Theorem 1. □

Proof of Proposition 7. Simply apply Proposition 8 for K large enough, with $\delta = 1/\sqrt{K}$, and set $V_K^\pm = V_K^{\frac{1}{\sqrt{K}},\pm}$. Checking that $\beta_K^\pm \to \beta_0^\pm$ is very similar to the proof of Theorem 1. □

6. Proof of ergodicity and construction of approximate correctors

In this Section, we prove the existence of exact correctors for approximate Hamiltonians, namely Proposition 8.

We first introduce the zeroth-order nonlocal operator:

$$\mathcal{I}(v)(Y) = \int_{\mathbf{R}^{N+1}} \mathrm{d}Z \; J(Z) \; (v(Y-Z) - v(Y)),$$

where the function J satisfies (13).

Next, we consider the solution of the following equation on $(0, +\infty) \times \mathbf{R}^{N+1}$

$$\begin{cases} \frac{\partial W_K^{\delta,\pm}}{\partial \tau} = \varepsilon\, \mathcal{I}\left(W_K^{\delta,\pm}(\tau,\cdot)\right) + \beta + H_K^{\delta,\pm}\left(P \cdot Y + W_K^{\delta,\pm},\; y,\; P + \nabla W_K^{\delta,\pm}\right) \\ W_K^{\delta,\pm}(0,Y) = 0 \quad \text{for} \quad Y \in \mathbf{R}^{N+1} \end{cases} \tag{59}$$

and prove the following result:

Proposition 9. (A priori estimate for the problem with initial conditions and $\varepsilon \geqq 0$) *Let $p \in \mathbf{R}^N$, $P = (p, 1) \in \mathbf{R}^N \times \mathbf{R}$. Let us consider the truncated Hamiltonians $H_K^{\delta,+}$ and $H_K^{\delta,-}$ defined by* (35) *and* (45) *for $K > \sqrt{1+|p|^2}$ large enough and for $\delta K > 0$ large enough, and H satisfying* (A1)–(A3). *For any given $\beta \in \mathbf{R}$, let us consider the solution $W_K^{\delta,\pm}$ to* (59).

$\mathbf{L}^\infty$ a priori bounds on the solution. *Let us define*

$$R_K^{\delta,\pm} = \inf\left\{R \geqq 0, \quad \Omega_K^{\delta,\pm} \subset B_R(0)\right\}$$

and $C_K^{\delta,\pm} := 1 + \sqrt{N+1} \cdot \left(|P| + R_K^{\delta,\pm}\right)$. Then we have for all $(\tau, Y, Y') \in \mathbf{R} \times \mathbf{R}^{N+1} \times \mathbf{R}^{N+1}$:

$$|W_K^{\delta,\pm}(\tau, Y') - W_K^{\delta,\pm}(\tau, Y)| \leqq C_K^{\delta,\pm}. \tag{60}$$

Moreover there exist real numbers $\lambda_K^{\delta,\pm}(\beta,\varepsilon)$ such that the maps $\beta \mapsto \lambda_K^{\delta,\pm}(\beta,\varepsilon)$ are continuous, nondecreasing and with $\lambda_K^{\delta,\pm} = \lambda_K^{\delta,\pm}(\beta,\varepsilon)$ and for any $\tau', \tau \geqq 0, \quad Y', Y \in \mathbf{R}^{N+1}$:

$$|W_K^{\delta,\pm}(\tau', Y') - W_K^{\delta,\pm}(\tau, Y) - \lambda_K^{\delta,\pm} \cdot \left(\tau' - \tau\right)| \leqq 7 C_K^{\delta,\pm}. \tag{61}$$

A priori bounds on the derivatives of the solution. *Moreover, $W_K^{\delta,+}$ is Lipschitz continuous, with respect to (τ, Y) and we have the following a priori bounds*:

$$\begin{cases} \left(P + \nabla W_K^{\delta,+}(\tau, Y)\right) \in \Omega_K^{\delta,+}, \\ 0 \leqq 1 + \frac{\partial W_K^{\delta,+}}{\partial y_{N+1}}(\tau, Y) \leqq \frac{2h^\delta(K) - \inf H}{\delta}, \end{cases} \tag{62}$$

$$-\varepsilon\, C_K^{\delta,+} + \beta + \inf H \leqq \frac{\partial W_K^{\delta,+}}{\partial \tau}(\tau, Y) \leqq \varepsilon\, C_K^{\delta,+} + \beta + M_K^{\delta,+}, \tag{63}$$

$$-\varepsilon\, C_K^{\delta,+} + \beta + \inf H \leqq \lambda_K^{\delta,+}(\beta,\varepsilon) \leqq \varepsilon\, C_K^{\delta,+} + \beta + M_K^{\delta,+} \tag{64}$$

and

$$\begin{cases} \left(P + \nabla W_K^{\delta,-}(\tau, Y)\right) \in \Omega_K^{\delta,-}, \\ 0 \leqq 1 + \frac{\partial W_K^{\delta,-}}{\partial y_{N+1}}(\tau, Y) \leqq \frac{2\check{h}^{\delta}(K)+2h(K)}{\delta}, \end{cases} \tag{65}$$

$$-\varepsilon\, C_K^{\delta,-} + \beta + m_K^{\delta,-} \leqq \frac{\partial W_K^{\delta,-}}{\partial \tau}(\tau, Y) \leqq \varepsilon\, C_K^{\delta,-} + \beta + 2h(K), \tag{66}$$

$$-\varepsilon\, C_K^{\delta,-} + \beta + m_K^{\delta,-} \leqq \lambda_K^{\delta,-}(\beta, \varepsilon) \leqq \varepsilon\, C_K^{\delta,-} + \beta + 2h(K), \tag{67}$$

where $m_K^{\delta,-}$ *and* $M_K^{\delta,+}$ *are defined by* (43) *and* (39)*, respectively.*

Further properties of the solution. *The correctors satisfy*

$$\begin{cases} W_K^{\delta,\pm}(\tau, y, y_{N+1} + 1) = W_K^{\delta,\pm}(\tau, y, y_{N+1}), \\ |W_K^{\delta,\pm}(\tau, y + k, y_{N+1}) - W_K^{\delta,\pm}(\tau, y, y_{N+1})| \leqq 1, \quad \textit{for every } k \in \mathbf{Z}^N. \end{cases} \tag{68}$$

If $p = P/Q$ *with* $P \in \mathbf{Z}^N$ *and* $Q \in \mathbf{N}\setminus\{0\}$*, then*

$$W_K^{\delta,\pm}(\tau, y + Qk, y_{N+1}) = W_K^{\delta,\pm}(\tau, y, y_{N+1}) \quad \textit{for every } k \in \mathbf{Z}^N. \tag{69}$$

Proof of Proposition 9. We perform the proof in several steps.

Step 1: Existence, uniqueness and a priori bounds on the gradient. By Proposition 4, we know that there exists a unique and continuous solution $W_K^{\delta,\pm}$ to (59).

We now remark that $U_K^{\delta,\pm} = P \cdot Y + W_K^{\delta,\pm}$ satisfies

$$\frac{\partial U_K^{\delta,\pm}}{\partial \tau} = \varepsilon\, \mathcal{I}\left(U_K^{\delta,\pm}(\tau, \cdot)\right) + \beta + H_K^{\delta,\pm}(U,\ y,\ \nabla U)\,. \tag{70}$$

We know that

$$\nabla U_K^{\delta,\pm}(0, Y) = P \in \Omega_K^{\delta,\pm} \quad \text{for every } Y \in \mathbf{R}^{N+1}$$

with $\Omega_K^{\delta,\pm}$ star shaped with respect to the origin, and

$$H_K^{\delta,+}(\cdot, \cdot, Q) = \text{ constant if } Q \in \mathbf{R}^{N+1}\setminus\Omega_K^{\delta,+}. \tag{71}$$

By Corollary 1 and the fact that the function r is nondecreasing, we deduce that

$$\nabla U_K^{\delta,+}(\tau, Y) \in \Omega_K^{\delta,+} \quad \text{for every } (\tau, Y) \in [0, +\infty) \times \mathbf{R}^{N+1}.$$

Similarly we conclude that

$$\nabla U_K^{\delta,-}(\tau, Y) \in \Omega_K^{\delta,-} \quad \text{for every } (\tau, Y) \in [0, +\infty) \times \mathbf{R}^{N+1},$$

where (71) is replaced by

$$H_K^{\delta,-}(\cdot,\cdot,Q) = \begin{cases} f_K^\delta(q_{N+1}) & \text{if } Q=(q,q_{N+1}) \in \mathbf{R}^{N+1}\backslash\Omega_K^{\delta,-} \\ \text{constant} & \text{if } |q_{N+1}| \geqq \frac{2\check{h}^\delta(K)+2h(K)}{\delta} \end{cases}$$

and we use the fact that

$$\Omega_K^{\delta,-} = B_{r_K}(0) \times \left\{ q_{N+1} \in \mathbf{R}, |q_{N+1}| \leqq \frac{2\check{h}^\delta(K)+2h(K)}{\delta} \right\}$$

Finally, we remark that

$$0 \leqq \frac{\partial U_K^{\delta,\pm}}{\partial y_{N+1}}(0,Y) = 1 \quad \text{for every } Y \in \mathbf{R}^{N+1}$$

and then by Proposition 6, we deduce that

$$0 \leqq \frac{\partial U_K^{\delta,\pm}}{\partial y_{N+1}}(\tau,Y) \quad \text{for every } (\tau,Y) \in [0,+\infty) \times \mathbf{R}^{N+1}$$

All these results on $U_K^{\delta,\pm}$ prove (62) and (65) for $W_K^{\delta,\pm}$. It is now easy to obtain (63) and (66).

Step 2: Properties of the solution by integer translations. We first remark that

$$U_K^{\delta,\pm}(0,y,y_{N+1}+1) - 1 = U_K^{\delta,\pm}(0,y,y_{N+1}).$$

From the invariance of (70) under translation in y_{N+1}, and by integer addition to the solution, we deduce that for all $\tau \geqq 0$:

$$U_K^{\delta,\pm}(\tau,y,y_{N+1}+1) - 1 = U_K^{\delta,\pm}(\tau,y,y_{N+1}).$$

This proves the first line of (68).
Moreover, for a given $k \in \mathbf{Z}^N$, we set $p \cdot k = l + \alpha$, with $l \in \mathbf{Z}$ and $\alpha \in [0,1)$. Then we have

$$l \leqq U_K^{\delta,\pm}(0,y+k,y_{N+1}) - U_K^{\delta,\pm}(0,y,y_{N+1}) \leqq l+1.$$

From the comparison principle, and the various invariances under integer translations of the equation, we deduce that for all $\tau \geqq 0$:

$$l \leqq U_K^{\delta,\pm}(\tau,y+k,y_{N+1}) - U_K^{\delta,\pm}(\tau,y,y_{N+1}) \leqq l+1.$$

This proves the second line of (68).
Finally if $p = P/Q$ with $P \in \mathbf{Z}^N$ and $Q \in \mathbf{N}\backslash\{0\}$, then

$$U_K^{\delta,\pm}(0,y+Qk,y_{N+1}) - P \cdot k = U_K^{\delta,\pm}(0,y,y_{N+1})$$

and then

$$U_K^{\delta,\pm}(\tau, y + Qk, y_{N+1}) - P \cdot k = U_K^{\delta,\pm}(\psi, y, y_{N+1}).$$

This proves (69).

Step 3: Control of the oscillations in space and consequences. Let us consider $Y, Y' \in \mathbf{R}^{N+1}$, and let us write $Y' - Y = L + \gamma$ with $L \in \mathbf{Z}^{N+1}$, $\gamma \in [0,1)^{N+1}$. Then

$$\begin{aligned}
&|W_K^{\delta,\pm}(\tau, Y') - W_K^{\delta,\pm}(\tau, Y)| \\
&\quad \leqq |W_K^{\delta,\pm}(\tau, Y + L + \gamma) - W_K^{\delta,\pm}(\tau, Y + \gamma)| \\
&\qquad + |W_K^{\delta,\pm}(\tau, Y + \gamma) - W_K^{\delta,\pm}(\tau, Y)| \\
&\quad \leqq 1 + \sqrt{N+1} \cdot \left(|P| + R_K^{\delta,\pm}\right) =: C_K^{\delta,\pm},
\end{aligned}$$

where in the last line, we used (68) for estimate by integer translations, and the estimates on the gradient (62)–(65) to bound the variation of $W_K^{\delta,\pm}$ on the cube $[0,1)^{N+1}$. As a consequence of estimates (60), estimates (63)–(66) are true in the viscosity sense (and then in the sense of Definition 1).

Step 4: Control of the oscillations in time. We first set $w = W_K^{\delta,\pm}$ to simplify the notation. In order to control the oscillations in time of w, we define two continuous functions by:

$$\begin{aligned}
\lambda_+(T) &= \sup_{\tau \geqq T} \frac{w(\tau + T, 0) - w(\tau - T, 0)}{2T} \\
\lambda_-(T) &= \inf_{\tau \geqq T} \frac{w(\tau + T, 0) - w(\tau - T, 0)}{2T},
\end{aligned}$$

which satisfy $\lambda_-(T) \leqq \lambda_+(T)$. From (63), (66), we deduce that

$$-\varepsilon\, C_K^{\delta,+} + \beta + \inf H \leqq \lambda_-(T) \leqq \lambda_+(T) \leqq \varepsilon\, C_K^{\delta,+} + \beta + M_K^{\delta,+} \tag{72}$$

if $w = W_K^{\delta,+}$, and

$$-\varepsilon\, C_K^{\delta,-} + \beta + m_K^{\delta,-} \leqq \lambda_-(T) \leqq \lambda_+(T) \leqq \varepsilon\, C_K^{\delta,-} + \beta + 2h(K) \tag{73}$$

if $w = W_K^{\delta,-}$. By definition of $\lambda_\pm(T)$, for any $\delta > 0$, there exists $\tau_\pm \geqq T$ such that:

$$\left| \lambda_\pm(T) - \frac{w(\tau_\pm + T, 0) - w(\tau_\pm - T, 0)}{2T} \right| \leqq \delta.$$

From (60), we see that w satisfies for $\tau \geqq T$:

$$|w(\tau - T, Y) - w(\tau - T, 0)| \leqq C_0 := C_K^{\delta,\pm} \geqq 1. \tag{74}$$

Let us define $k \in \mathbf{Z}$ such that $2C_0 < w(\tau_- - T, 0) + k - w(\tau_+ - T, 0) \leqq 3C_0$. Then from (74), we deduce that for every $Y \in \mathbf{R}^N$

$$0 < w(\tau_- - T, Y) + k - w(\tau_+ - T, Y) \leqq 5C_0.$$

From the comparison principle, we deduce that for every $Y \in \mathbf{R}^N$

$$0 \leqq w(\tau_- + T, Y) + k - w(\tau_+ + T, Y) \leqq 5C_0.$$

Therefore we deduce

$$\begin{aligned} -5C_0 &\leqq (w(\tau_- + T, Y) - w(\tau_- - T, Y)) - (w(\tau_+ + T, Y) - w(\tau_+ - T, Y)) \\ &\leqq 5C_0 \end{aligned}$$

and then

$$|\lambda_+(T) - \lambda_-(T)| \leqq 2\delta + \frac{5C_0}{2T}$$

and because $\delta > 0$ is arbitrarily small we deduce that

$$|\lambda_+(T) - \lambda_-(T)| \leqq \frac{5C_0}{2T}. \tag{75}$$

Now let us consider $T_1 > 0$ and $T_2 > 0$ such that $T_2/T_1 = P/Q$ with $P, Q \in \mathbf{N}\setminus\{0\}$. Remark that the following inequality holds true:

$$\begin{aligned} \lambda_+(PT_1) &= \sup_{\tau \geqq PT_1} \sum_{i=1}^{P} \frac{w(\tau + 2iT_1 - PT_1, 0) - w(\tau + 2(i-1)T_1 - PT_1, 0)}{2PT_1} \\ &\leqq \sum_{i=1}^{P} \frac{\lambda_+(T_1)}{P} = \lambda_+(T_1). \end{aligned}$$

Similarly, we get $\lambda_-(QT_2) \geqq \lambda_-(T_2)$. Then we have

$$\lambda_+(T_1) \geqq \lambda_+(PT_1) = \lambda_+(QT_2) \geqq \lambda_-(QT_2) \geqq \lambda_-(T_2) \geqq \lambda_+(T_2) - \frac{5C_0}{2T_2}.$$

By symmetry we deduce that

$$|\lambda_+(T_2) - \lambda_+(T_1)| \leqq \max\left(\frac{5C_0}{2T_2}, \frac{5C_0}{2T_1}\right) \tag{76}$$

and similarly

$$|\lambda_-(T_2) - \lambda_-(T_1)| \leqq \max\left(\frac{5C_0}{2T_2}, \frac{5C_0}{2T_1}\right). \tag{77}$$

Since $\lambda_\pm$ are continuous, we can extend inequalities (76)–(77) to the case $T_2/T_1 \notin \mathbf{Q}$. Eventually, inequalities (76), (77) and (75) imply the existence of the following limits

$$\lim_{T \to +\infty} \lambda_+(T) = \lim_{T \to +\infty} \lambda_-(T) = \lambda$$

and we deduce that

$$|\lambda_\pm(T) - \lambda| \leqq \frac{5C_0}{2T}. \tag{78}$$

Combining (78) and (74), we conclude that for any $\tau, \sigma \geqq 0$ and $Y, Z \in \mathbf{R}^{N+1}$,

$$|w(\tau, Y) - w(\sigma, Z) - \lambda(\tau - \sigma)| \leqq 7C_0.$$

This proves (60) with $w = W_K^{\delta,\pm}$, $C_0 = C_K^{\delta,\pm}$, and $\lambda := \lambda_K^{\delta,\pm}$. Moreover, by (72)–(73), we deduce (64)–(67).
This ends the proof of Proposition 9. □

Proof of Proposition 8. We perform the proof in several steps.

Step 1: Construction of a global solution. We first consider the sequence of functions for $n \in \mathbf{N}$

$$w^n(\tau, Y) = W_K^{\delta,\pm}(\tau + n, Y) - k^n \text{ with } k^n \in \mathbf{Z} \text{ such that } W_K^{\delta,\pm}(n, 0) - k^n \in [0, 1)$$

where $W_K^{\delta,\pm}$ is given by Proposition 9 for $\varepsilon > 0$. The first derivatives of w^n with respect to space and time are then bounded independently on n, and we can extract a subsequence that converges to a function w^∞ that is defined on the whole space and for all time, and satisfies for $(\tau, Y) \in \mathbf{R} \times \mathbf{R}^{N+1}$

$$\frac{\partial w^\infty}{\partial \tau} = \varepsilon\, \mathcal{I}\left(w^\infty(\tau, \cdot)\right) + \beta + H_K^{\delta,\pm}\left(P \cdot Y + w^\infty,\ y,\ P + \nabla w^\infty\right).$$

Moreover w^∞ is globally Lipschitz continuous in space and time, with a priori estimates given in Proposition 9.

Step 2: Construction of a periodic-in-time solution. We consider the vector $e^0 = (\lambda, 0, \dots, 0, 1) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}$ and an arbitrary vector $\nu^0 = (\nu^0_\tau, 0, \dots, 0, \nu^0_{N+1}) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}$ such that $e^0 \cdot \nu^0 = \lambda \nu^0_\tau + \nu^0_{N+1} > 0$, and define

$$u^\infty(\tau, Y) = w^\infty(\tau, Y) + P \cdot Y$$

which satisfies

$$\frac{\partial u^\infty}{\partial \tau} = \varepsilon\, \mathcal{I}\left(u^\infty(\tau, \cdot)\right) + \beta + H_K^{\delta,\pm}\left(u^\infty,\ y,\ \nabla u^\infty\right) \quad \text{for } (\tau, Y) \in \mathbf{R} \times \mathbf{R}^{N+1} \tag{79}$$

and for $\sigma \in \mathbf{R}$

$$\overline{u}^\sigma(\tau, Y) = u^\infty\left((\tau, Y) - \sigma \nu^0\right) + 1.$$

By construction we have $\overline{u}^0 = u^\infty + 1$. We define

$$\sigma^0 = \sup\left\{\sigma \geqq 0, \overline{u}^{\sigma'} \geqq u^\infty \text{ for } 0 \leqq \sigma' \leqq \sigma\right\}.$$

By the uniform Lipschitz continuity of u^∞ in space and time, we deduce that $\sigma^0 > 0$. Moreover, w^∞ satisfies (61), that is,

$$|w^\infty(\tau', Y') - w^\infty(\tau, Y) - \lambda \cdot \left(\tau' - \tau\right)| \leqq 7C_K^{\delta,\pm}$$

for every $\tau', \tau \in \mathbf{R}$, $Y', Y \in \mathbf{R}^{N+1}$ and then

$$|u^\infty(\tau', Y') - u^\infty(\tau, Y) - \lambda \cdot \left(\tau' - \tau\right) - P \cdot (Y' - Y)| \leqq 7C_K^{\delta,\pm}$$

for every $\tau', \tau \in \mathbf{R}$, $Y', Y \in \mathbf{R}^{N+1}$. Then using the fact that $P = (p, 1) \in \mathbf{R}^N \times \mathbf{R}$, we get:

$$|u^\infty((\tau, Y) - \sigma\nu^0) - u^\infty(\tau, Y) + e^0 \cdot \sigma\nu^0| \leqq 7C_K^{\delta,\pm} \tag{80}$$

for every $\tau \in \mathbf{R}$, $Y \in \mathbf{R}^{N+1}$ and

$$u^\infty(\cdot - \sigma\nu^0) \leqq u^\infty - \sigma\ (e^0 \cdot \nu^0) + 7C_K^{\delta,\pm}. \tag{81}$$

Therefore, for σ large enough, we deduce that $\overline{u}^\sigma < u^\infty$. We conclude that $\sigma^0 < +\infty$.

From the definition of σ^0, we deduce that there exists a sequence $P^n = (\tau^n, Y^n) \in \mathbf{R} \times \mathbf{R}^{N+1}$ such that

$$\begin{cases} u^\infty(P^n) - \overline{u}^{\sigma^0}(P^n) \to 0, \\ u^\infty \leqq \overline{u}^{\sigma^0}, \\ u^\infty \text{ satisfies (79).} \end{cases}$$

We set $Y^n = L^n + Z^n$ with $L^n \in \mathbf{Z}^{N+1}$ and $Z^n \in [0, 1)^{N+1}$, and define

$$u^{\infty,n}(\tau, Y) = u^\infty(\tau + \tau^n, Y + L^n) - k^n \quad \text{with } k^n \in \mathbf{Z}$$

such that $u^\infty(\tau^n, L^n) - k^n \in [0, 1)$. Up to extraction of a subsequence, we can assume that $u^{\infty,n}$ converges to a function $u^{\infty,\infty}$ and Z^n converges to $Z^\infty \in [0, 1]^{N+1}$ such that:

$$\begin{cases} u^{\infty,\infty}(0, Z^\infty) = u^{\infty,\infty}\left((0, Z^\infty) - \sigma^0\nu^0\right) + 1, \\ u^{\infty,\infty} \leqq u^{\infty,\infty}\left(\cdot - \sigma^0\nu^0\right) + 1, \\ \text{the functions } u^{\infty,\infty} \text{ and } u^{\infty,\infty}\left(\cdot - \sigma^0\nu^0\right) + 1 \text{ are solutions of (79),} \end{cases}$$

where we used at the last line the invariance of (79) by translations in τ and y_{N+1}, and its invariance by addition of integers to the solution. From the strong maximum principle (only applied for $\varepsilon > 0$), we deduce that

$$u^{\infty,\infty} = u^{\infty,\infty}\left(\cdot - \sigma^0\nu^0\right) + 1.$$

From (80), we deduce that for $k \in \mathbf{Z}$

$$|u^{\infty,\infty}((\tau, Y) - k\,\sigma^0\nu^0) - u^{\infty,\infty}(\tau, Y) + k\,\sigma^0(e^0 \cdot \nu^0)| \leqq 7C_K^{\delta,\pm} \tag{82}$$

for every $\tau \in \mathbf{R}$, $Y \in \mathbf{R}^{N+1}$ and then we deduce (taking $k \to +\infty$) that $\sigma^0\ (e^0 \cdot \nu^0) = 1$. As a consequence of our proof, we get that

$$v^{\infty,\infty}(\tau, Y) = u^{\infty,\infty}(\tau, Y) - \lambda\tau - P \cdot Y$$

is $(\sigma^0 e^0)$-periodic.

In particular, when $\lambda \neq 0$, we can choose $\nu^0 = (\text{sign}(\lambda), 0, \ldots, 0, 0) \in \mathbf{R} \times \mathbf{R}^{N+1}$. This shows that $v^{\infty,\infty}(\tau, Y)$ is $\frac{1}{|\lambda|}$-periodic in τ and 1-periodic in y_{N+1} (which was already known for $W_K^{\delta,\pm}$ in Proposition 9).

In the general case, we simply consider $\nu^0 = (\nu_\tau^0, 0, \ldots, 0, \nu_{N+1}^0) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}$ with $\nu_\tau^0 \neq 0$ and $e^0 \cdot \nu^0 > 0$, which implies that $v^{\infty,\infty}(\tau, Y)$ is $(\sigma^0\nu^0)$-periodic and 1-periodic in y_{N+1}.

Step 3: Improving the periodicity by the sliding method. First, for a reference on the sliding method, see in particular Berestycki and Nirenberg [11]. For any vector $\nu = (\nu_\tau, 0, \ldots, 0, \nu_{N+1}) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}$ such that $e^0 \cdot \nu > 0$, we define

$$\tilde{u}^\sigma(\tau, Y) = u^{\infty,\infty}((\tau, Y) - \sigma\nu).$$

We set

$$\sigma_* = \inf\left\{\sigma \geqq 0, \tilde{u}^{\sigma'} \leqq \tilde{u} \text{ for } \sigma' \geqq \sigma\right\}.$$

The estimate (81) is still true for $u^{\infty,\infty}$ with ν^0 replaced by ν, and so we deduce that $\sigma_* < +\infty$.

As in Step 2, we get the existence of a sequence P^n such that $\tilde{u}^{\sigma_*}(P^n) - u^{\infty,\infty}(P^n) \to 0$, and can define a new sequence of functions (which are translations of $u^{\infty,\infty}$) such that the limit $u^{\infty,\infty,\infty}$ has the same properties as $u^{\infty,\infty}$ and satisfies moreover

$$\begin{aligned} &u^{\infty,\infty,\infty}(\cdot - \sigma_*\nu) = u^{\infty,\infty,\infty} \\ &u^{\infty,\infty,\infty}(\cdot - \sigma\nu) \leqq u^{\infty,\infty,\infty} \quad \text{for every} \quad \sigma \geqq \sigma_*. \end{aligned}$$

From estimate (82) applied to $u^{\infty,\infty,\infty}$, ν^0 replaced by ν, and σ^0 replaced by σ_*, for integers $k \to +\infty$, we deduce that $\sigma_* = 0$, and that $u^{\infty,\infty,\infty}$ is nondecreasing in the direction ν.

We can now follow this construction for a sequence of vectors

$$\nu^n = (\nu_\tau^n, 0, \ldots, 0, \nu_{N+1}^n) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}$$

with $e^0 \cdot \nu^n > 0$ and $\nu^n \to \pm(e^0)^\perp$ with

$$(e^0)^\perp = (-1, 0, \ldots, 0, \lambda) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}.$$

Up to the extraction of a subsequence of the translated functions at each step, we get at the limit of this process a function that we simply denote by u, such that u is nondecreasing in each direction $\pm(e^0)^\perp$. This means that u is independent on the direction $(e^0)^\perp$, that is, only depends on the coordinates $(\tau, Y) - \frac{((\tau,Y)\cdot(e^0)^\perp)\,(e^0)^\perp}{|(e^0)^\perp|^2}$, that is,

$$u(\tau, y, y_{N+1}) = u(0, y, y_{N+1} + \lambda\tau). \tag{83}$$

In particular the function $u(\tau, Y) - P \cdot Y$ satisfies on $\mathbf{R} \times \mathbf{R}^{N+1}$ all the properties given in Proposition 9. Finally we define the corrector as

$$V_K^{\delta,\pm}(\tau, Y) := u(\tau, Y) - \lambda\tau - P \cdot Y.$$

Step 4: The limit $\varepsilon \to 0$. We now take the limit $\varepsilon \to 0$ and up to the extraction of a subsequence, we get a limit corrector still denoted by $V_K^{\delta,\pm}(\tau, Y)$ and a limit $\lambda_K^{\delta,\pm} =: \lambda_K^{\delta,\pm}(\beta)$, such that $w^\pm(\tau, Y) = V_K^{\delta,\pm}(\tau, Y) + \lambda_K^{\delta,\pm}\,\tau$ satisfies on $\mathbf{R}\times\mathbf{R}^{N+1}$ all the properties given in Proposition 9 for $\varepsilon = 0$. Moreover, from (83), we deduce the following invariance of the corrector

$$V_K^{\delta,\pm}(\tau, y, y_{N+1}) = V_K^{\delta,\pm}(0, y, y_{N+1} + \lambda_K^{\delta,\pm}\tau) \tag{84}$$

and we have

$$\lambda_K^{\delta,\pm} + \frac{\partial V_K^{\delta,\pm}}{\partial \tau} = \beta + H_K^{\delta,\pm}\left(\lambda_K^{\delta,\pm}\tau + P \cdot Y + V_K^{\delta,\pm},\ y,\ P + \nabla V_K^{\delta,\pm}\right). \tag{85}$$

Step 5: New estimates on $\lambda_K^{\delta,\pm}$ and consequences. Let us consider the supremum of $V_K^{\delta,\pm}$. We have two cases: either the supremum is reached at a point P^0 or "it is reached at infinity".

CASE 1: $\sup_{\mathbf{R}\times\mathbf{R}^{N+1}} V_K^{\delta,\pm} = V_K^{\delta,\pm}(P^0)$. Let us set $P^0 = (\tau^0, Y^0)$ with $Y^0 = (y^0, y_{N+1}^0)$. Looking at this point, we deduce that

$$\lambda_K^{\delta,\pm} \leqq \beta + H_K^{\delta,\pm}(\lambda_K^{\delta,\pm}\tau^0 + P \cdot Y^0 + V_K^{\delta,\pm}(P^0), y^0, P).$$

Using the fact that

$$H_K^{\delta,\pm}(\cdot, \cdot, Q) = H^{\pm\delta}(\cdot, \cdot, Q) \quad \text{for } |Q| \leqq K$$

and $K > |P|$, we deduce that

$$\lambda_K^{\delta,\pm} \leqq \beta \pm \delta + h(|p|), \tag{86}$$

where h is defined by (33).

CASE 2: There exists $(P^n)_n$ such that $V_K^{\delta,\pm}(P^n) \to \sup_{\mathbf{R}\times\mathbf{R}^{N+1}} V_K^{\delta,\pm}$ with $|P^n| \to +\infty$. In this case, up to consider the limit of translated functions, we come back to Case 1 for this limit function which is still a solution of the PDE (85). This proves (86) in Case 2.

Similarly, looking at the infimum of $V_K^{\delta,\pm}$, we get:

$$\beta \pm \delta + \inf_{(v,y)\in\mathbf{R}\times\mathbf{R}^N} H(v,y,p) \leqq \lambda_K^{\delta,\pm}. \tag{87}$$

Then (86) and (87) implies (51). By subtraction of (51) to (63) and (66), we get, respectively, (54) and (55).

Step 6: New L^∞ bound on the corrector. Let us consider

$$\overline{v}(y) = \inf_{y_{N+1}\in\mathbf{R}} V_K^{\delta,\pm}(\tau, y, y_{N+1}) = V_K^{\delta,\pm}(0, y, \overline{y}_{N+1}(y))$$

for some $\overline{y}_{N+1}(y) \in [0,1)$ The function $\overline{v}$ only depends on y (and not on τ) because of (84). By construction $\overline{v}$ is a supersolution of (85). Therefore we have

$$\lambda_K^{\delta,\pm} \geqq \beta + H_K^{\delta,\pm}\left(P\cdot(y,\overline{y}_{N+1}(y)) + \overline{v}(y),\ y,\ P + (\nabla_y\overline{v}, 0)\right) \quad \text{for all } y \in \mathbf{R}^N$$

and then by (86), we get

$$h(|p|) \pm \delta \geqq H_K^{\delta,\pm}\left(P\cdot(y,\overline{y}_{N+1}(y)) + \overline{v}(y),\ y,\ P + (\nabla_y\overline{v}, 0)\right) \quad \text{for all } y \in \mathbf{R}^N.$$

Estimate (47) implies

$$h(|p|) \geqq H\left(P\cdot(y,\overline{y}_{N+1}(y)) + \overline{v}(y),\ y,\ p + \nabla_y\overline{v}\right)$$

and then the subdifferential of $p\cdot y + \overline{v}(y)$ satisfies

$$|p + \nabla_y\overline{v}| \leqq r(h(|p|)). \tag{88}$$

We deduce that (88) is true almost everywhere. From the fact that $V_K^{\delta,\pm}$ is 1-periodic in y_{N+1} and satisfies $0 \leqq 1 + \frac{\partial V_K^{\delta,\pm}}{\partial y_{N+1}}$, we have for every $y'_{N+1}, y_{N+1} \in \mathbf{R}$ and every $\tau \in \mathbf{R},\ y \in \mathbf{R}^N$

$$|V_K^{\delta,\pm}(\tau, y, y'_{N+1}) - V_K^{\delta,\pm}(\tau, y, y_{N+1})| \leqq 1. \tag{89}$$

Now, for $\tau', \tau \in \mathbf{R}$ and $Y' = (y', y'_{N+1}),\ Y = (y, y_{N+1}) \in \mathbf{R}^{N+1}$, we set $k \in \mathbf{Z}^N$ such that $y' - (y+k) \in [0,1)^N$, and we get with $v = V_K^{\delta,\pm}$, $\lambda = \lambda_K^{\delta,\pm}$:

$$\begin{aligned}
|v(\tau',Y') - v(\tau,Y)| &= |v(0,y',y'_{N+1}+\lambda\tau') - v(0,y,y_{N+1}+\lambda\tau)|\\
&\leqq |v(0,y',y'_{N+1}+\lambda\tau') - v(0,y',\overline{y}_{N+1}(y'))|\\
&\quad + |v(0,y',\overline{y}_{N+1}(y')) - v(0,y,\overline{y}_{N+1}(y))|\\
&\quad + |v(0,y,\overline{y}_{N+1}(y)) - v(0,y,y_{N+1}+\lambda\tau)|\\
&\leqq 2 + |\overline{v}(y') - \overline{v}(y)| \leqq 3 + |\overline{v}(y') - \overline{v}(y+k)|\\
&\leqq 3 + \sqrt{N}\Big(|p| + r(h(|p|))\Big),
\end{aligned}$$

where we used successively (84), (89) twice, (57) and (88). Finally, up to the subtraction of an integer from $v(0,0)$, we can assume that $|v(0,0)| \leqq 1$ and we then get with $\tau' = 0$, $Y' = 0$

$$|v(\tau, Y)| \leqq 4 + \sqrt{N}\,(|p| + r(h(|p|)))\,,$$

which proves (53).

Step 7: Monotonicity and continuity of $\lambda_K^{\delta,\pm}(\beta)$. The proof of the monotonicity and the continuity of $\lambda_K^{\delta,\pm}(\beta)$ is similar to the proof of Theorem 1.

This ends the proof of Proposition 8. □

Appendix A: Ergodicity (again) and construction of super and subcorrectors for the original Hamiltonian

Theorem 3. (Existence of sub and supercorrectors)
Under Assumptions (A1)–(A3)*, consider $p \in \mathbf{R}^N$ and let $\lambda = \overline{H}^0(p)$. Then there exists a bounded supersolution v_+* (*resp. subsolution v_-*) *of* (4).

Before to make the proof of Theorem 3, let us show a simple corollary:

Proof. (*A second proof of Theorem 1*) Let us consider the solution w of

$$\begin{cases} w_\tau = H(p \cdot y + w, y, p + \nabla w) & \text{for } (\tau, y) \in (0, +\infty) \times \mathbf{R}^N, \\ w(0, y) = 0 & \text{for } y \in \mathbf{R}^N. \end{cases} \tag{A.90}$$

With $v_\pm$ given by Theorem 3 and an integer $k \geqq C \geqq |v_\pm|$, the comparison principle implies

$$v_- + \lambda\tau - k \leqq w \leqq v_+ + \lambda\tau + k,$$

which proves that $\frac{w(\tau,y)}{\tau} \to \lambda$ as $\tau \to +\infty$ uniformly for $y \in \mathbf{R}^N$. This ends the proof of Theorem 1. □

Proof of Theorem 3.
Case 1 : $\lambda \neq 0$
Let us apply Proposition 8 with $\beta = 0$. We have

$$|\lambda_K^{\delta,\pm}| \leqq C, |V_K^{\delta,\pm}| \leqq C$$

for some constant C independent of δ small enough and K large enough. Let us call $\lambda = \lambda^\pm$ the limit of $\lambda_K^{\delta,\pm}$ (see the proof of Theorem 1 to check the equality $\lambda = \lambda^+ = \lambda^-$) for a subsequence of $(\delta, K) \to (0, +\infty)$. If $\lambda \neq 0$, we know that, in this limit, $\lambda_K^{\delta,\pm} \neq 0$, and we can define

$$v_{K,+}^{\delta,+}(\tau, y) = \inf_{\left\{y_{N+1} \in \mathbf{R},\, \tau = \sigma + \frac{y_{N+1}}{\lambda_K^{\delta,+}}\right\}} V_K^{\delta,+}(\sigma, y, y_{N+1}),$$

$$v_{K,-}^{\delta,-}(\tau, y) = \sup_{\left\{y_{N+1} \in \mathbf{R},\, \tau = \sigma + \frac{y_{N+1}}{\lambda_K^{\delta,-}}\right\}} V_K^{\delta,-}(\sigma, y, y_{N+1})$$

which are respectively super and subsolutions, that is, satisfy

$$\lambda_K^{\delta,+} + \frac{\partial v_{K,+}^{\delta,+}}{\partial \tau} \geqq H_K^{\delta,+}\left(\lambda_K^{\delta,+}\tau + p\cdot y + v_{K,+}^{\delta,+},\ y, p + \nabla v_{K,+}^{\delta,+}, 1 + \lambda_K^{\delta,+}\frac{\partial v_{K,+}^{\delta,+}}{\partial \tau}\right)$$

$$\lambda_K^{\delta,-} + \frac{\partial v_{K,-}^{\delta,-}}{\partial \tau} \leqq H_K^{\delta,-}\left(\lambda_K^{\delta,-}\tau + p\cdot y + v_{K,-}^{\delta,-},\ y, p + \nabla v_{K,-}^{\delta,-}, 1 + \lambda_K^{\delta,-}\frac{\partial v_{K,-}^{\delta,-}}{\partial \tau}\right)$$

In the limit $(\delta, K) \to (0, +\infty)$, we set

$$v_+ = \liminf{}_* v_{K,+}^{\delta,+}, \quad v_- = \limsup{}^* v_{K,-}^{\delta,-}$$

which still satisfy

$$|v_\pm| \leqq C.$$

Therefore, we get a bounded supersolution v_+ and a bounded subsolution v_-.
Case 2 : $\lambda = 0$
We proceed as in the second proof of Theorem 1, choosing some $\beta > 0$ such that $\lambda(\beta) = \lim \lambda_K^{\delta,\pm}(\beta) > 0$. For $\lambda(\beta) > 0$ arbitrarily small, this shows (by the comparison principle and the limit $\lambda(\beta) \to 0$) that the solution w to the initial value problem is bounded from above. Similarly, for $\lambda(\beta) < 0$ arbitrarily close to zero, we get that the solution w is bounded from below. Finally $w(\tau, y)$ is bounded for all $\tau > 0$ and $y \in \mathbf{R}^N$. We then define $w^n(\tau, y) = w(n + \tau, y)$ and as $n \to +\infty$, we define

$$v_+ = \liminf{}_* w^n, \quad v_- = \limsup{}^* w^n,$$

which are respectively super and subsolutions on the whole space and time. This ends the proof of Theorem 3. □

Appendix B: Proof of Proposition 1

The following proof is classical but we provide it for the reader's convenience.

When $\varepsilon = 0$, the result is classical and when $\varepsilon > 0$, we do not restrict ourselves by assuming that $\varepsilon = 1$.

We proceed in two steps. First, we prove that there exists a constant $K > 0$ such that:

$$\forall t \in (0, T), x, y \in \mathbf{R}^N, u(t, x) - v(t, y) \leqq K(1 + |x - y|). \tag{B.91}$$

To obtain such a result, it suffices to obtain the following inequality:

$$\forall t \in (0, T), x, y \in \mathbf{R}^N, u(t, x) - v(t, y) \leqq K\zeta(x - y) \tag{B.92}$$

where $\zeta(z) = \sqrt{1 + |z|^2}$. Since u and v are at most of linear growth, there exists $L > 0$ such that:

$$\forall t, s \in (0, T), x, y \in \mathbf{R}^N, u(t, x) - v(s, y) \leqq L(1 + |x| + |y|).$$

Let us consider the family of functions $\beta_R \in C^2(\mathbf{R}^N)$ parameterized by $R \geqq 1$ and introduced in [18]; we assume that they satisfy for some $C > 0$, which does not depend on R:

$$\begin{cases} \beta_R \geqq 0, \\ \liminf\limits_{|x|\to+\infty} \dfrac{\beta_R(x)}{|x|} \geqq 3L, \\ |\nabla \beta_R(x)| \leqq C, \\ \lim\limits_{R\to+\infty} \beta_R(x) = 0. \end{cases}$$

For any $K > 0$, consider the following penalized supremum:

$$M_K = \sup_{t\in(0,T],x,y\in\mathbf{R}^N} \{u(t,x) - v(t,y) - Ke^{\mu t}\zeta(x-y) - \beta_R(x)\}$$

with $\mu > 0$ to be chosen later. It suffices to prove that $M_K \leqq 0$ for some K large enough, not depending on R; indeed, by letting $R \to +\infty$ pointwise, we can conclude. In order to prove the existence of such a K, we argue by contradiction (as usual) and we suppose that $M_K > 0$ for any $K > 0$. Hence, we dedouble the time variable: for any $\nu > 0$, consider:

$$M_{K,\nu} = \sup_{t,s\in(0,T],x,y\in\mathbf{R}^N} \{u(t,x) - v(s,y) - \frac{(s-t)^2}{2\nu} - Ke^{\mu t}\zeta(x-y) - \beta_R(x)\}$$

and we see that $M_{K,\nu} \geqq M_K > 0$. This supremum is attained at $(\bar{t},\bar{s},\bar{x},\bar{y})$ and $(\bar{t},\bar{s},\bar{x},\bar{y}) \to (\tilde{t},\tilde{t},\tilde{x},\tilde{y})$ as $\nu \to 0$ with $(\tilde{t},\tilde{x},\tilde{y})$ that realizes M_K and with $\tilde{t} > 0$ if K large enough (use the fact that u_0 is uniformly continuous). Hence we are sure that $\bar{t}, \bar{s} > 0$. Let us now write the two viscosity inequalities:

$$\begin{aligned} \mu K e^{\mu\bar{t}}\zeta(\bar{x}-\bar{y}) + \frac{\bar{t}-\bar{s}}{\nu} & \\ \leqq F(u(\bar{t},\bar{x}),\bar{x},\bar{p}+\nabla\beta_R(\bar{x})) &+ \int \mathrm{d}z J(z)(u(\bar{t},\bar{x}-z) - u(\bar{t},\bar{x})), \\ \frac{\bar{t}-\bar{s}}{\nu} \geqq F(v(\bar{s},\bar{y}),\bar{y},\bar{p}) &+ \int \mathrm{d}z J(z)(v(\bar{s},\bar{y}-z) - v(\bar{s},\bar{y})) \end{aligned}$$

where $\bar{p} = Ke^{\mu\bar{t}}\nabla\zeta(\bar{x}-\bar{y})$. Notice that we can still write the viscosity inequalities if $\bar{t} = T$ or $\bar{s} = T$. Subtracting both inequalities yields:

$$\begin{aligned} Ke^{\mu\bar{t}}\zeta(\bar{x}-\bar{y}) \leqq{} & F(u(\bar{t},\bar{x}),\bar{x},\bar{p}+\nabla\beta_R(\bar{x})) - F(v(\bar{s},\bar{y}),\bar{y},\bar{p}) \\ & + \int \mathrm{d}z J(z)(u(\bar{t},\bar{x}-z) - v(\bar{s},\bar{y}-z) - (u(\bar{t},\bar{x}) - v(\bar{s},\bar{y}))). \end{aligned}$$

We deduce from $M_{K,\nu} \geqq 0$ that $u(\bar{t},\bar{x}) \geqq v(\bar{s},\bar{y})$ and by construction,

$$u(\bar{t},\bar{x}-z) - v(\bar{s},\bar{y}-z) - (u(\bar{t},\bar{x}) - v(\bar{s},\bar{y})) \leqq \beta_R(\bar{x}-z) - \beta_R(\bar{x}).$$

Using these estimates and (A1) permits one to obtain:

$$\begin{aligned}\frac{K}{\sqrt{2}}(1+|\overline{x}-\overline{y}|) \leqq K\zeta(\overline{x}-\overline{y}) \leqq Ke^{\mu\bar{t}}\zeta(\overline{x}-\overline{y}) &\leqq \gamma(1+|\overline{p}|)|\overline{x}-\overline{y}| + \gamma|\nabla\beta_R(\overline{x})| + \mathcal{I}[\beta_R](\overline{x}) \\ &\leqq \gamma|\overline{x}-\overline{y}| + C\gamma + C\mathcal{I}_1 \leqq \tilde{C}(1+|\overline{x}-\overline{y}|).\end{aligned}$$

Choosing K large enough yields the desired contradiction.

The second step consists of adapting the classical proof of the comparison principle, that is, by considering the following penalized supremum:

$$M_{\alpha,\varepsilon,\nu} = \sup_{t,s\in(0,T],x,y\in\mathbf{R}^N} \left\{ u(t,x) - v(s,y) - \frac{(t-s)^2}{2\nu} - \frac{|x-y|^2}{2\varepsilon} - \frac{\alpha}{2}|x|^2 - \eta t \right\}.$$

Let us demonstrate a contradiction if $0 < M = \sup_{\mathbf{R}^N}\{u-v\} \in (-\infty; +\infty]$. In such a case, for α, ε, and ν small enough, we have $M_{\alpha,\varepsilon,\nu} \geqq M > 0$. The supremum is attained at $(\overline{t}, \overline{s}, \overline{x}, \overline{y})$ and the initial condition ensures that $\overline{t} > 0, \overline{s} > 0$. From $M_{\alpha,\varepsilon,\nu} \geqq 0$ and (B.91), we deduce that:

$$\frac{\alpha}{2}|\overline{x}|^2 \leqq K(1+|\overline{x}-\overline{y}|) - \frac{|\overline{x}-\overline{y}|^2}{2\varepsilon} \leqq K + \sup_{r\geqq 0}\left\{Kr - \frac{1}{2\varepsilon}r^2\right\} = K + C_\varepsilon.$$

We conclude that, for a fixed $\varepsilon > 0$, $\alpha\overline{x} \to 0$ as $\alpha \to 0$. Writing both viscosity inequalities and subtracting them yields

$$\eta \leqq \gamma|\alpha\overline{x}| + \gamma\left(|\overline{x}-\overline{y}| + \frac{|\overline{x}-\overline{y}|^2}{\varepsilon}\right) + \frac{\alpha}{2}\mathcal{I}_2.$$

Letting ν, α, and ε successively go to 0, usual penalization results permit one to obtain the contradiction: $\eta \leqq 0$ and we are done.

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