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ASYMPTOTIC PROPERTIES OF ENTROPY SOLUTIONS TO FRACTAL BURGERS EQUATION*

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Abstract. We study properties of solutions of the initial value problem for the nonlinear and nonlocal equation $u_t + (-\partial_x^2)^{\alpha/2} u + uu_x = 0$ with $\alpha \in (0, 1]$, supplemented with an initial datum approaching the constant states $u_\pm$ ($u_- < u_+$) as $x \to \pm\infty$, respectively. It was shown by Karch, Miao, and Xu [*SIAM J. Math. Anal.*, 39 (2008), pp. 1536–1549] that, for $\alpha \in (1, 2)$, the large time asymptotics of solutions is described by rarefaction waves. The goal of this paper is to show that the asymptotic profile of solutions changes for $\alpha \le 1$. If $\alpha = 1$, there exists a self-similar solution to the equation which describes the large time asymptotics of other solutions. In the case $\alpha \in (0, 1)$, we show that the nonlinearity of the equation is negligible in the large time asymptotic expansion of solutions.

Key words. fractal Burgers equation, asymptotic behavior of solutions, self-similar solutions, entropy solutions

AMS subject classifications. 35K05, 35K15

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1. Introduction.

In this work, we continue the study of asymptotic properties of solutions of the Cauchy problem for the following nonlocal conservation law:

$$u_t + \Lambda^\alpha u + uu_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

$$u(0, x) = u_0(x), \tag{1.2}$$

where $\Lambda^\alpha = (-\partial^2/\partial x^2)^{\alpha/2}$ is the pseudodifferential operator defined via the Fourier transform $\widehat{(\Lambda^\alpha v)}(\xi) = |\xi|^\alpha\, \widehat{v}(\xi)$. This equation is referred to as the fractal Burgers equation.

Motivated by the recent probabilistic approach to problem (1.1)–(1.2) by Jourdain, Méléard, and Woyczyński [7, 8], we assume that the initial datum u_0 is a function with bounded variation on $\mathbb{R}$:

$$u_0(x) = c + \int_{-\infty}^{x} m(dy) \tag{1.3}$$

with $c \in \mathbb{R}$ and m being a finite signed measure on $\mathbb{R}$. Moreover, we require that

$$u_0 - u_- \in L^1((-\infty, 0)) \quad \text{and} \quad u_0 - u_+ \in L^1((0, +\infty)), \tag{1.4}$$

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where

$$u_- = c \quad \text{and} \quad u_+ - u_- = \int_{\mathbb{R}} m(dx) \quad \text{satisfy} \quad u_- < u_+. \tag{1.5}$$

Here, we are going to use the notation $\|m\| = \int_{\mathbb{R}} |m|(dx)$, where $|m|$ is the total variation of the measure m.

If $c = 0$ and if m is a probability measure, the function $u_0(x)$ defined in (1.3) is the *cumulative distribution function* and this property is shared by the solution $u(t)$ for every $t > 0$ (see [7, 8]). As a consequence of our results, we describe the asymptotic behavior of the family $\{u(t)\}_{t\geq 0}$ of probability distribution functions as $t \to +\infty$ (see the summary at the end of this section).

It was shown in [9] that, under assumptions (1.3)–(1.4) and for $1 < \alpha \leq 2$, the large time asymptotics of solutions to (1.1)–(1.2) is described by the so-called *rarefaction waves*. The goal of this paper is to complete these results and to obtain universal asymptotic profiles of solutions for $0 < \alpha \leq 1$.

1.1. Known results. Let us first recall the results obtained in [9]. For $\alpha \in (1, 2]$, the initial value problem for the fractal Burgers equation (1.1)–(1.2) with $u_0 \in L^\infty(\mathbb{R})$ has the unique, smooth, global-in-time solution (cf. [5, Thm. 1.1], [6, Thm. 7]). If, moreover, the initial datum is of the form (1.3) and satisfies (1.4)–(1.5), the corresponding solution $u = u(x,t)$ behaves asymptotically when $t \to +\infty$ as the rarefaction wave (cf. [9, Thm. 1.1]). More precisely, for every $p \in (\frac{3-\alpha}{\alpha-1}, +\infty]$ there exists a constant $C > 0$ such that for all $t > 0$

$$\|u(t) - w^R(t)\|_p \leq C t^{-\frac{1}{2}[\alpha - 1 - \frac{3-\alpha}{p}]} \log(2+t) \tag{1.6}$$

($\|\cdot\|_p$ is the standard norm in $L^p(\mathbb{R})$). Here, the rarefaction wave $w^R = w^R(x,t)$ is the explicit function

$$w^R(x,t) = W^R\left(\frac{x}{t}\right) = \begin{cases} u_-, & \frac{x}{t} \leq u_-, \\ \frac{x}{t}, & u_- \leq \frac{x}{t} \leq u_+, \\ u_+, & \frac{x}{t} \geq u_+. \end{cases} \tag{1.7}$$

It is well known that w^R is the unique entropy solution of the Riemann problem for the nonviscous Burgers equation $w^R_t + w^R w^R_x = 0$.

The goal of the work is to show that, for $\alpha \in (0, 1]$, one should expect completely different asymptotic profiles of solutions. Let us notice that the initial value problem (1.1)–(1.2) has the unique global-in-time entropy solution for every $u_0 \in L^\infty(\mathbb{R})$ and $\alpha \in (0, 1]$ due to the recent work by the first author [1]. We recall this result in section 2.

1.2. Main results. Our two main results are Theorems 1.3 and 1.7, stated below. Both of them are a consequence of the following L^p-estimate of the difference of two entropy solutions.

THEOREM 1.1 (asymptotic stability). *Let $0 < \alpha \leq 1$. Assume that u and $\widetilde{u}$ are two entropy solutions of* (1.1)–(1.2) *with initial conditions u_0 and $\widetilde{u}_0$ of the form* (1.3) *with finite signed measures m and $\widetilde{m}$. Suppose, moreover, that the measure $\widetilde{m}$ of $\widetilde{u}_0$ is nonnegative and $u_0 - \widetilde{u}_0 \in L^1(\mathbb{R})$. Then for every $p \in [1, +\infty]$ there exists a constant $C > 0$ such that for all $t > 0$*

$$\|u(t) - \widetilde{u}(t)\|_p \leq C t^{-\frac{1}{\alpha}(1-\frac{1}{p})} \|u_0 - \widetilde{u}_0\|_1. \tag{1.8}$$

Remark 1.2. It is worth mentioning that this estimate is sharper than the one obtained by interpolating the L^1-contraction principle and L^∞-bounds on the solutions.

In the case $\alpha < 1$, the linear part of the fractal Burgers equation dominates the nonlinear one for large times. In the case $\alpha = 1$, both parts are balanced; indeed, self-similar solutions exist. Let us be more precise now.

For $\alpha < 1$, the Duhamel principle (see (3.3)) shows that the nonlinearity in (1.1) is negligible in the asymptotic expansion of solutions.

THEOREM 1.3 (asymptotic behavior as the linear part). *Let $0 < \alpha < 1$ and $u = u(x,t)$ be the entropy solution to* (1.1)–(1.2) *corresponding to the initial condition u_0 of the form* (1.3) *satisfying* (1.4)–(1.5). *Denote by $S_\alpha(t)U_0$ the solution of the linear equation $u_t + \Lambda^\alpha u = 0$ with the initial condition*

$$u(x,0) = U_0(x) \equiv \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \tag{1.9}$$

For every $p \in \left(\frac{1}{1-\alpha}, +\infty\right]$ there exists a constant $C > 0$ such that for all $t > 0$

$$\begin{aligned} \|u(t) - S_\alpha(t)U_0\|_p \le{}& Ct^{-\frac{1}{\alpha}(1-\frac{1}{p})}\|u_0 - U_0\|_1 \\ &+ C(u_+ - u_-)\max\{|u_+|, |u_-|\}\, t^{1-\frac{1}{\alpha}(1-\frac{1}{p})}. \end{aligned} \tag{1.10}$$

Remark 1.4. It follows from the proof of Theorem 1.3 that inequality (1.10) is valid for every $p \in [1, +\infty]$. However, its right-hand side decays only for $p \in \left(\frac{1}{1-\alpha}, +\infty\right]$.

Remark 1.5. Let us recall here the formula $S_\alpha(t)U_0 = p_\alpha(t) * U_0$, where $p_\alpha = p_\alpha(x,t)$ denotes the fundamental solution of the equation $u_t + \Lambda^\alpha u = 0$ (cf. the beginning of section 3 for its properties). Hence, changing variables in the convolution $p_\alpha(t) * U_0$, one can write the asymptotic term in (1.10) in the self-similar form $(S_\alpha(t)U_0)(x) = H_\alpha(xt^{-1/\alpha})$, where $H_\alpha(x) = (p_\alpha(1) * U_0)(x)$ is a smooth and nondecreasing function satisfying $\lim_{x\to\pm\infty} H_\alpha(x) = u_\pm$ and $\partial_x H_\alpha(x) = (u_+ - u_-)p_\alpha(x,1)$.

In the case $\alpha = 1$, we use the uniqueness result from [1] combined with a standard scaling technique to show that (1.1) has self-similar solutions. In section 4, we recall this well-known reasoning, which leads to the proof of the following theorem.

THEOREM 1.6 (existence of self-similar solutions). *Assume $\alpha = 1$. The unique entropy solution $U = U(x,t)$ of the initial value problem* (1.1)–(1.2) *with the initial condition* (1.9) *is self-similar; i.e., it has the form $U(x,t) = U\left(\frac{x}{t}, 1\right)$ for all $x \in \mathbb{R}$ and all $t > 0$.*

Our second main convergence result states that the self-similar solution $U = U(x,t)$ describes the large time asymptotics of other solutions to (1.1)–(1.2).

THEOREM 1.7 (asymptotic behavior as the self-similar solution). *Let $\alpha = 1$. Let $u = u(x,t)$ be the entropy solution to problem* (1.1)–(1.2) *corresponding to the initial condition of the form* (1.3) *satisfying* (1.4)–(1.5). *Denote by $U = U(x,t)$ the self-similar solution from Theorem* 1.6. *For every $p \in [1, +\infty]$ there exists a constant $C > 0$ such that for all $t > 0$*

$$\|u(t) - U(t)\|_p \le Ct^{-\left(1-\frac{1}{p}\right)}\|u_0 - U_0\|_1. \tag{1.11}$$

Remark 1.8 (technical remark on the BV assumption). The computations show that the constants C in (1.6), (1.8), (1.10), and (1.11) do not depend on $\|m\|$. Consequently, it is not necessary to assume that u_0 is of the form (1.3); more precisely, all our results hold true for $u_0 \in L^\infty(\mathbb{R})$ satisfying only (1.4) with given reals $u_- < u_+$. Nevertheless, we have chosen to assume that u_0 has a bounded variation, since it is natural in the probabilistic approach of [7, 8] and it simplifies the proofs; see Remark 4.2 for more details.

1.3. Properties of self-similar solutions. Let us complete the result stated in Theorem 1.7 by listing main qualitative properties of the profile $U(1)$.

THEOREM 1.9 (qualitative properties of the self-similar profile). *The self-similar solution* $U(x,t) = U\left(\frac{x}{t}, 1\right)$ *from Theorem* 1.6 *enjoys the following properties:*

p1. (Regularity) *The function* $U(1) = U(x,1)$ *is Lipschitz-continuous.*

p2. (Monotonicity and limits) $U(1)$ *is increasing and satisfies*

$$\lim_{x\to\pm\infty} U(x,1) = u_\pm.$$

p3. (Symmetry) *For all* $y \in \mathbb{R}$*, we have*

$$U(\overline{c}+y,1) = 2\overline{c} - U(\overline{c}-y,1), \quad \textit{where} \quad \overline{c} \equiv \frac{u_- + u_+}{2}.$$

p4. (Convexity) $U(1)$ *is convex (resp., concave) on* $(-\infty, \overline{c}]$ *(resp., on* $[\overline{c}, +\infty)$*).*

p5. (Decay at infinity) *We have*

$$U_x(x,1) \sim \frac{u_+ - u_-}{2\pi^2}\,|x|^{-2} \quad \textit{as} \quad |x| \to +\infty.$$

Actually, the profile $U(1) = U(x,1)$ is expected to be C_b^∞ or analytic, due to recent regularity results [11, 4, 12] for the critical fractal Burgers equation with $\alpha = 1$. It was shown that the solution is smooth whenever u_0 either is periodic or is from $L^2(\mathbb{R})$ or from a critical Besov space. Unfortunately, we do not know if those results can be adapted to any initial condition from $L^\infty(\mathbb{R})$.

Property p3 implies that $U(x(t),t)$ is a constant equal to $\overline{c}$ along the characteristic $x(t) = \overline{c}t$, with the symmetry

$$U(\overline{c}t + y, t) = 2\overline{c} - U(\overline{c}t - y, t)$$

for all $t > 0$ and $y \in \mathbb{R}$. Thus, the real number $\overline{c}$ can be interpreted as a mean celerity of the profile $U(t)$, which is the same mean celerity as for the rarefaction wave in (1.7).

In property p5, we obtain the decay at infinity which is the same as for the fundamental solutions $p_1(x,t) = t^{-1}p_1\left(xt^{-1},1\right)$ of the linear equation $u_t + \Lambda^1 u = 0$, given by the explicit formula

$$p_1(x,1) = \frac{2}{1+4\pi^2x^2}. \tag{1.12}$$

Following the terminology introduced in [3], one may say that property p5 expresses a far field asymptotics and is somewhere in relation with the results in [3] for fractal conservation laws with $\alpha \in (1,2)$, where the Duhamel principle plays a crucial role. This principle is less convenient in the critical case $\alpha = 1$, and our proof of p5 does not use it.

Finally, if $u_- = 0$ and $u_+ - u_- = 1$, property p2 means that $U(1)$ is the cumulative distribution function of some probability law $\mathcal{L}$ with density $U_x(1)$. Property p3 ensures that $\mathcal{L}$ is symmetrically distributed around its median $\overline{c}$; notice that any random variable with law $\mathcal{L}$ has no expectation, because of property p5. Properties p4–p5 make precise that the density of $\mathcal{L}$ decays around $\overline{c}$ with the same rate at infinity as for the Cauchy law with density $p_1(x, 1)$.

The probability distributions of both laws around their respective medians can be compared as follows.

THEOREM 1.10 (comparison with the Cauchy law). *Let $\mathcal{L}$ be the probability law with density $U_x(1)$, where $U = U(x, t)$ is the self-similar solution defined in Theorem 1.6, with $u_- = 0$ and $u_+ = 1$. Let X (resp., Y) be a real random variable on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with law $\mathcal{L}$ (resp., the Cauchy law* (1.12) *(with zero median)). Then, we have for all $r > 0$*

$$\mathbb{P}(|X - \overline{c}| < r) < \mathbb{P}(|Y - 0| < r),$$

where $\overline{c}$ denotes the median of X.

Remark 1.11. More can be said in order to compare random variables $X - \overline{c}$ and Y. Indeed, their cumulative distribution functions satisfy $F_{X-\overline{c}}(x) = F_Y(x) - g(x)$, where g is an explicit positive function (on the positive axis) depending on the self-similar solution of (1.1) (see (6.26)).

1.4. Probabilistic interpretation of results for $\boldsymbol{\alpha \in (0, 2]}$. To summarize, let us emphasize the probabilistic meaning of the complete asymptotic study of the fractal Burgers equation we now have in hand. We have already mentioned that the solution u of (1.1)–(1.2) supplemented with the initial datum of the form (1.3) with $c = 0$ and with a probability measure m on $\mathbb{R}$ is the cumulative distribution function for every $t \geq 0$. This family of probabilities defined by problem (1.1)–(1.2) behaves asymptotically when $t \to +\infty$ as

- the uniform distribution on the interval $[0, t]$ if $1 < \alpha \leq 2$ (see the result from [9] recalled in inequality (1.6)),
- the family of laws $\{\mathcal{L}_t\}_{t \geq 0}$ constructed in Theorem 1.6 if $\alpha = 1$ (see Theorem 1.7), and
- the symmetric α-stable laws $p_\alpha(t)$ if $0 < \alpha < 1$ (cf. Theorem 1.3 and Remark 1.5).

1.5. Organization of the article. The remainder of this paper is organized as follows. In the next section, we recall the notion of entropy solutions to (1.1)–(1.2) with $\alpha \in (0, 1]$. Results on the regularized equation (i.e., (1.1) with an additional term $-\varepsilon u_{xx}$ on the left-hand side) are gathered in section 3. The convergence of solutions as $\varepsilon \to 0$ to the regularized problem is discussed in section 4. The main asymptotic results for (1.1)–(1.2) are proved in section 5 by passage to the limit as ε goes to zero. Section 6 is devoted to the qualitative study of the self-similar profile for $\alpha = 1$. Finally, the proof of Theorem 4.1 and another technical lemma are gathered in the appendices.

2. Entropy solutions for $0 < \alpha \leq 1$.

2.1. The Lévy–Khintchine representation of $\boldsymbol{\Lambda^\alpha}$. It is well known that the operator $\Lambda^\alpha = (-\partial^2/\partial x^2)^{\alpha/2}$ for $\alpha \in (0, 2)$ has an integral representation: for every Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$ and each $r > 0$, we have

$$\Lambda^\alpha \varphi = \Lambda_r^{(\alpha)} \varphi + \Lambda_r^{(0)} \varphi, \tag{2.1}$$

where the integro-differential operators $\Lambda_r^{(\alpha)}$ and $\Lambda_r^{(0)}$ are defined by

$$\Lambda_r^{(\alpha)}\varphi(x) \equiv -G_\alpha \int_{|z|\le r} \frac{\varphi(x+z)-\varphi(x)-\varphi_x(x)z}{|z|^{1+\alpha}}\,dz, \tag{2.2}$$

$$\Lambda_r^{(0)}\varphi(x) \equiv -G_\alpha \int_{|z|> r} \frac{\varphi(x+z)-\varphi(x)}{|z|^{1+\alpha}}\,dz, \tag{2.3}$$

where $G_\alpha = \frac{\alpha\Gamma\left(\frac{1+\alpha}{2}\right)}{2\pi^{\frac{1}{2}+\alpha}\Gamma\left(1-\frac{\alpha}{2}\right)} > 0$ and Γ is Euler's function. On the basis of this formula, we can extend the domain of definition of Λ^α and consider $\Lambda_r^{(0)}$ and $\Lambda_r^{(\alpha)}$ as the operators

$$\Lambda_r^{(0)} : C_b(\mathbb{R}) \to C_b(\mathbb{R}) \quad \text{and} \quad \Lambda_r^{(\alpha)} : C_b^2(\mathbb{R}) \to C_b(\mathbb{R});$$

hence, $\Lambda^\alpha : C_b^2(\mathbb{R}) \to C_b(\mathbb{R})$.

Let us recall some properties on these operators. First, the so-called Kato inequality can be generalized to Λ^α for each $\alpha \in (0,2]$: let $\eta \in C^2(\mathbb{R})$ be convex and $\varphi \in C_b^2(\mathbb{R})$; then

$$\Lambda^\alpha \eta(u) \le \eta'(u)\Lambda^\alpha u. \tag{2.4}$$

Note that for $\alpha = 2$ we have

$$-(\eta(u))_{xx} = -\eta''(u)u_x^2 - \eta'(u)u_{xx} \le -\eta'(u)u_{xx} \quad \text{since} \quad \eta'' \ge 0.$$

If $\alpha \in (0,2)$, inequality (2.4) is the direct consequence of the integral representation (2.1)–(2.3) and of the following inequalities:

$$\Lambda_r^{(0)}\eta(u) \le \eta'(u)\Lambda_r^{(0)}u \quad \text{and} \quad \Lambda_r^{(\alpha)}\eta(u) \le \eta'(u)\Lambda_r^{(\alpha)}u, \tag{2.5}$$

resulting from the convexity of the function η.

Finally, these operators satisfy the integration by parts formula: for all $u \in C_b^2(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \varphi \Lambda u\,dx = \int_{\mathbb{R}} u \Lambda \varphi\,dx, \tag{2.6}$$

where $\Lambda \in \{\Lambda_r^{(0)}, \Lambda_r^{(\alpha)}, \Lambda^\alpha\}$ for every $\alpha \in (0,2]$ and all $r > 0$. Notice that $\Lambda\varphi \in L^1(\mathbb{R})$, since it is obvious from (2.2)–(2.3) that $\Lambda_r^{(\alpha)} : W^{2,1}(\mathbb{R}) \to L^1(\mathbb{R})$ and $\Lambda_r^{(0)} : L^1(\mathbb{R}) \to L^1(\mathbb{R})$.

Detailed proofs of all these properties are based on the representation (2.1)–(2.3) and are written, e.g., in [1].

2.2. Existence and uniqueness of entropy solutions. It was shown in [2] (see also [11]) that solutions of the initial value problem for the fractal conservation law

$$u_t + \Lambda^\alpha u + (f(u))_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{2.7}$$

$$u(0,x) = u_0(x), \tag{2.8}$$

where $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz-continuous, can become discontinuous in finite time if $0 < \alpha < 1$. Hence, in order to deal with discontinuous solutions, the notion of entropy solution in the sense of Kruzhkov was extended in [1] to fractal conservation laws (2.7)–(2.8) (see also [10] for the recent generalization to Lévy mixed hyperbolic/parabolic equations). Here, the crucial role is played by the Lévy–Khintchine representation (2.1)–(2.3) of the operator Λ^α.

DEFINITION 2.1. *Let $0 < \alpha \leq 1$ and $u_0 \in L^\infty(\mathbb{R})$. A function $u \in L^\infty(\mathbb{R} \times (0, +\infty))$ is an* entropy solution *to* (2.7)–(2.8) *if for all $\varphi \in \mathcal{D}(\mathbb{R} \times [0, +\infty))$, $\varphi \geq 0$, $\eta \in C^2(\mathbb{R})$ convex, $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi' = \eta' f'$, and $r > 0$ we have*

$$\int_{\mathbb{R}} \int_0^{+\infty} \left(\eta(u)\varphi_t + \phi(u)\varphi_x - \eta(u)\Lambda_r^{(\alpha)}\varphi - \varphi\eta'(u)\, \Lambda_r^{(0)} u \right) dxdt + \int_{\mathbb{R}} \eta(u_0(x))\varphi(x, 0)\, dx \geq 0.$$

Note that, due to formula (2.3), the quantity $\Lambda_r^{(0)} u$ in the above inequality is well defined for any bounded function u.

The notion of entropy solutions allows us to solve the fractal Burgers equation for the range of exponent $\alpha \in (0, 1]$.

THEOREM 2.2 (see [1]). *Assume that $0 < \alpha \leq 1$ and $u_0 \in L^\infty(\mathbb{R})$. There exists a unique entropy solution $u = u(x, t)$ to problem* (2.7)–(2.8)*. This solution u belongs to $C([0, +\infty); L^1_{loc}(\mathbb{R}))$ and satisfies $u(0) = u_0$. Moreover, we have the following maximum principle:* $\operatorname{ess\,inf} u_0 \leq u \leq \operatorname{ess\,sup} u_0$.

If $\alpha \in (1, 2]$, all solutions to (2.7)–(2.8) with bounded initial conditions are smooth and global-in-time (see [5, 11, 13]). On the other hand, the occurrence of discontinuities in finite time of entropy solutions to (2.7)–(2.8) with $\alpha = 1$ seems to be unclear. As mentioned in the introduction, regularity results have recently been obtained [11, 4, 12] for a large class of initial conditions which, unfortunately, does not include general L^∞-initial data. Nevertheless, Theorem 2.2 provides the existence and the uniqueness of a global-in-time entropy solution even for the critical case $\alpha = 1$.

3. Regularized problem. In this section, we gather properties of solutions to the Cauchy problem for the regularized fractal Burgers equation with $\varepsilon > 0$:

$$u_t^\varepsilon + \Lambda^\alpha u^\varepsilon - \varepsilon u_{xx}^\varepsilon + u^\varepsilon u_x^\varepsilon = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1}$$

$$u^\varepsilon(x, 0) = u_0(x). \tag{3.2}$$

Our purpose is to derive asymptotic stability estimates of a solution $u^\varepsilon = u^\varepsilon(x, t)$ (uniform in ε) that will be valid for (1.1)–(1.2) after passing to the limit $\varepsilon \to 0$. Most of the results of this section are inspired from [9], and, when it is the case, the reader is referred to the corresponding proofs of that work.

Below, we will use the following integral formulation of the initial value problem (3.1)–(3.2):

$$u^\varepsilon(t) = S_\alpha^\varepsilon(t)u_0 - \int_0^t S_\alpha^\varepsilon(t - \tau)u^\varepsilon(\tau)u_x^\varepsilon(\tau)\, d\tau, \tag{3.3}$$

where $\{S_\alpha^\varepsilon(t)\}_{t>0}$ denotes the semigroup of linear operators whose infinitesimal generator is $-\Lambda^\alpha u^\varepsilon + \varepsilon u_{xx}^\varepsilon$.

If, for each $\alpha \in (0, 2]$, the function $p_\alpha(x, t)$ denotes the fundamental solution of the linear equation $u_t + \Lambda^\alpha u = 0$, then

$$S_\alpha^\varepsilon(t)u_0 = p_\alpha(t) * p_2(\varepsilon t) * u_0. \tag{3.4}$$

It is well known that $p_\alpha(x,t)$ can be represented via the Fourier transform $\widehat{p}_\alpha(\xi,t) = e^{-t|\xi|^\alpha}$. In particular,

$$p_\alpha(x,t) = t^{-\frac{1}{\alpha}} P_\alpha(xt^{-\frac{1}{\alpha}}), \tag{3.5}$$

where P_α is the inverse Fourier transform of $e^{-|\xi|^\alpha}$. For every $\alpha \in (0,2]$ the function P_α is smooth and nonnegative, $\int_{\mathbb{R}} P_\alpha(y)\,dy = 1$, and satisfies the estimates (optimal for $\alpha \neq 2$)

$$0 < P_\alpha(x) \le C(1+|x|)^{-(\alpha+1)} \quad \text{and} \quad |(P_\alpha)_x(x)| \le C(1+|x|)^{-(\alpha+2)} \tag{3.6}$$

for a constant C and all $x \in \mathbb{R}$.

One can see that problem (3.1)–(3.2) admits a unique global-in-time smooth solution that satisfies the maximum principle.

THEOREM 3.1. *Let $\alpha \in (0,2]$, $\varepsilon > 0$, and $u_0 \in L^\infty(\mathbb{R})$. There exists the unique solution $u^\varepsilon = u^\varepsilon(x,t)$ to problem* (3.1)–(3.2) *in the following sense:*

- *$u^\varepsilon \in C_b(\mathbb{R} \times (0,+\infty)) \cap C_b^\infty(\mathbb{R} \times (a,+\infty))$ for all $a > 0$;*
- *u^ε satisfies* (3.1) *on $\mathbb{R} \times (0,+\infty)$;*
- *$\lim_{t\to 0} u^\varepsilon(t) = u_0$ in $L^\infty(\mathbb{R})$ weak-$*$ and in $L^p_{loc}(\mathbb{R})$ for all $p \in [1,+\infty)$.*

Moreover, the following inequalities hold true:

$$\operatorname{ess\,inf} u_0 \le u^\varepsilon(t) \le \operatorname{ess\,sup} u_0 \quad \textit{for all} \quad t > 0. \tag{3.7}$$

Proof. Here, the results from [5] can be easily modified in order to get the existence and the regularity of solutions to (3.1)–(3.2) with $\varepsilon > 0$. ◻

Remark 3.2. Let us mention that the fractal conservation law (3.1) also satisfies the comparison and L^1-contraction principles and does not increase the BV seminorm; that is to say, if u^ε and $\widetilde{u^\varepsilon}$ are solutions to (3.1)–(3.2) with respective initial data u_0 and $\widetilde{u_0}$, then the following hold:

- $u^\varepsilon \le \widetilde{u^\varepsilon}$ whenever $u_0 \le \widetilde{u_0}$;
- $\|u^\varepsilon(t) - \widetilde{u^\varepsilon}(t)\|_1 \le \|u_0 - \widetilde{u_0}\|_1$ for all $t > 0$ whenever $u_0 - \widetilde{u_0} \in L^1(\mathbb{R})$;
- $\|u^\varepsilon_x(t)\|_1 \le \|m\|$ for all $t > 0$ whenever u_0 is of the form (1.3).

As explained in [5, Remarks 1.2 and 6.2], these properties are immediate consequences of the splitting method developed in [5] and the fact that both the hyperbolic equation $u_t + uu_x = 0$ and the fractal equation $u_t + \Lambda^\alpha u - \varepsilon u_{xx} = 0$ satisfy these properties.

The next proposition provides an estimate on the gradient of u^ε.

PROPOSITION 3.3. *Let $0 < \alpha \le 1$, let $\varepsilon > 0$, and let u_0 be of the form* (1.3) *with a finite nonnegative measure m on $\mathbb{R}$. Denote by $u^\varepsilon = u^\varepsilon(x,t)$ the unique solution of problem* (3.1)–(3.2). *Then the following hold:*

- *$u^\varepsilon_x(x,t) \ge 0$ for all $x \in \mathbb{R}$ and $t > 0$;*
- *for every $p \in [1,+\infty]$ and $t > 0$, we have $u^\varepsilon_x(t) \in L^p(\mathbb{R})$; moreover, there exists a constant $C > 0$ independent of $\varepsilon > 0$ such that for all $t > 0$*

$$\|u^\varepsilon_x(t)\|_p \le Ct^{-\frac{1}{\alpha}(1-\frac{1}{p})}\|m\|. \tag{3.8}$$

Proof. For a fixed real h, the function $u^\varepsilon(\cdot+h,\cdot)$ is the solution to (3.1)–(3.2) with the initial datum $u_0(\cdot+h)$. Consequently, for nondecreasing u_0 and for $h > 0$, the inequality $u_0(\cdot+h) \ge u_0(\cdot)$ and the comparison principle imply $u^\varepsilon(\cdot+h,\cdot) \ge u^\varepsilon(\cdot,\cdot)$, which gives $u^\varepsilon_x \ge 0$.

Moreover, $v \equiv u^\varepsilon_x$ is bounded outside some neighborhood of $t = 0$ by Theorem 3.1 and integrable w.r.t. the space variable by Remark 3.2; by interpolation of the L^1- and L^∞-norms, one has $v(t) \in L^p(\mathbb{R})$ for all $t > 0$. In particular, one has $\lim_{|x|\to+\infty} v(x,t) = 0$ since $v(t) \in C_b^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$; this will be useful later on.

To show the decay estimates (3.8), it suffices to modify the argument from [9] slightly. In particular, we multiply the equation for $v = u^\varepsilon_x \geq 0$

$$v_t + \Lambda^\alpha v - \varepsilon v_{xx} + (u^\varepsilon u^\varepsilon_x)_x = 0$$

by v^{p-1} to obtain , at least formally,

$$\frac{1}{p}\frac{d}{dt}\|v(t)\|_p^p + \int_{\mathbb{R}} v^{p-1}\Lambda^\alpha v\,dx - \varepsilon \int_{\mathbb{R}} v^{p-1} v_{xx}\,dx + \frac{p-1}{p}\int_{\mathbb{R}} v^{p+1}\,dx = 0. \tag{3.9}$$

Here, one should be sure that all terms are integrable. It is the case if $p \geq 2$, since $v^{p-1}(t) \in L^1(\mathbb{R})$ and all other terms are bounded by the regularity described in Theorem 3.1. Notice also that we have skipped all boundary terms obtained from integration by parts, because $\lim_{|x|\to+\infty} v(x,t) = 0$.

Note that the third term on the left-hand side of (3.9) is nonnegative because integrating by parts we have

$$-\varepsilon \int_{\mathbb{R}} v_{xx}(x,t)\Phi(v(x,t))\,dx \geq 0 \tag{3.10}$$

for any nondecreasing function $\Phi \in C^1(\mathbb{R})$ with $\Phi(0) = 0$.

Hence, one gets for all $p \geq 2$ and all $t > 0$

$$\frac{1}{p}\frac{d}{dt}\|v(t)\|_p^p + \int_{\mathbb{R}} v^{p-1}\Lambda^\alpha v\,dx \leq 0.$$

Now for the proof of the decay estimate in (3.8), one should follow estimates from [9, Lemma 3.1]; let us mention that the arguments used only the inequalities above for $p = 2^n$ $(n \in \mathbb{N})$, so we lose no information by taking $p \geq 2$. ◻

We can now give asymptotic stability estimates uniform in ε.

THEOREM 3.4. *Let $\alpha \in (0,2]$ and $\varepsilon > 0$. Assume that u^ε and $\widetilde{u^\varepsilon}$ are two solutions of the regularized problem* (3.1)–(3.2) *with initial conditions u_0 and $\widetilde{u}_0$ of the form* (1.3) *associated with finite signed measures m and $\widetilde{m}$, respectively. Suppose, moreover, that the measure $\widetilde{m}$ of $\widetilde{u}_0$ is nonnegative and $u_0 - \widetilde{u}_0 \in L^1(\mathbb{R})$. Then, for every $p \in [1,+\infty]$ there exists a constant $C > 0$ independent of ε such that for all $t > 0$*

$$\|u^\varepsilon(t) - \widetilde{u^\varepsilon}(t)\|_p \leq Ct^{-\frac{1}{\alpha}(1-\frac{1}{p})}\|u_0 - \widetilde{u}_0\|_1\,. \tag{3.11}$$

Proof. First, inequality (3.11) for $p = 1$ reduces to the L^1-contraction principle (see Remark 3.2). Next, the proof of (3.11) for $p > 1$ follows the arguments from [9, Proof of Lemma 3.1]. In this reasoning, in the case of solutions of the regularized problem (3.1)–(3.2), we have to deal with the additional term $-\varepsilon v_{xx} = -\varepsilon(u^\varepsilon - \widetilde{u^\varepsilon})_{xx}$, which can always be skipped in calculations thanks to inequality (3.10). ◻

THEOREM 3.5. *Let $0 < \alpha < 1$ and $\varepsilon > 0$. Assume that u_0 is of the form* (1.3) *with a finite nonnegative measure m on $\mathbb{R}$. Then, for every $p \in [1,+\infty]$ there exists $C > 0$ independent of ε such that the solution u^ε to* (3.1)–(3.2) *satisfies for all $t > 0$*

$$\|u^\varepsilon(t) - S^\varepsilon_\alpha(t)u_0\|_p \leq C\|u_0\|_\infty\|m\|t^{1-\frac{1}{\alpha}(1-\frac{1}{p})}$$

(where $\{S^\varepsilon_\alpha(t)\}_{t\geq 0}$ is the semigroup of linear operators generated by $-\Lambda^\alpha + \varepsilon\partial_x^2$).

Proof. Using the integral equation (3.3) we immediately obtain

$$\|u^\varepsilon(t) - S_\alpha^\varepsilon(t)u_0\|_p \leq \int_0^t \|S_\alpha^\varepsilon(t-\tau)u^\varepsilon(\tau)u_x^\varepsilon(\tau)\|_p \, d\tau. \tag{3.12}$$

Now, we estimate the integral in the right-hand side of (3.12) using the L^p-decay of the semigroup $S_\alpha^\varepsilon(t)$ as well as inequalities (3.7) and (3.8). Indeed, it follows from (3.5)–(3.6) that

$$\|p_2(\varepsilon t)\|_1 = 1 \quad \text{and} \quad \|p_\alpha(t)\|_r = t^{-\frac{1}{\alpha}(1-\frac{1}{r})}\|p_\alpha(1)\|_r$$

for every $r \in [1, +\infty]$. Hence, by the Young inequality for the convolution and inequalities (3.7), (3.8), we obtain

$$\begin{aligned} &\|S_\alpha^\varepsilon(t-\tau)u^\varepsilon(\tau)u_x^\varepsilon(\tau)\|_p \\ &\qquad \leq \|p_\alpha(t-\tau) * (u^\varepsilon(\tau)u_x^\varepsilon(\tau))\|_p \\ &\qquad \leq C(t-\tau)^{-\frac{1}{\alpha}(\frac{1}{q}-\frac{1}{p})}\|u^\varepsilon(\tau)\|_\infty \|u_x^\varepsilon(\tau)\|_q \\ &\qquad \leq C(t-\tau)^{-\frac{1}{\alpha}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_\infty \|m\| \tau^{-\frac{1}{\alpha}(1-\frac{1}{q})} \end{aligned} \tag{3.13}$$

for all $1 \leq q \leq p \leq +\infty$, $t > 0$, $\tau \in (0,t)$, and the constant $C > 0$ independent of t, τ, ε.

Next, we decompose the integral on the right-hand side of (3.12) as follows: $\int_0^t \ldots \, d\tau = \int_0^{t/2} \ldots \, d\tau + \int_{t/2}^t \ldots d\tau$, and we bound both integrands by using inequality (3.13) either with $q = 1$ or with $q = p$. This leads to the following inequality:

$$\begin{aligned} &\|u^\varepsilon(t) - S_\alpha^\varepsilon(t)u_0\|_p \\ &\quad \leq C\|u_0\|_\infty \|m\| \left(\int_0^{t/2} (t-\tau)^{-\frac{1}{\alpha}(1-\frac{1}{p})} \, d\tau + \int_{t/2}^t \tau^{-\frac{1}{\alpha}(1-\frac{1}{p})} \, \tau \right). \end{aligned} \tag{3.14}$$

Computing both integrals w.r.t. τ, we complete the proof of Theorem 3.5. $\square$

4. Entropy solution: Parabolic approximation and self-similarity. In this section, we state the result on the convergence, as $\varepsilon \to 0$, of solutions u^ε of (3.1)–(3.2) toward the entropy solution u of (1.1)–(1.2). We also prove Theorem 1.6, which is about self-similar entropy solutions in the case $\alpha = 1$.

Together with the general fractal conservation law (2.7)–(2.8), we study the associated regularized problem

$$u_t^\varepsilon + \Lambda^\alpha u^\varepsilon - \varepsilon u_{xx}^\varepsilon + (f(u^\varepsilon))_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{4.1}$$

$$u^\varepsilon(x, 0) = u_0(x), \tag{4.2}$$

where $f \in C^\infty(\mathbb{R})$. Hence, by the results of [5] (see also Theorem 3.1), problem (4.1)–(4.2) admits the unique, global-in-time, smooth solution u^ε.

THEOREM 4.1. *Assume that u_0 is of the form* (1.3) *and $\varepsilon > 0$. Let $u^\varepsilon = u^\varepsilon(x,t)$ be the solution to* (4.1)–(4.2) *and $u = u(x,t)$ be the entropy solution to* (2.7)–(2.8). *Then, for every $T > 0$, $u^\varepsilon \to u$ in $C([0,T]; L^1_{loc}(\mathbb{R}))$ as $\varepsilon \to 0$.*

The proof of Theorem 4.1 is given in Appendix A.

Remark 4.2. This result actually holds true for u_0 only essentially bounded and f locally Lipschitz-continuous; more generally, multidimensional fractal conservation laws with source terms $h = h(u,x,t)$ and fluxes $f = f(u,x,t)$ (see [6, 5]) can be considered. Nevertheless, for the sake of simplicity, we assume that u_0 has a bounded variation and $f \in C^\infty$. Indeed, as pointed out to us by the referee, the BV assumption simplifies the proof of Theorem 4.1 by using the Aubin–Simon compactness theorem.

Proof of Theorem 1.6. The existence of the solution $U = U(x,t)$ to (1.1) with $\alpha = 1$ supplemented with the initial condition (1.9) is provided by Theorem 2.2. To obtain the self-similar form of U, we follow a standard argument based on the uniqueness result from Theorem 2.2. Observe that if U is the solution to (1.1), the rescaled function $U^\lambda(x,t) = U(\lambda x, \lambda t)$ is the solution for every $\lambda > 0$, too. Since, the initial datum (1.9) is invariant under the rescaling $U_0^\lambda(x) = U_0(\lambda x)$, by the uniqueness, we obtain that for all $\lambda > 0$, $U(x,t) = U(\lambda x, \lambda t)$ for a.e. $(x,t) \in \mathbb{R} \times (0,+\infty)$. □

5. Passage to the limit $\varepsilon \to 0$ and asymptotic study. In this section, we prove Theorems 1.1, 1.3, and 1.7.

Proof of Theorem 1.1. Denote by u^ε and $\widetilde{u}^\varepsilon$ the solutions to the regularized equation (3.1) with the initial conditions u_0 and $\widetilde{u}_0$. By Theorem 4.1 and the maximum principle (3.7), we know that $\lim_{\varepsilon\to 0} u^\varepsilon(t) = u(t)$ and $\lim_{\varepsilon\to 0} \widetilde{u^\varepsilon}(t) = \widetilde{u}(t)$ in $L^p_{loc}(\mathbb{R})$ for every $p \in [1,+\infty)$ and in $L^\infty(\mathbb{R})$ weak-$*$. Hence, for each $R > 0$ and $p \in [1,+\infty]$, using Theorem 3.4 we have

$$
\begin{aligned}
\|u(t) - \widetilde{u}(t)\|_{L^p((-R,R))} &\le \liminf_{\varepsilon\to 0} \|u^\varepsilon(t) - \widetilde{u}^\varepsilon(t)\|_{L^p((-R,R))} \\
&\le C t^{-\frac{1}{\alpha}(1-\frac{1}{p})} \|u_0 - \widetilde{u}_0\|_1.
\end{aligned}
$$

Since $R > 0$ is arbitrary and the right-hand side of this inequality does not depend on R, we complete the proof of inequality (1.8). □

Proof of Theorem 1.3. In view of Theorem 1.1, it suffices to show the following inequality:

$$
\|\widetilde{u}(t) - S_\alpha(t)u_0\|_p \le C\|U_0\|_\infty \|m\| t^{1-\frac{1}{\alpha}(1-\frac{1}{p})},
$$

where $\widetilde{u}$ is the solution to (1.1) with U_0 as the initial condition. Notice that $\|U_0\|_\infty = u_+ - u_-$ and $\|m\| = \max\{|u_+|, |u_-|\}$ in this case.

Here, we argue exactly as in the proof of Theorem 1.1, since we can assume that $\lim_{\varepsilon\to 0} \widetilde{u}^\varepsilon(t) = \widetilde{u}(t)$ in $L^p_{loc}(\mathbb{R})$ for every $p \in [1,+\infty)$ and in $L^\infty(\mathbb{R})$ weak-$*$. Moreover, it is well known that for fixed $t > 0$

$$
\lim_{\varepsilon\to 0} S_\alpha^\varepsilon(t)U_0 = \lim_{\varepsilon\to 0} p_2(\varepsilon t) * \big(p_\alpha(t) * U_0\big) = S_\alpha(t)U_0 \quad \text{in} \quad L^p(\mathbb{R})
$$

for all $p \in [1,+\infty]$. Hence, for every $R > 0$ and $p \in [1,+\infty]$, by Theorem 3.5, we obtain

$$
\begin{aligned}
\|\widetilde{u}(t) - S_\alpha(t)U_0\|_{L^p((-R,R))} &\le \liminf_{\varepsilon\to 0} \|\widetilde{u}^\varepsilon(t) - S_\alpha^\varepsilon(t)U_0\|_{L^p((-R,R))} \\
&\le C\|U_0\|_\infty \|m\| t^{1-\frac{1}{\alpha}(1-\frac{1}{p})}.
\end{aligned}
$$

The proof is completed by letting $R \to +\infty$. □

Proof of Theorem 1.7. Apply Theorem 1.1 with $\alpha = 1$ and $\widetilde{u}_0 = U_0$. □

6. Qualitative study of the self-similar profile for $\alpha = 1$. This section is devoted to the proofs of Theorems 1.9 and 1.10.

6.1. Proofs of properties p1–p4 from Theorem 1.9. The Lipschitz continuity stated in p1 is an immediate consequence of Proposition 3.3 and Theorem 4.1. Indeed, $U(1)$ is the limit in $L^1_{loc}(\mathbb{R})$ of $u^\varepsilon(1)$ as $\varepsilon \to 0$, where $u^\varepsilon = u^\varepsilon(x,t)$ is the solution to (3.1)–(3.2) with $u_0 = U_0$ defined in (1.9). Moreover, by (3.8), the family $\{u^\varepsilon(1) : \varepsilon > 0\}$ is equi-Lipschitz-continuous, which implies that the limit $U(1)$ is Lipschitz-continuous.

Before proving properties p2–p4, let us reduce the problem to a simpler one. We remark that (1.1) is invariant under the transformation

$$V(x,t) \equiv U\left(x + \overline{c}t, t\right) - \overline{c}, \quad \text{where} \quad \overline{c} \equiv \frac{u_- + u_+}{2}; \tag{6.1}$$

that is to say, if U is a solution to (1.1) with $U(x,0) = U_0(x)$ defined in (1.9), then V is a solution to (1.1) with the initial datum

$$V(x,0) = V_0(x) \equiv \begin{cases} v_+, & x < 0, \\ v_-, & x > 0, \end{cases} \tag{6.2}$$

where $v_- = -v_+$ and $v_+ \equiv |\overline{c}| \geq 0$. It is clear that U satisfies p2–p4 whenever V enjoys these properties. In what follows, we thus assume without loss of generality that $u_- = -u_+$ and $u_+ > 0$.

It has been shown in [2, Lemma 3.1] that if $u_0 \in L^\infty(\mathbb{R})$ is nonincreasing, odd, and convex on $(0,+\infty)$, then the solution $u = u(x,t)$ of (1.1)–(1.2) shares these properties w.r.t. x for all $t > 0$. The proof is based on a splitting method and on the fact that the "odd, concave/convex" property is conserved by both the hyperbolic equation $u_t + uu_x$ and the fractal equation $u_t + \Lambda^1 u = 0$. The same proof works with minor modifications to show that if u_0 is nondecreasing, odd, and convex on $(-\infty,0)$, then these properties are preserved by problem (1.1)–(1.2). Details are left to the reader since in that case no shock can be created by the Burgers part and the proof is even easier. By the hypothesis $u_- = -u_+ < 0$ made above, the initial datum in (1.9) is nondecreasing, odd, and convex on $(-\infty,0)$. We conclude that so is the profile $U(1)$. The proofs of properties p3–p4 are now complete.

What is left to prove is the limit in property p2. By Theorem 2.2, we have $U(t) \to U_0$ in $L^1_{loc}(\mathbb{R})$ as $t \to 0$. In particular, the convergence holds true a.e. along a subsequence $t_n \to 0$ as $n \to +\infty$, and there exists $\pm x_\pm > 0$ such that $U(x_\pm, t_n) \to u_\pm$. By the self-similarity of U, we get $U(\frac{x_\pm}{t_n}, 1) \to u_\pm$ as $n \to +\infty$. Since $\frac{x_\pm}{t_n} \to \pm\infty$ and $U(1)$ is nondecreasing, we deduce property p2.

6.2. Some technical lemmata. The last property of Theorem 1.9 is the most difficult part to prove. In this preparatory subsection, we state and prove technical results that shall be needed in our reasoning.

LEMMA 6.1. *Let $v \in L^\infty(\mathbb{R})$ be nonnegative, even, and nonincreasing on $(0,+\infty)$. Assume that there exists $\ell > 0$ such that for all $x_0 > 1/2$*

$$\lim_{n\to+\infty} n^{-1} \int_{n(x_0-1/2)}^{n(x_0+1/2)} y^2 v(y)dy = \ell. \tag{6.3}$$

Then, we have $y^2 v(y) \to_{|y|\to+\infty} \ell$.

Proof. For all $x_0 > 1/2$, we have

$$n^{-1} \int_{n(x_0-1/2)}^{n(x_0+1/2)} y^2 v(y)dy \geq n^2 (x_0 - 1/2)^2 v(n(x_0+1/2)),$$

thanks to the fact that v is nonincreasing on $(0,+\infty)$. Hence, we have

$$n^2\left(x_0+\frac{1}{2}\right)^2 v\left(n\left(x_0+\frac{1}{2}\right)\right) \quad\le\quad \frac{n^2(x_0+1/2)^2}{n^2(x_0-1/2)^2} \quad n^{-1}\int_{n(x_0-1/2)}^{n(x_0+1/2)} y^2v(y)dy.$$

Taking the upper semilimit, we get for all $x_0>1/2$

$$\limsup_{n\to+\infty} n^2\left(x_0+\frac{1}{2}\right)^2 v\left(n\left(x_0+\frac{1}{2}\right)\right)\le \ell\left(\frac{x_0+1/2}{x_0-1/2}\right)^2, \tag{6.4}$$

thanks to (6.3). In the same way, one can show that for all $x_0>1/2$

$$\ell\left(\frac{x_0-1/2}{x_0+1/2}\right)^2 \le \liminf_{n\to+\infty} n^2\left(x_0-\frac{1}{2}\right)^2 v\left(n\left(x_0-\frac{1}{2}\right)\right). \tag{6.5}$$

Moreover, for fixed $x_0>1/2$ and all $y\ge x_0+1/2$, there exists a unique integer n_y such that

$$n_y(x_0+1/2)\le y<(n_y+1)(x_0+1/2).$$

Using again that v is nonincreasing on $[0,+\infty)$, we infer that

$$\begin{aligned} y^2v(y) &\le (n_y+1)^2\left(x_0+\frac{1}{2}\right)^2 v\left(n_y\left(x_0+\frac{1}{2}\right)\right)\\ &= \frac{(n_y+1)^2(x_0+1/2)^2}{n_y^2(x_0+1/2)^2}\; n_y^2\left(x_0+\frac{1}{2}\right)^2 v\left(n_y\left(x_0+\frac{1}{2}\right)\right). \end{aligned}$$

Notice that $n_y\to+\infty$ as $y\to+\infty$. Therefore, passing to the upper semilimit as $y\to+\infty$ in the inequality above, one can show that for all $x_0>1/2$

$$\limsup_{y\to+\infty} y^2v(y)\le \ell\left(\frac{x_0+1/2}{x_0-1/2}\right)^2,$$

thanks to (6.4). In the same way, we deduce from (6.5) that for all $x_0>1/2$

$$\ell\left(\frac{x_0-1/2}{x_0+1/2}\right)^2\le \liminf_{y\to+\infty} y^2v(y).$$

Finally, letting $x_0\to+\infty$ in both inequalities above implies that

$$\ell\le \liminf_{y\to+\infty} y^2v(y)\le \limsup_{y\to+\infty} y^2v(y)\le \ell.$$

Since v is even, we have completed the proof of the lemma. ∎

For all $r>0$, the operator Λ^1 is the sum of $\Lambda_r^{(0)}$ and $\Lambda_r^{(1)}$. As far as $\Lambda_r^{(1)}$ is concerned, we have the following lemma.

LEMMA 6.2. *Let $u\in L^\infty(\mathbb{R})$ be nondecreasing, odd, and convex on $(-\infty,0)$. Then, for the operator defined in* (2.2) *we have $\Lambda_r^{(1)}u\in L^1_{loc}(\mathbb{R}_*)$ together with the inequality*

$$\int_{|x|>R}|\Lambda_r^{(1)}u(x)|\,dx\le \frac{4G_1r}{R-r}\|u\|_\infty \tag{6.6}$$

for all $R > r > 0$.

Proof. The proof is divided into a sequence of steps.

Step 1. Estimates of u_x. The convex function u on $(-\infty, 0)$ is locally Lipschitz-continuous on $(-\infty, 0)$ and a fortiori a.e. differentiable. Since $u(0) = 0$, we have for $x < 0$

$$|u_x(x)| \leq \|u\|_\infty \, |x|^{-1}; \tag{6.7}$$

remark that this estimate holds true for $x \in \mathbb{R}$ since u is odd.

Step 2. Estimates of u_{xx}. By convexity of u, u_{xx} is a nonnegative Radon measure on $(-\infty, 0)$ in the distribution sense. Hence, $u_x \in BV_{loc}((-\infty, 0))$ satisfies $\int_{(\widetilde{x},x]} u_{yy}(dy) = u_x(x) - u_x(\widetilde{x})$ for a.e. $\widetilde{x} < x < 0$. Using (6.7) and letting $\widetilde{x} \to -\infty$, we conclude that for a.e. $x < 0$

$$\int_{(-\infty,x]} u_{yy}(dy) = u_x(x), \tag{6.8}$$

thanks to the sup-continuity of nonnegative measures. Again by (6.7) and the oddity of u_{xx}, this shows that for a.e. $x \neq 0$

$$\int_{|y| \geq |x|} |u_{yy}|(dy) \leq 2\|u\|_\infty |x|^{-1}; \tag{6.9}$$

notice that by the inf-continuity of nonnegative measures this inequality holds for all $x \neq 0$.

Step 3. Estimate of $\Lambda_r^{(1)} u$. Let us prove that $\Lambda_r^{(1)} u$ is well defined by formula (2.2) for a.e. $x \neq 0$. By the preceding steps, we know that $u \in L^\infty(\mathbb{R}) \cap W^{1,\infty}_{loc}(\mathbb{R}_*)$ and $u_x \in BV_{loc}(\mathbb{R}_*)$. By Taylor's formula (see Lemma B.1 in Appendix B), we infer that for all $R > r > 0$

$$\begin{aligned} I &\equiv \int_{|x|>R} \int_{|z| \leq r} \frac{|u(x+z) - u(x) - u_x(x) z|}{|z|^2} \, dx \, dz \\ &\leq \int_{|x|>R} \int_{|z| \leq r} |z|^{-2} \left| \int_{I_{x,z}} |x + z - y| u_{yy}(dy) \right| \, dx \, dz, \end{aligned}$$

where $I_{x,z} \equiv (x, x+z)$ if $z > 0$ and $I_{x,z} \equiv (x+z, x)$ in the opposite case. Therefore, we see that

$$\begin{aligned} I &\leq \int_{|x|>R} \int_{|z| \leq r} |z|^{-1} \int_{I_{x,z}} |u_{yy}|(dy) \, dx \, dz \\ &= \int_{\mathbb{R}_*} \int_{\mathbb{R}} |z|^{-1} \mathbf{1}_{\{|z| \leq r\}} \int_{|x|>R} \mathbf{1}_{I_{x,z}}(y) \, dx \, |u_{yy}|(dy) \, dz \end{aligned}$$

by integrating first w.r.t. x; notice that all the integrands are measurable by Fubini's theorem, since the Radon measure $|u_{yy}|(dy)$ is σ-finite on $\mathbb{R}_*$. For fixed $(y, z) \in \mathbb{R}_* \times \mathbb{R}$, we have

$$\mathbf{1}_{\{|z| \leq r\}} \int_{|x|>R} \mathbf{1}_{I_{x,z}}(y) \, dx \leq |z| \; \mathbf{1}_{\{|z| \leq r\}} \; \mathbf{1}_{\{|y| \geq R-r\}},$$

because the measure of the set $\{x : y \in I_{x,z}\}$ can be estimated by $|z|$, and if $|z| \leq r$, then $\mathbf{1}_{I_{x,z}}(y) = 0$ for all $|x| > R$ whenever $|y| < R - r$. It follows that

$$I \leq \int_{\mathbb{R}_*} \int_{\mathbb{R}} \mathbf{1}_{\{|z| \leq r\}} \; \mathbf{1}_{\{|y| \geq R-r\}} \; |u_{yy}|(dy) \, dz = 2r \int_{|y| \geq R-r} |u_{yy}|(dy).$$

Recalling the definition of I above and estimate (6.9), we have shown that

$$\int_{|x|>R}\int_{|z|\le r}\frac{|u(x+z)-u(x)-u_x(x)z|}{|z|^2}\,dx\,dz \le 4r\|u\|_\infty (R-r)^{-1}. \tag{6.10}$$

Fubini's theorem then implies that $\Lambda_r^{(1)}u(x)$ is well defined by (2.2) for a.e. $|x|>R>r$ by satisfying the desired estimate (6.6).

Step 4. Local integrability on $\mathbb{R}_*$. Estimate (6.6) implies that $\Lambda_r^{(1)}u \in L^1_{loc}(\mathbb{R}\setminus[-r,r])$. In fact, $\Lambda_r^{(1)}u$ is locally integrable on all $\mathbb{R}_*$. Indeed, simple computations show that for all $r>\widetilde{r}>0$

$$\Lambda_r^{(1)}u+\Lambda_r^{(0)}u=\Lambda_{\widetilde{r}}^{(1)}u+\Lambda_{\widetilde{r}}^{(0)}u, \tag{6.11}$$

since their difference evaluated at some x is equal to $\int_{\widetilde{r}\le|z|\le r}\frac{-u_x(x)z}{|z|^2}$, which is null by the oddity of the function $z\to -u_x(x)z$. By Step 3, it follows that $\Lambda_r^{(1)}u=\Lambda_{\widetilde{r}}^{(1)}u+\Lambda_{\widetilde{r}}^{(0)}u-\Lambda_r^{(0)}u\in L^1_{loc}(\mathbb{R}\setminus[-\widetilde{r},\widetilde{r}])$, which completes the proof. ∎

It is clear that $\Lambda_r^{(0)}$ maps $L^\infty(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and if $\{u_n\}_{n\in\mathbb{N}}$ is uniformly essentially bounded and $u_n\to u$ in $L^1_{loc}(\mathbb{R})$, then $\Lambda_r^{(0)}u_n\to\Lambda_r^{(0)}u$ in $L^1_{loc}(\mathbb{R})$ as $n\to+\infty$.

Remark 6.3. Lemma 6.2 implies that $\Lambda^1 u\in L^1_{loc}(\mathbb{R}_*)$ whenever $u\in L^\infty(\mathbb{R})$ is nondecreasing, odd, and convex on $(-\infty,0)$. This sum does not depend on $r>0$ by (6.11). Moreover, one sees from (6.10), Fubini's theorem, and (2.1) that for all $\varphi\in\mathcal{D}(\mathbb{R}_*)$, $\int_{\mathbb{R}}\varphi\Lambda^1u\,dx=\int_{\mathbb{R}}u\Lambda^1\varphi\,dx$. This means that this sum corresponds to the distribution fractional Laplacian of u on $\mathbb{R}_*$.

We deduce from the previous lemma the following one.

LEMMA 6.4. *Let $u\in C_b(\mathbb{R})$ be nondecreasing, odd, and convex on* $(-\infty,0)$. *Then, the function $\Lambda^1u\in L^1_{loc}(\mathbb{R}_*)$ satisfies for all $x_0>1/2$*

$$\lim_{n\to+\infty} n^{-1}\int_{n(x_0-1/2)}^{n(x_0+1/2)}|\Lambda^1u(y)|dy=0.$$

Proof. By Remark 6.3, one has $\Lambda^1u\in L^1_{loc}(\mathbb{R}_*)$. Let $r>0$ be fixed. One has

$$\begin{aligned}
I_n &\equiv n^{-1}\int_{n(x_0-1/2)}^{n(x_0+1/2)}|\Lambda^1u(y)|dy\\
&\le n^{-1}\int_{n(x_0-1/2)}^{n(x_0+1/2)}|\Lambda_r^{(1)}u(y)|dy+n^{-1}\int_{n(x_0-1/2)}^{n(x_0+1/2)}|\Lambda_r^{(0)}u(y)|dy\\
&\le \frac{2G_1r}{n^2(x_0-1/2)-nr}\|u\|_\infty+\sup\left\{|\Lambda_r^{(0)}(y)| : n\left(x_0-\frac12\right)<y<n\left(x_0+\frac12\right)\right\},
\end{aligned}$$

thanks to (6.6). Moreover, $\Lambda_r^{(0)}u$ is continuous, and hence the supremum above is achieved at some $y_n\ge n(x_0-1/2)$; hence, one has

$$I_n\le\frac{2G_1r}{n^2(x_0-1/2)-nr}\|u\|_\infty+G_1\int_{|z|>r}\frac{|u(y_n+z)-u(y_n)|}{|z|^2}\,dz,$$

where $\lim_{n\to+\infty}y_n=+\infty$. Since u is nondecreasing and bounded, it has a limit at infinity; the dominated convergence theorem then implies that the integral term above tends to zero as $n\to+\infty$. It follows that $\lim_{n\to+\infty}I_n=0$. ∎

6.3. Proof of property p5 from Theorem 1.9. We assume again without loss of generality that $u_- = -u_+ < 0$, thanks to the transformation (6.1); hence, $U_0 \in L^\infty(\mathbb{R})$ is nondecreasing, odd, and convex on $(-\infty, 0)$, and so is $U(t)$ for all $t > 0$ by properties p2–p4 of Theorem 1.9. We proceed again in several steps.

Step 1. Study of $\Lambda^1 U$. Before deriving the equation satisfied by $U(1)$, we study $\Lambda^1 U$.

LEMMA 6.5. *Let $\alpha = 1$, and let $U = U(x,t)$ be the self-similar solution from Theorem* 1.6 *with initial datum U_0 in* (1.9) *for some $u_- = -u_+ < 0$. Then, for all $t \geq 0$ one has $\Lambda^1 U(t) \in L^1_{loc}(\mathbb{R}_*)$. Moreover, $\Lambda^1 U(t)$ converges toward $\Lambda^1 U_0$ in $L^1_{loc}(\mathbb{R}_*)$ as $t \to 0$, where for all $x \neq 0$*

$$\Lambda^1 U_0(x) = \frac{u_+ - u_-}{2\pi^2}\, x^{-1}.$$

Proof. By properties p2–p4 of Theorem 1.9, $U(t) \in L^\infty(\mathbb{R})$ is nondecreasing, odd, and convex on $(-\infty, 0)$ for all $t \geq 0$. By Remark 6.3, $\Lambda^1 U(t)$ and $\Lambda^1 U_0$ belong to $L^1_{loc}(\mathbb{R}_*)$. By taking $0 < r < |x|$, simple computations show that

$$\Lambda_r^{(1)} U_0(x) = 0 \quad \text{and} \quad \Lambda_r^{(0)} U_0(x) = \frac{u_+ - u_-}{2\pi^2}\, x^{-1}, \tag{6.12}$$

so that

$$\Lambda^1 U_0(x) = \frac{u_+ - u_-}{2\pi^2}\, x^{-1};$$

here, we have used the equalities $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$ in order to get $G_1 = (2\pi^2)^{-1}$ in (2.2)–(2.3). Moreover, Theorem 2.2 implies that $U(t) \to U_0$ as $t \to 0$ in $L^1_{loc}(\mathbb{R})$ with $\|U(t)\|_\infty \leq \|U_0\|_\infty$. We remark that for fixed $r > 0$, $\Lambda_r^{(0)} U(t) \to \Lambda_r^{(0)} U_0$ in $L^1_{loc}(\mathbb{R})$ as $t \to 0$. It follows that for all $\widetilde{R} > R > r$

$$\begin{aligned}
&\limsup_{t \to 0} \int_{R<|x|<\widetilde{R}} |\Lambda^1 U(t) - \Lambda^1 U_0|\, dx \\
&\leq \limsup_{t \to 0} \int_{R<|x|<\widetilde{R}} |\Lambda_r^{(1)} U(t) - \Lambda_r^{(1)} U_0|\, dx \\
&= \limsup_{t \to 0} \int_{R<|x|<\widetilde{R}} |\Lambda_r^{(1)} U(t)|\, dx \quad \text{by (6.12)} \\
&\leq \limsup_{t \to 0} 4G_1 r \|U(t)\|_\infty (R - r)^{-1} \quad \text{by (6.6) in Lemma 6.2} \\
&\leq 4G_1 r \|U_0\|_\infty (R - r)^{-1}.
\end{aligned}$$

The proof is completed by letting $r \to 0$. ∎

Step 2. Equation satisfied by $U(1)$. By using $\eta(r) = \pm r$ in Definition 2.1, we obtain (in a classical way) that entropy solutions to (1.1) are distribution solutions, i.e.,

$$U_t + UU_x + \Lambda^1 U = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0, +\infty)). \tag{6.13}$$

By property p1 of Theorem 1.9, one has $U(1) \in W^{1,\infty}(\mathbb{R})$. By the self-similarity $U(x,t) = U\left(\frac{x}{t}, 1\right)$, one has at least $U_t, U_x \in L^\infty_{loc}(\mathbb{R} \times (0, +\infty))$ together with the following equalities for a.e. $t > 0$ and $x \in \mathbb{R}$:

$$U_t(x,t) = -xt^{-2} U_x\Big(\frac{x}{t}, 1\Big), \qquad U_x(x,t) = t^{-1} U_x\Big(\frac{x}{t}, 1\Big).$$

By Lemma 6.5, we also have $\Lambda^1 U(1) \in L^1_{loc}(\mathbb{R}_*)$. Using again the self-similarity, it is easy to deduce from (2.1) that $\Lambda^1 U \in L^1_{loc}(\mathbb{R}_* \times (0,+\infty))$ with for a.e. $t > 0$ and $x \in \mathbb{R}_*$

$$\Lambda^1 U(x,t) = t^{-1}\Lambda^1 U\Big(\frac{x}{t},1\Big)$$

(in fact, $\Lambda^1 U \in L^\infty_{loc}(\mathbb{R}\times(0,+\infty))$ by (6.13) so that $\Lambda^1 U(1) \in L^\infty_{loc}(\mathbb{R})$). Putting these formulas into (6.13), we get for a.e. $t > 0$ and $x \in \mathbb{R}$

$$-xt^{-2}U_x\Big(\frac{x}{t},1\Big) + t^{-1}U\Big(\frac{x}{t},1\Big)U_x\Big(\frac{x}{t},1\Big) + t^{-1}\Lambda^1 U\Big(\frac{x}{t},1\Big) = 0.$$

Multiplying by t and changing the variable by $y = t^{-1}x$, one infers that the profile $\mathcal{U}(y) \equiv U(y,1)$ satisfies for a.e. $y \in \mathbb{R}$

$$(\mathcal{U}(y) - y)\mathcal{U}_y(y) + \Lambda^1\mathcal{U}(y) = 0. \tag{6.14}$$

Step 3. Reduction of the problem. By properties p1–p4, the function $\mathcal{U}_y \in L^\infty(\mathbb{R})$ is nonnegative, even, and nondecreasing on $(-\infty,0)$. Then, Lemma 6.1 shows that the proof of p5 can be reduced to the proof of the following property:

$$\forall x_0 > \frac{1}{2} \quad \lim_{n\to+\infty} n^{-1}\int_{n(x_0-1/2)}^{n(x_0+1/2)} y^2\mathcal{U}_y(y)dy = \frac{u_+ - u_-}{2\pi^2}. \tag{6.15}$$

Moreover, equality (6.14) implies that $\mathcal{U}_y(y) = \frac{\Lambda^1\mathcal{U}(y)}{y-\mathcal{U}(y)}$ (for a.e. $y > \|\mathcal{U}\|_\infty$), and Lemma 6.4 implies that

$$\lim_{n\to+\infty} n^{-1}\int_{n(x_0-1/2)}^{n(x_0+1/2)} |\Lambda^1\mathcal{U}(y)|\,dy = 0;$$

hence, since $\frac{y^2}{y-\mathcal{U}(y)} = y + \mathcal{O}(1)$ as $|y| \to +\infty$, one deduces that (6.15) is equivalent to the following property:

$$\forall x_0 > \frac{1}{2} \quad \lim_{n\to+\infty} n^{-1}\int_{n(x_0-1/2)}^{n(x_0+1/2)} y\Lambda^1\mathcal{U}(y)\,dy = \frac{u_+ - u_-}{2\pi^2}. \tag{6.16}$$

Conclusion: Proof of (6.16). Let us change the variable by $y = nx$. Easy computations show that

$$\begin{aligned} n^{-1}\int_{n(x_0-1/2)}^{n(x_0+1/2)} y\Lambda^1\mathcal{U}(y)\,dy &= n^{-1}\int_{x_0-1/2}^{x_0+1/2} nx\Lambda^1 U\left(\frac{x}{n^{-1}},1\right) ndx \\ &= \int_{x_0-1/2}^{x_0+1/2} x\,\Lambda^1 U(x,n^{-1})\,dx. \end{aligned}$$

Since Lemma 6.5 implies that $\{\Lambda^1 U(x,n^{-1})\}_{n\in\mathbb{N}}$ converges in $L^1((x_0-1/2,x_0+1/2))$ toward $\frac{u_+-u_-}{2\pi^2}$ as $n \to +\infty$, the proofs of (6.16) and thus of property p5 are complete.

6.4. Duhamel's representation of the self-similar profile. It remains to prove Theorem 1.10, for which we need the following result.

PROPOSITION 6.6. *Let $\alpha = 1$, and let $U = U(x,t)$ be the self-similar solution of Theorem* 1.6 *with $u_\pm = \pm 1/2$. Then, for all $x \in \mathbb{R}$, we have*

$$\begin{aligned}(6.17)\quad U(x,1) = -\frac{1}{2} &+ H_1(x,1)\\ &- \int_0^{1/2} \partial_x p_1(1-\tau) * \frac{U^2(\cdot/\tau,1)}{2}(x)\,d\tau\\ &- \int_{1/2}^1 \tau^{-1}\, p_1(1-\tau) * \left(U\left(\frac{\cdot}{\tau},1\right) U_x\left(\frac{\cdot}{\tau},1\right)\right)(x)\,d\tau\end{aligned}$$

(where $H_1(x,1) = \int_{-\infty}^x p_1(y,1)dy$).

Proof. The proof proceeds in several steps.

Step 1. Duhamel's representation of the approximate solution. Notice that formula (6.17) makes sense. Indeed, by the homogeneity property (3.5), we have for all $t > 0$

$$\|\partial_x p_1(t)\|_1 = C_0 t^{-1}, \tag{6.18}$$

where $C_0 \equiv \|\partial_x P_1(1)\|_1$ is finite by (3.6). Hence, the integral $\int_0^{1/2} \dots d\tau$ in (6.17) is well defined since the integration variable τ is far from the singularity at $\tau = 1$. In the same way, since $U(1) \in W^{1,\infty}(\mathbb{R})$, the integral $\int_{1/2}^1 \dots d\tau$ is also well defined.

Now let $u^\varepsilon = u^\varepsilon(x,t)$ be the solution to the regularized equation (3.1), with initial datum U_0 in (1.9). The goal is to pass to the limit in formula (3.3) at time $t = 1$, namely,

$$\begin{aligned}(6.19)\quad u^\varepsilon(x,1) = S_1^\varepsilon&(1)U_0(x)\\ &- \int_0^{1/2} p_2(\varepsilon(1-\tau)) * \partial_x p_1(1-\tau) * \frac{(u^\varepsilon(\tau))^2}{2}(x)\,d\tau\\ &- \int_{1/2}^1 p_2(\varepsilon(1-\tau)) * p_1(1-\tau) * (u^\varepsilon(\tau) u_x^\varepsilon(\tau))(x)\,d\tau\end{aligned}$$

for all $x \in \mathbb{R}$.

Step 2. Pointwise limits and bounds of the integrands. We first remark that

$$\lim_{x\to\pm\infty} u^\varepsilon(x,t) = u^\pm\,.$$

Indeed, we know that u^ε is nondecreasing and it can be shown, for instance, that $u^\varepsilon - U_0 \in L^1(\mathbb{R})$. This fact can be proved by splitting methods, for instance.

Hence, thanks to the Dini theorem for cumulative distribution functions, we know that for fixed $t > 0$, $\lim_{\varepsilon\to 0} u^\varepsilon(t)$ converges toward $U(t)$ uniformly on $\mathbb{R}$.

Let us next recall that $\partial_x p_1(t) \in L^1(\mathbb{R})$, so that for fixed $\tau \in (0,1)$

$$\lim_{\varepsilon\to 0} \partial_x p_1(1-\tau) * \frac{(u^\varepsilon(\tau))^2}{2} = \partial_x p_1(1-\tau) * \frac{(U(\tau))^2}{2} \quad \text{uniformly on } \mathbb{R}.$$

It follows from classical approximate unit properties of the heat kernel $p_2(x,t)$ that for all $\tau \in (0,1)$

$$\lim_{\varepsilon\to 0} p_2(\varepsilon(1-\tau)) * \partial_x p_1(1-\tau) * \frac{(u^\varepsilon(\tau))^2}{2} = \partial_x p_1(1-\tau) * \frac{(U(\tau))^2}{2} \tag{6.20}$$

uniformly on $\mathbb{R}$. In particular, for all $\tau \in (0,1)$, we also have

$$\lim_{\varepsilon \to 0} p_2(\varepsilon(1-\tau)) * p_1(1-\tau) * (u^\varepsilon(\tau) u_x^\varepsilon(\tau)) = p_1(1-\tau) * (U(\tau)U_x(\tau)) \tag{6.21}$$

uniformly on $\mathbb{R}$, since

$$p_2(\varepsilon(1-\tau)) * \partial_x p_1(1-\tau) * \frac{(u^\varepsilon(\tau))^2}{2} = p_2(\varepsilon(1-\tau)) * p_1(1-\tau) * (u^\varepsilon(\tau) u_x^\varepsilon(\tau))$$

and $\partial_x p_1(1-\tau) * \frac{U^2(\tau)}{2} = p_1(1-\tau) * (U(\tau)U_x(\tau))$.

Moreover, by (3.7), (3.8) with $p = +\infty$ and (6.18), one can see that the integrands of (6.19) are pointwise bounded by

$$\left\| p_2(\varepsilon(1-\tau)) * \partial_x p_1(1-\tau) * \frac{(u^\varepsilon(\tau))^2}{2} \right\|_\infty \le C_0 (1-\tau)^{-1} \frac{\|u_0\|_\infty^2}{2} \tag{6.22}$$

and

$$\left\| p_2(\varepsilon(1-\tau)) * p_1(1-\tau) * (u^\varepsilon(\tau) u_x^\varepsilon(\tau)) \right\|_\infty \le \tau^{-1} \|u_0\|_\infty. \tag{6.23}$$

Step 3. Passage to the limit. Recall that

$$\lim_{\varepsilon \to 0} S_1^\varepsilon(1) U_0 = S_1(1) U_0 = p_1(1) * U_0$$

in $L^p(\mathbb{R})$ for all $p \in [1, +\infty]$. Let us recall that $U_0(x) = \pm 1/2$ for $\pm x \ge 0$ and $\int_\mathbb{R} p_1(y,1)dy = 1$, so that for all $x \in \mathbb{R}$

$$1/2 + p_1(1) * U_0(x) = p_1(1) * (U_0 + 1/2)(x) = \int_{-\infty}^x p_1(y,1)dy = H_1(x,1).$$

We have proved in particular that $\lim_{\varepsilon \to 0} S_1^\varepsilon(1) U_0 = -1/2 + H_1(1)$ pointwise on $\mathbb{R}$.

In order to pass to the limit in the integral terms of (6.19), we use the Lebesgue dominated convergence theorem. We deduce from (6.20) and (6.22) that for all $x \in \mathbb{R}$ the first integral term converges toward

$$\int_0^{1/2} \partial_x p_1(1-\tau) * \frac{(U(\tau))^2}{2}(x)\, d\tau$$

as $\varepsilon \to 0$. In the same way, we deduce from (6.21) and (6.23) that the last integral term converges toward

$$\int_{1/2}^1 p_1(1-\tau) * (U(\tau)U_x(\tau))(x)\, d\tau.$$

The limit as $\varepsilon \to 0$ in (6.19) then implies that for all $x \in \mathbb{R}$

$$U(x,1) = -\frac{1}{2} + H_1(x,1) - \int_0^{1/2} \partial_x p_1(1-\tau) * \frac{U^2(\tau)}{2}(x)\, d\tau \\ - \int_{1/2}^1 p_1(1-\tau) * (U(\tau)U_x(\tau))(x)\, d\tau.$$

This completes the proof of (6.17), thanks to the self-similarity of U. $\square$

Proof of Theorem 1.10. We have to prove that for all $r > 0$

$$\mathbb{P}(|X - \overline{c}| < r) < \mathbb{P}(|Y - 0| < r). \tag{6.24}$$

Let us verify that $\overline{c}$ and 0 are the medians of X and Y, respectively. First, a simple computation allows us to see that $p_1(x, 1)$, defined by the Fourier transform by $\widehat{p_1}(\xi, 1) = e^{-|\xi|}$, also satisfies formula (1.12). This density of probability is even, and the median of Y is null. Second, by property p3 of Theorem 1.9, $U_x(1)$ is symmetric w.r.t. to the axis $\{x = \overline{c}\}$ and the median of X is $\overline{c} = \frac{u_- + u_+}{2}$.

In particular, the centered random variable $X - \overline{c}$ admits a density being the even function

$$f_{X-\overline{c}}(x) = U_x(x + \overline{c}, 1).$$

It becomes clear that (6.24) is equivalent to the following property:

$$\forall x > 0 \quad F_{X-\overline{c}}(x) < F_Y(x), \tag{6.25}$$

where $F_{X-\overline{c}}$ and F_Y are the cumulative distribution functions of $X - \overline{c}$ and Y, respectively.

Let us compute these functions. First, we have seen above that $f_{X-\overline{c}}(x) = V_x(x, 1)$, where V is defined by the transformation (6.1). Let us recall that V is the self-similar solution to (1.1) with initial datum $V(x, 0) = \pm 1/2$ for $\pm x > 0$. Hence, $F_{X-\overline{c}}$ is equal to $V(\cdot, 1)$ up to an additive constant, which has to be $1/2$ by property p2 of Theorem 1.9; that is to say, we have $F_{X-\overline{c}}(x) = 1/2 + V(x, 1)$ for all $x \in \mathbb{R}$. Second, we defined H_1 in Proposition 6.6 such that $F_Y(x) = H_1(x, 1)$. By this proposition, we have for all $x \in \mathbb{R}$

$$F_{X-\overline{c}}(x) = F_Y(x) - g(x),$$

where $g(x)$ is defined by

$$\begin{aligned} g(x) \equiv \int_0^{1/2} \partial_x p_1(1 - \tau) * \frac{V^2(\cdot/\tau, 1)}{2}(x)\, d\tau \\ + \int_{1/2}^1 \tau^{-1} p_1(1 - \tau) * \left(V\left(\frac{\cdot}{\tau}, 1\right) V_x\left(\frac{\cdot}{\tau}, 1\right)\right)(x)\, d\tau. \end{aligned} \tag{6.26}$$

One concludes that the proof of (6.25), and thus of (6.24), is equivalent to the proof of the positivity of $g(x)$ for positive x. But, by the definition of g, it suffices to prove that for each $\tau \in (0, 1)$ and $x > 0$

$$p_1(1 - \tau) * (V(\cdot/\tau, 1)V_x(\cdot/\tau, 1))(x) > 0. \tag{6.27}$$

Indeed, the second integral term in (6.26) would be positive, and the first integral term also, since for fixed τ

$$\partial_x p_1(1 - \tau) * \frac{V^2(\cdot/\tau, 1)}{2}(x) = \tau^{-1} p_1(1 - \tau) * \left(V\left(\frac{\cdot}{\tau}, 1\right) V_x\left(\frac{\cdot}{\tau}, 1\right)\right)(x).$$

Let us end by proving inequality (6.27), thus concluding Theorem 1.10. It is clear that the function $V(\cdot/\tau,1)V_x(\cdot/\tau,1)$ is odd, since $V(1)$ is odd. Moreover, we already know that $V_x(1)$ is nonnegative, even, and nonincreasing on $(0,+\infty)$, since $V(1)$ is nondecreasing, odd, and concave on $[0,+\infty)$. By property p5, we conclude that $V_x(1)$ is positive a.e. on $(0,+\infty)$ and thus on $\mathbb{R}$ as the even function. In particular, $V(1)$ is increasing and, for all $x>0$, $V(x,1)>V(0,1)=0$.

To summarize, $V(\cdot/\tau,1)V_x(\cdot/\tau,1)$ is odd and positive on $(0,+\infty)$. Moreover, it is clear that $p_1(1-\tau)$ is positive, even, and decreasing on $(0,+\infty)$; see (1.12). A simple computation then implies that the convolution product in (6.27) is effectively positive for positive x. The proof of Theorem 1.10 is complete. ∎

Appendix A. Proof of Theorem 4.1. The proof relies on a form of the Aubin–Simon compactness result that we recall below.

THEOREM A.1 (see [14]). *Let $T>0$, $1<p\le+\infty$, and $1\le q\le+\infty$, and assume that V, E, and F are Banach spaces such that V is compactly embedded in E and E is continuously embedded in F. If A is a bounded subset of $W^{1,p}(0,T;F)$ and of $L^q(0,T;V)$, then A is relatively compact in $C([0,T];F)$ and in $L^q(0,T;E)$.*

Let us now prove the convergence result of section 4.

Proof of Theorem 4.1. The proof follows in two steps: first, we show the relative compactness of the family of functions $\mathcal{F}\equiv\{u^\varepsilon:\varepsilon\in(0,1]\}$ in the space $C([0,T];L^1_{loc}(\mathbb{R}))$, and, next, we pass to the limit in entropy inequalities.

Step 1. Compactness. By the nonincrease of the L^∞-norm and BV-seminorm in Remark 3.2, one has $\|u^\varepsilon(t)\|_\infty\le\|u_0\|_\infty$ and $\|u^\varepsilon(t)\|_1\le\|m\|$ for all $t>0$. The family $\mathcal{F}$ is thus bounded in $L^\infty(\mathbb{R}\times(0,T))$ and in $L^\infty(0,T;BV(\mathbb{R}))$ for any $T>0$. By (4.1), u^ε_t is bounded in $L^\infty\left(0,T;(C^2_K(\mathbb{R}))'\right)$ for any compact $K\subset\mathbb{R}$, where $C^2_K(\mathbb{R})$ is the set of C^2 functions with compact support in K (with its natural Banach norm) and $(C^2_K(\mathbb{R}))'$ is its topological dual. The Aubin–Simon theorem, applied with $q=p=+\infty$, $V=L^1(K)\cap BV(K)$, $E=L^1(K)$, $F=(C^2_K(\mathbb{R}))'$, and $A=\mathbf{1}_{K\times[0,T]}\mathcal{F}$, ensures that $\mathcal{F}$ is relatively compact in $L^\infty(0,T;L^1(K))$ for all $T>0$ and all compact $K\subset\mathbb{R}$; since u^ε is continuous with values in $L^1_{loc}(\mathbb{R})$, this gives the compactness of $\mathcal{F}$ in $C([0,T];L^1_{loc}(\mathbb{R}))$.

Step 2. Passage to the limit. We deduce from Step 1 that there exists $v\in C([0,+\infty);L^1_{loc}(\mathbb{R}))$ such that u^ε converges toward v as $\varepsilon\to0$ (up to a subsequence) in $C([0,T];L^1_{loc}(\mathbb{R}))$ for all $T>0$. Up to another subsequence if necessary, we can assume that $u^\varepsilon\to v$ a.e. From inequality (3.7), we deduce that $v\in L^\infty(\mathbb{R}\times(0,+\infty))$. We have to prove that $v=u$; by the uniqueness of entropy solutions (cf. Theorem 2.2), it suffices to show that v is an entropy solution to (2.7)–(2.8).

To do so, observe that u^ε satisfies

$$\int_{\mathbb{R}}\int_a^{+\infty}\Big(\eta(u^\varepsilon)\varphi_t+\phi(u^\varepsilon)\varphi_x\Big)\,dxdt + \int_{\mathbb{R}}\int_a^{+\infty}\Big(-\eta(u^\varepsilon)\Lambda_r^{(\alpha)}\varphi-\varphi\eta'(u^\varepsilon)\,\Lambda_r^{(0)}u^\varepsilon\Big)\,dxdt + \varepsilon\int_{\mathbb{R}}\int_a^{+\infty}\eta(u^\varepsilon)\,\varphi_{xx}\,dxdt+\int_{\mathbb{R}}\eta(u^\varepsilon(x,a))\varphi(x,a)\,dx\ge0$$

for all $\varphi\in\mathcal{D}(\mathbb{R}\times[0,+\infty))$ nonnegative, $\eta\in C^2(\mathbb{R})$ convex, $\phi'=\eta'f'$, and $a,r>0$. This inequality derives from the multiplication of (4.1) by $\eta'(u^\varepsilon)\varphi$, the Kato inequalities (2.4), and integration by parts over the domain $\mathbb{R}\times[a,+\infty)$. Passing to the limit $a \to 0$ in this inequality, we get

$$\int_{\mathbb{R}} \int_0^{+\infty} \Big(\eta(u^\varepsilon)\varphi_t + \phi(u^\varepsilon)\varphi_x - \eta(u^\varepsilon)\Lambda_r^{(\alpha)}\varphi - \varphi\eta'(u^\varepsilon)\ \Lambda_r^{(0)} u^\varepsilon \Big)\, dxdt \\ + \int_{\mathbb{R}} \eta(u_0(x))\varphi(x,0)\, dx \geq -\varepsilon \int_{\mathbb{R}} \int_0^{+\infty} \eta(u^\varepsilon)\varphi_{xx}\, dxdt.$$

Finally, let us recall that $u^\varepsilon \to v$ a.e. as $\varepsilon \to 0$ and that u^ε is bounded in the L^∞-norm by $\|u_0\|_\infty$. The Lebesgue dominated convergence theorem allows us to pass to the limit, as $\varepsilon \to 0$, in the inequality above and to deduce that

$$\int_{\mathbb{R}} \int_0^{+\infty} \Big(\eta(v)\varphi_t + \phi(v)\varphi_x - \eta(v)\Lambda_r^{(\alpha)}\varphi - \varphi\eta'(v)\ \Lambda_r^{(0)} v \Big)\, dxdt \\ + \int_{\mathbb{R}} \eta(u_0(x))\varphi(x,0)\, dx \geq 0.$$

Hence, according to Definition 2.1 and Theorem 2.2, the function v is the unique entropy solution to (2.7)–(2.8). The proof of Theorem 4.1 is complete. ∎

Appendix B. An additional technical lemma.

LEMMA B.1. *Let I be an open interval of $\mathbb{R}$ and $u \in W^{1,\infty}(I)$ be such that $u_x \in BV(I)$. Then, for a.e. $x \in I$ and all $z \in I - x$ we have*

$$u(x+z) = u(x) + u_x(x)z + \int_{I_{x,z}} |x+z-y|\ u_{yy}(dy),$$

where $I_{x,z} \equiv (x, x+z)$ if $z > 0$ and $I_{x,z} \equiv (x+z, x)$ if not.

Proof. We can reduce to the case $I = (a,b)$ with $a, b \in \mathbb{R}$. Let us assume without loss of generality that $z > 0$. Since $u_x \in BV(I)$, the function $\widetilde{u}_x(x) \equiv c + \int_{(a,x]} u_{yy}(dy)$ is an a.e. representative of u_x, where c is the trace of u_x on the left boundary of I. The trace of $u_x \in BV(I_{x,z})$ onto $\{x\}$ is equal to $\widetilde{u}_x(x)$, because $\{x\}$ is the left boundary of I_{x+z}. A simple integration by parts formula now gives

$$\begin{aligned} u(x+z) &= u(x) + \int_{I_{x,z}} u_y(y)dy \\ &= u(x) - \int_{I_{x,z}} (y-x-z)u_{yy}(y)dy + \widetilde{u}_x(x)z. \end{aligned}$$

The proof is complete. ∎

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