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Self-Similar Solutions for a Fractional Thin Film Equation Governing Hydraulic Fractures

C. Imbert[1], **A. Mellet**[2,3]

[1] CNRS, UMR 7580, Université Paris-Est Créteil, 61 avenue du Général de Gaulle, 94 010 Créteil Cedex, France
[2] Department of Mathematics, University of Maryland, College Park, MD 20742, USA.
E-mail: mellet@math.umd.edu
[3] Fondation Sciences Mathématiques de Paris, 11 rue Pierre et Marie Curie, 75231 Paris Cedex 05, France

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Abstract: In this paper, self-similar solutions for a fractional thin film equation governing hydraulic fractures are constructed. One of the boundary conditions, which accounts for the energy required to break the rock, involves the toughness coefficient $K \geq 0$. Mathematically, this condition plays the same role as the contact angle condition in the thin film equation. We consider two situations: The zero toughness ($K = 0$) and the finite toughness $K \in (0, \infty)$ cases. In the first case, we prove the existence of self-similar solutions with constant mass. In the second case, we prove that for all $K > 0$ there exists an injection rate for the fluid such that self-similar solutions exist.

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1. Introduction

1.1. The model. The following third order degenerate parabolic equation arises in the modeling of hydraulic fractures:

$$\partial_t u + \partial_x (u^3 \partial_x I(u)) = S, \tag{1}$$

where the operator I denotes the square root of the Laplace operator:

$$I(u) = -(-\Delta)^{\frac{1}{2}} u.$$

This equation can be seen as a fractional version of the thin film equation (which corresponds to $I(u) = \Delta u$). It is also reminiscent of the porous media equation, which corresponds to $I(u) = -u$.

In the context of hydraulic fractures, the unknown $u(x, t)$ represents the opening of a rock fracture that is propagated in an elastic material due to the pressure exerted by a viscous fluid that fills the fracture. Such fractures occur naturally, for instance in volcanic dikes where magma causes fracture propagation below the surface of the Earth, or can be deliberately propagated in oil or gas reservoirs to increase production. The term S on the right hand side of the equation is a source term which models the injection of fluid into the fracture. It is usually assumed to be 0 or of the form $h(t)\delta(x)$ (corresponding at the injection of fluid into the fracture at a rate $h(t)$ through a pipe located at $x = 0$).

There is a significant amount of work involving the mathematical modeling of hydraulic fractures (see for instance Barenblatt [4] and references therein). The model that we consider in our paper, which corresponds to a very simple fracture geometry, was developed independently by Geertsma and De Klerk [13] and Zheltov and Kristianovich [25]. Note that the profile of the self-similar solution of the porous medium equation exhibited independently by Zeldovitch and Kompaneets [24] and Barenblatt [3] is a stationary solution of (1). Spence and Sharp [23] initiated the work on self-similar solutions and formal asymptotic analysis of the solutions of (1) near the tip of the fracture (i.e. the boundary of the support of u). There is now an abundant literature that has extended this formal analysis to various regimes (see for instance [1,2,16] and references therein). Several numerical methods have also been developed for this model (see in particular Peirce et al. [18–21]).

In a recent paper [14], we established the existence of a weak solution to this equation in a bounded interval. To our knowledge, this was the first rigorous existence result for this equation. In fact, in that paper, we considered the more general equation

$$\partial_t u + \partial_x (u^n \partial_x I(u)) = 0$$

for any $n \geq 1$. Indeed, as shown in [14], the particular value $n = 3$ in (1) follows from the choice of no-slip Navier boundary conditions for the fluid in contact with the rock. However, as for the thin film equation, other values of n (namely $n = 1$ and $n = 2$) are also of interest when other types of fluid boundary conditions are considered in the derivation of the equation (see Sect. 1.3 below and [14]). Mathematically, the properties of the solutions depend strongly on the value of the parameter n, as is the case for the thin film equation. In fact, the results of [14] show that many aspects of the mathematical analysis of (1) are similar to the theory of the thin film equation; however, the fact that I is a non-local operator introduces many new difficulties to the problem. It was also pointed out in [14] that the value $n = 4$ is critical for this equation, in the same way that the value $n = 3$ is known to be critical for the thin film equation. As we will see, the main results in our paper will indeed require that $n < 4$.

Before going any further, we need to determine the appropriate boundary conditions. Equation (1) is satisfied within the fracture, that is in the region $\{u > 0\}$ (note that u has to be defined in whole of $\mathbb{R}$ so that the non-local square root of the Laplacian can be defined). At the tip of the fracture, that is on $\partial\{u > 0\}$, it is natural to assume a null flux boundary condition (no leak of fluid through the rock):

$$u^n \partial_x I(u) = 0 \quad \text{on } \partial\{u > 0\}$$

(models involving leak at the tip of the fracture are also of interest for applications, but will not be discussed in this paper). Together with the fact that $u = 0$ in $\mathbb{R}^n \setminus \{u > 0\}$, this gives us two boundary conditions. Since we are dealing with a free boundary problem of order three, these two conditions are not enough to have a well posed problem. The missing condition takes into account the energy required to break the rock and takes the form (see for instance [16]):

$$u(t, x) = K\sqrt{|x - x_0|} + o(|x - x_0|^{1/2}) \quad \text{as } x \to x_0 \tag{2}$$

for all $x_0 \in \partial\{u(t, \cdot) > 0\}$, where the coefficient K is related to the toughness of the rock and is assumed to be known. From a mathematical point of view, we note that condition (2) plays the same role as the contact angle condition for the thin film equation.

The particular case $K = 0$ is mathematically interesting (it corresponds to the "zero contact angle" condition—or complete wetting regime—often studied in the thin film literature). In the framework of hydraulic fracture, this *zero toughness* condition can be interpreted as modeling the expansion of a fracture in a pre-cracked rock.

Note that in [14], we did not include a free boundary condition, and instead considered that Eq. (1) was satisfied everywhere. The solutions that we constructed there belonged to $L^2_t(H^{3/2}_x)$ and thus satisfied $u(t, \cdot) \in C^\alpha$ for a.e. $t > 0$ for all $\alpha < 1$. In particular, compactly supported solutions would satisfy (2) on the boundary of their support (or tip of the fracture) with $K = 0$. In other words, the solutions constructed in [14] correspond to the zero toughness (or pre-cracked) regime. In the present paper, we consider the full free boundary problem with $K > 0$ and we will prove the existence of self-similar solutions in both the zero toughness case (without injection of fluid) and the non-zero toughness case (with specific injection rate). These are thus the first rigorous existence results for solutions satisfying the free boundary condition (2) with $K > 0$. We also rigorously investigate the behavior of the solution at the tip of the fracture.

In the case of the thin film equation ($I(u) = \Delta u$), the existence of self-similar solutions has been proved in the zero contact angle case (which corresponds to the case $K = 0$ here) in particular by Bernis et al. [5] in dimension 1, and by Ferreira and Bernis [11] in dimension greater than 2. It is worth noticing that while our result concerns only the dimension 1, the proofs will be somewhat more similar to the higher dimensional case for the thin film equation.

1.2. Main results. To summarize the introduction above, the equation under consideration in this paper is the following:

$$\partial_t u + \partial_x(u^n \partial_x I(u)) = h(t)\delta, \quad t > 0, \quad \text{in } \{u > 0\}, \tag{3}$$

where $n \geq 1$, together with the boundary conditions

$$u^n \partial_x I(u) = 0 \quad \text{on } \partial\{u > 0\} \tag{4}$$

and

$$u(t, x) = K\sqrt{|x - x_0|} + o\left(\sqrt{|x - x_0|}\right) \quad \text{as } x \sim x_0 \tag{5}$$

for all $x_0 \in \partial\{u(t, \cdot) > 0\}$.

The two main parameters are the function $h(t)$, which corresponds to the injection rate of the fluid into the fracture, and the constant K, which describes the toughness of the rock. Note that when $h = 0$ (no injection of fluid) and $K \neq 0$, then (3)–(5) has a stationary solution supported in $(-1, 1)$ given by

$$V(x) = \frac{K}{\sqrt{2}}\sqrt{(1 - x^2)_+}.$$

(this is checked easily using the fact that $I(\sqrt{(1 - x^2)_+}) = -\frac{2}{\sqrt{\pi}}$). Clearly, the function $\sqrt{a}\, V(x/a)$ is also a stationary solution supported in $(-a, a)$ for any $a > 0$.

The goal of this paper is to prove the existence of another type of particular solutions of (3)–(5): compactly supported self-similar solutions. More precisely, we are looking for solutions of the form

$$u(t, x) = t^{-\alpha} U(t^{-\beta} x) \tag{6}$$

for some *profile function* U, which is even and supported in an interval $[-a, a]$ for some $a > 0$.

Inserting (6) into (3), we find that U must solve

$$\begin{aligned} &-\alpha U(t^{-\beta} x) - \beta t^{-\beta} x U'(t^{-\beta} x) + t^{-n\alpha+1-3\beta}(U^n I(U)')'(t^{-\beta} x) \\ &\quad = t^{1+\alpha} h(t) t^{-\beta} \delta(t^{-\beta} x) \end{aligned}$$

(using the fact that $t^{-\beta}\delta(t^{-\beta} x) = \delta(x)$). So we must take α and β such that

$$1 - 3\beta = n\alpha \tag{7}$$

and the injection rate $h(t)$ given by

$$h(t) = \lambda t^{-\alpha-1+\beta} \tag{8}$$

for some constant $\lambda \in \mathbb{R}$. Then the profile $y \mapsto U(y)$ is solution to the equation

$$-\alpha U - \beta y U' + (U^n I(U)')' = \lambda\delta \quad \text{in } (-a, a). \tag{9}$$

The profile function U must also satisfy appropriate boundary conditions. Clearly, if U satisfies

$$U^n I(U)' = 0 \quad \text{on } \partial\{U > 0\} \tag{10}$$

then u will satisfy (4). The boundary condition (5), however, is more delicate. Indeed, we notice that if U satisfies

$$U(x) = K\sqrt{|x-a|} + o\left(\sqrt{|x-a|}\right),$$

then the function $u(t,x)$ defined by (6) satisfies

$$u(t,x) = Kt^{-\alpha-\beta/2}|x-a(t)|^{1/2} + o(|x-a(t)|^{1/2}) \tag{11}$$

with $a(t) = t^\beta a \in \partial\{u(t,\cdot) > 0\}$. So a self-similar solution $u(t,x)$ can only satisfy the free boundary condition (5) with given, time independent, toughness coefficient K if either $K = 0$ (zero toughness) or if $\alpha = -\frac{\beta}{2}$

We will thus construct two types of self-similar solutions:

- In the case where no fluid is injected ($h(t) = 0$), we will show that there exist self-similar solutions satisfying (5) with $K = 0$ (zero toughness case) and constant mass m (in particular $\alpha = \beta$);
- For given toughness coefficient $K > 0$, we will show that there exists an injection rate $h(t)$ (of the form (8)) such that there exists a self-similar solution satisfying (5) for all t.

More precisely, our main result is the following:

Theorem 1.1. *Assume that $n \in [1, 4)$.*

(i) *Assume that $K = 0$ and $h(t) = 0$. Then, for any $m > 0$ there exists a self-similar solution of (3)–(5) of the form*

$$u(t,x) = t^{-\frac{1}{n+3}} U\left(t^{-\frac{1}{n+3}} x\right)$$

satisfying $\int_{\mathbb{R}} u(t,x)\,dx = m$ for all $t > 0$. The profile function $x \mapsto U(x)$ is a non-negative, even function with supp $U = [-a, a]$ *for some $a > 0$ (depending on m). Furthermore, for all $t > 0$, there exists a constant $C(t) > 0$ such that u satisfies*

$$u(t,x) = \begin{cases} C(t)|x-x_0|^{\frac{3}{2}} + \mathcal{O}\left(|x-x_0|^{\frac{2}{n}}\right) & \text{if } n \in [1, \frac{4}{3}) \\ C(t)|x-x_0|^{\frac{3}{2}}\left|\ln|x-x_0|\right|^{\frac{3}{4}} + \mathcal{O}\left(|x-x_0|^{\frac{3}{2}}\right) & \text{if } n = \frac{4}{3} \\ C(t)|x-x_0|^{\frac{2}{n}} + o\left(|x-x_0|^{\frac{2}{n}}\right) & \text{if } n \in \left(\frac{4}{3}, 4\right) \end{cases}$$

when $x \to x_0$, for any $x_0 \in \partial\{u(t,\cdot) > 0\}$.

(ii) *For any $K > 0$ and for any $a > 0$ there exists $\lambda > 0$ such that Eqs. (3)–(5) has a self-similar solution when $h(t) = \lambda t^{\frac{n-3}{6-n}}$. This solution has the form*

$$u(t,x) = t^{\frac{1}{6-n}} U\left(t^{-\frac{2}{6-n}} x\right)$$

where U *is a non-negative, even function with* $\text{supp}\ U = [-a, a]$*. Furthermore,* u *satisfies*

$$u(t,x) = K\sqrt{|x-x_0|} + \begin{cases} \mathcal{O}\left(|x-x_0|^{\frac{3}{2}}\right) & \text{if } n \in [1,2) \\ \mathcal{O}\left(|x-x_0|^{\frac{3}{2}} \ln\left(\frac{1}{|x-x_0|}\right)\right) & \text{if } n = 2 \\ \mathcal{O}\left(|x-x_0|^{\frac{5-n}{2}}\right) & \text{if } n \in (2,4) \end{cases}$$

when $x \to x_0$*, for any* $x_0 \in \partial\{u(t,\cdot) > 0\}$.

Remark 1.2. 1. Note that in the physical case, that is when $n = 3$, we find $h(t) = \lambda$, so self-similar solutions in that case correspond to a constant injection rate.

2. Note also that in the first part of the theorem ($K = 0$), the self-similar solution satisfies

$$\lim_{t \to 0^+} u(t,x) = m\delta$$

in the sense of distributions. Such a solution is also sometimes called a Source-type solution. On the other hand, in the second part ($K \neq 0$), we clearly have

$$\lim_{t \to 0^+} ||u(t,x)||_{L^\infty} = 0.$$

3. In the case $n = 3$, $K > 0$, we recover here known (formal) results concerning the rate of growth of hydraulic fractures (see [1,10,12,15]): the length of the fracture is proportional to $t^{2/3}$ and its width ($= u(t,0)$) is proportional to $t^{1/3}$. We also recover the following asymptotic at the tip of the fracture (see [15])

$$u(t,x) = K\sqrt{|x-x_0|} + \mathcal{O}(|x-x_0|).$$

4. In the second part of the theorem, we fix K and a (which is half the length of the support of u at time $t = 1$), and find the appropriate value of λ for a solution to exist. It would be more satisfactory to show that a solution exists for all values of $K > 0$ and $\lambda > 0$. We will see in the next section that the constant λ satisfies

$$\lambda = \frac{3}{6-n} \int_{-a}^{a} U(x)dx.$$

Using this relation, we can then show that for a given K, we have $\lambda(a) \to 0$ as $a \to 0$ and $\lambda(a) \to \infty$ as $a \to \infty$. It seems thus reasonable to expect that for all K and for all $\lambda > 0$, there exists a self similar solution of (3)–(5) (which is obtained for an appropriate choice of a). However, to prove this rigorously, one needs to show that the function $a \mapsto \lambda(a)$ is continuous, and such a result should typically follow from some uniqueness principle for U.

Unfortunately the question of the uniqueness of the self-similar solution for this problem, which is of independent interest, is notoriously hard to obtain for such non-linear higher order equations. In [11], Ferreira and Bernis prove the uniqueness of self similar solutions for the thin film equation in the zero contact angle case. However, even in the zero toughness case, such a proof does not seem to extend to our case, mainly because of the nonlocal character of the fractional Laplacian. The question of the uniqueness of self similar solutions, both in the case $K = 0$ and $K > 0$ is thus left as an interesting and challenging open problem here.

In the next section, we will set up the equations to be solved by the profile $U(x)$ in both cases of Theorem 1.1. At the end of that section (see Sect. 2.3 below), we describe the general strategy to be used, which is reminiscent of the approach of Bernis and Ferreira [11] for the thin film equation in dimension greater than or equal to 2. In particular, this strategy relies on an integral formulation and a fixed point argument, which requires a detailed knowledge of the Green function associated to the operator $u \mapsto I(u)'$. The properties of this Green function are discussed in Sect. 3, which is the core of this paper. In particular, very detailed results concerning the boundary behavior of the solution of the equation $I(u)' = f$ are given in that section. These results play a fundamental role in the proof of our main result, which is given in Sect. 4.

1.3. Derivation of the model. As mentioned in the introduction, when $n = 3$, Eq. (1) was introduced to describe the propagation of an impermeable KGD fracture (named after Kristianovich, Geertsma and De Klerk) driven by a viscous fluid in a uniform elastic medium under condition of plane strain. We recall in this section the main steps of this derivation (see [13,25]). Everything in this section can be found in the literature, and is recalled here for the reader's sake. We denote by (x, y, z) the standard coordinates in $\mathbb{R}^3$; we consider a fracture which is invariant with respect to one variable (z) and symmetric with respect to another direction (y). The fracture can then be entirely described by its opening $u(x, t)$ in the y direction. Since it assumes that the fracture is an infinite strip whose cross-sections are in a state of plane strain, this model is only applicable to rectangular planar fracture with large aspect ratio.

Lubrication approximation. Under the lubrication approximation, the conservation of mass for the fluid inside the fracture leads to the following equation:

$$\partial_t u - \partial_x \left(\frac{u^3}{12\mu} \partial_x p \right) = 0,$$

where $p(x)$ denotes the pressure exerted on the fluid by the rock and μ is the viscosity coefficient of the fluid (see [14] for more details about the lubrication approximation).

The pressure law. In the very simple geometry that we consider here, the elasticity equation expresses the pressure as a function of the fracture opening. More precisely, after a rather involved computation, which is recalled in Appendix C [9,17], we obtain:

$$p(x) = \frac{E}{4(1-\nu^2)}(-\Delta)^{1/2} u \tag{12}$$

where the square root of the Laplacian $(-\Delta)^{1/2}$ is defined using Fourier transform by

$$\mathcal{F}((-\Delta)^{1/2} u)(k) = |k| \mathcal{F}(u)(k),$$

and E denotes Young's modulus and ν is Poisson's ratio. We use the following convention for the Fourier transform,

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx.$$

Propagation condition (*Free boundary condition*). Equation (1) is satisfied only inside the fracture, that is in the support of u. It must be supplemented with boundary condition on $\partial\{u > 0\}$ (the free boundary). Naturally, we impose

$$u = 0, \quad u^3 \partial_x p = 0 \quad \text{on } \partial\{u > 0\}$$

which ensures zero width and zero fluid loss at the tip of the fracture. However, because we have an equation of order three, and the support is not known a priori, we need an additional condition to fully determine the solution. This additional condition is a propagation condition which requires the rock toughness K_{IC} (which is given) to be equal to the stress intensity factor K_I at the tip of the fracture. If $\{u > 0\} = (a(t), b(t))$, then the stress intensity factor at $x = b(t)$ is defined by

$$K_I := \lim_{x \to \partial b^+} \sqrt{2\pi}\sqrt{x-b}\,\sigma_{yy}(x,0)$$

where σ_{yy} is the yy component of the stress tensor given by (see Appendix C):

$$\sigma_{yy}(x,0) = -p(x).$$

So the propagation condition prescribes the behavior of the pressure at the tip of the fracture (outside of the fracture). A simple but technical lemma (see Appendix D for a proof) shows that this is related to the behavior of u inside the fracture:

Lemma 1.3. *Assume that* $\{u > 0\} = (-1, 1)$ *and recall that* p *is defined by* (12). *Then we have the following relations*

$$\lim_{x \to 1^+} -\sqrt{x-1}\,p(x) = \frac{1}{\pi\sqrt{2}} \int_{-1}^{1} \frac{\sqrt{1+z}}{\sqrt{1-z}} p(z)\,dz \tag{13}$$

and

$$\lim_{x \to 1^-} u'(x)\sqrt{1-x} = -\frac{4(1-\nu^2)}{\pi\sqrt{2}E} \int_{-1}^{1} \frac{\sqrt{1+z}}{\sqrt{1-z}} p(z)\,dz \tag{14}$$

In view of this lemma, the propagation condition $K_I = K_{IC}$ is thus equivalent to (assuming $b = 1$):

$$u(t,x) \sim \sqrt{\frac{2}{\pi}} \frac{4(1-\nu^2)}{E} K_{IC}\sqrt{1-x} \quad \text{as } x \to 1^-$$

which is the free boundary condition (2).

In the literature (see for instance [15,16]), this relation is often written as

$$u(t,x) \sim \frac{K'}{E'}\sqrt{1-x} \quad \text{as } x \to 1^-$$

where $K' = 4\sqrt{\frac{2}{\pi}} K_{IC}$ and $E' = \frac{E}{1-\nu^2}$.

2. Preliminary

2.1. The zero toughness case [Theorem 1.1-(i)]. When $h(t) = 0$ (no injection of fluid), Eq. (3) preserves the total mass, and so in order to find a self-similar solution of the form (6) we must take $\alpha = \beta$. According to (11), the free boundary condition (5) can then only be satisfied for all time if we have $K = 0$ (there also exist solutions with $K \neq 0$ depending on t, but the physical meaning of such solutions is not clear).

Next, we note that the condition (7), with $\alpha = \beta$, implies

$$\alpha = \beta = \frac{1}{n+3},$$

and Eq. (9) becomes

$$-(xU)' + (n+3)(U^n I(U)')' = 0 \quad \text{in } (-a, a).$$

We can integrate this equation once, and using the null flux boundary condition (10), we find

$$(n+3)U^n (I(U))' = xU \quad \text{in } [-a, a]. \tag{15}$$

At the end points $\pm a$, we have the obvious condition $U(\pm a) = 0$, and condition (5) (with $K = 0$) can also be written as

$$U(x) = o\left(\sqrt{|a^2 - x^2|}\right) \quad \text{as } x \to \pm a.$$

We recall that we also have the mass condition $\int_{-a}^{a} U(x)\,dx = m$. However, instead of fixing the mass, we will fix $a = 1$ and ignore the mass condition. Indeed, if U solves (15) in $(-1, 1)$, then $V(x) = a^{3/n} U(x/a)$ solves (15) in $(-a, a)$ and satisfies $\int_{-a}^{a} V(x)\,dx = m$ provided we choose $a^{\frac{3+n}{n}} = m / \int_{-1}^{1} U(x)dx$.

We can also remove the multiplicative factor $n + 3$ (consider the function $V(x) = bU(x)$ with $b = (n+3)^{\frac{1}{n}}$).

In conclusion, our task will be to prove that there exists a profile function $x \mapsto U(x)$ solution of

$$\begin{cases} U^n I(U)' = xU & \text{for } x \in (-1, 1) \\ U = 0 & \text{for } x \notin (-1, 1) \\ U = o((1 - x^2)^{\frac{1}{2}}) & \text{for } x \sim \pm 1. \end{cases} \tag{16}$$

Remark 2.1. Note that for $n = 1$, the equation reduces to $I(U)' = x$, which has an explicit solution (see [6]):

$$U(x) = \frac{4}{9}(1 - x^2)_+^{\frac{3}{2}}.$$

See Lemma A.1 in Appendix for a proof of this fact.

So the first part of Theorem 1.1 is a consequence of the following proposition:

Proposition 2.2. *For all $n \in [1, 4)$, there exists a non-negative even function $U \in C^1(\mathbb{R}) \cap C^\infty_{loc}(-1, 1)$ such that $U > 0$ in $(-1, 1)$ and solving (16).*

Furthermore, U satisfies

$$U(x) = \begin{cases} C^*(1 - x^2)^{\frac{3}{2}} + \mathcal{O}\left(|1 - x^2|^{\frac{2}{n}}\right) & \text{if } n \in [1, \frac{4}{3}) \\ C^*(1 - x^2)^{\frac{3}{2}} |\ln(1 - x^2)|^{\frac{3}{4}} + \mathcal{O}\left(|1 - x^2|^{\frac{3}{2}}\right) & \text{if } n = \frac{4}{3} \\ C^*(1 - x^2)^{\frac{2}{n}} + o\left(|1 - x^2|^{\frac{2}{n}}\right) & \text{if } n \in \left(\frac{4}{3}, 4\right) \end{cases} \tag{17}$$

when $x \to \pm 1$ for some positive constant $C^* > 0$.

2.2. The finite toughness case [Theorem 1.1-(ii)]. When the toughness coefficient K is not zero, then (11) imposes

$$\alpha = -\frac{\beta}{2},$$

and using (7) we see that we must have $n \neq 6$ and

$$\alpha = -\frac{\beta}{2} = -\frac{1}{6-n}.$$

In particular, in view of (8) we see that a self-similar solution can only exists in that case if the injection rate has the form

$$h(t) = \lambda t^{\frac{n-3}{6-n}}.$$

Equation (9) can then be written as

$$(-\beta x U + U^n I(U)')' = \lambda\delta - \frac{3}{2}\beta U. \tag{18}$$

We now choose $a > 0$ and try to solve (18) on the interval $(-a, a)$. If we integrate this equation on $(-a, a)$, we see that the null-flux boundary condition (10) implies a compatibility condition between λ and the mass of U:

$$\lambda = \frac{3}{2}\beta m \quad \text{with} \quad m = \int_{-a}^{a} U(x)dx.$$

We can now eliminate λ from (18): the profile $U(x)$ must solve the following equation:

$$(-\beta x U + U^n I(U)')' = \frac{3}{2}\beta(m\delta - U) \quad \text{with} \quad m = \int_{-a}^{a} U(x)\,dx.$$

Integrating and using (10), we thus find

$$U^n I(U)' = \beta x U + \frac{3}{2}\beta\mathcal{U} \quad \text{in } (-a, a) \tag{19}$$

where

$$\mathcal{U}(x) = \begin{cases} \int_x^a U(y)\,dy & \text{if } x > 0, \\ -\int_{-a}^x U(y)\,dy & \text{if } x < 0. \end{cases}$$

We thus need to construct a solution of (19) satisfying

$$U(x) = K|x-a|^{1/2} + o(|x-a|^{1/2}), \tag{20}$$

for a given $K > 0$. Any such solution will solve (18) for the particular choice of λ given by

$$\lambda(a) = \frac{3}{2}\beta \int_{-a}^{a} U(x)dx. \tag{21}$$

As before, we see that we can always take $a = 1$ and get rid of the parameter β in the equation by considering the function $V(x) = bU(ax)$ with b such that

$$\beta b^n a^3 = 1.$$

Note that condition (20) can then be written as

$$V(x) = K'\sqrt{1-x^2} + o(\sqrt{1-x^2})$$

with $K' = \frac{Kb\sqrt{a}}{\sqrt{2}}$.

In Sect. 4.2 (see Proposition 2.3 below), we will prove the existence of such a $V(x)$. This implies that for any $K > 0$ and $a > 0$, Eq. (18) has a solution for a particular value of λ (given by (21)). As noted in Remark 1.2, we would like to say that for given $K > 0$ and λ_0, we can always find $a > 0$ such that $\lambda(a) = \lambda_0$. While we are unable to prove that fact, we do want to point out that Lemma 4.5 will give

$$V(x) \geq K'(1-x^2)^{\frac{1}{2}} \quad \text{for all } x \in (-1, 1)$$

and

$$V(x) \leq C({K'}^{1-n} + K')(1-x^2)^{\frac{1}{2}}$$

for a constant C depending only on n. We deduce

$$C^{-1}K' \leq \int_{-1}^{1} V(x)\,dx \leq C({K'}^{1-n} + K'),$$

and the corresponding function U will thus satisfies

$$C^{-1}Ka^{\frac{3}{2}} \leq \int_{-a}^{a} U(x)\,dx \leq C(a^{\frac{9-n}{2}}K^{1-n} + a^{3/2}K).$$

Using (21), we deduce that for $K > 0$ fixed we have

$$\lim_{a\to 0} \lambda(a) = 0 \quad \text{and} \quad \lim_{a\to\infty} \lambda(a) = \infty.$$

It is thus reasonable to expect that $\lambda(a) = \lambda_0$ for some a (but, as noted in Remark 1.2, one needs to establish the continuity of $a \mapsto \lambda(a)$ in order to conclude).

In conclusion, it is enough to solve (19) when $a = 1$ and $\beta = 1$. So we have to construct, for any $K > 0$, a solution $U(x)$ of

$$\begin{cases} U^n I(U)' = xU + \frac{3}{2}\mathcal{U} & \text{for } x \in (-1, 1) \\ U = 0 & \text{for } x \notin (-1, 1) \\ U = K\sqrt{1-x^2} + o((1-x^2)^{\frac{1}{2}}) & \text{when } x \to \pm 1 \end{cases} \tag{22}$$

where

$$\mathcal{U}(x) = \begin{cases} \int_x^1 U(y)\,dy & \text{if } x > 0, \\ -\int_{-1}^x U(y)\,dy & \text{if } x < 0. \end{cases}$$

The second part of Theorem 1.1 is thus an immediate consequence of the following proposition:

Proposition 2.3. *For all* $n \in [1, 4)$, *there exists a non-negative even function* $U \in C^{1/2}(\mathbb{R}) \cap C^{\infty}_{loc}(-1, 1)$ *such that* $U > 0$ *in* $(-1, 1)$ *and* U *solves* (22). *Furthermore,* U *satisfies*

$$U(x) = K\sqrt{1-x^2} + \begin{cases} \mathcal{O}\left((1-x^2)^{\frac{3}{2}}\right) & \text{if } n \in [1,2) \\ \mathcal{O}\left((1-x^2)^{\frac{3}{2}}|\ln(1-x^2)|\right) & \text{if } n = 2 \\ \mathcal{O}\left((1-x^2)^{\frac{5-n}{2}}\right) & \text{if } n \in (2,4) \end{cases} \tag{23}$$

when $x \to \pm 1$.

2.3. General strategy. In order to show the existence of even solutions to (16) and (22), we will follow the general approach used in [11] to prove the existence of source-type solutions for the thin film equation. The first step is to rewrite these equations as integral equations by introducing an appropriate Green function. More precisely, we consider the function $x \mapsto g(x, z)$ solution of (for all $z \in [-1, 1]$)

$$\begin{cases} I(g(\cdot, z))' = \frac{1}{2}\left[\delta_z - \delta_{-z}\right] & \text{for } x \in (-1,1) \\ g(x,z) = 0 & \text{for } x \notin (-1,1) \\ g(x,z) = \mathcal{O}\left((1-x^2)^{\frac{3}{2}}\right) & \text{for } x \sim \pm 1. \end{cases} \tag{24}$$

In particular, formally at least, for any even function $V(x)$ satisfying $V(x) = 0$ for all $x \notin (-1, 1)$, the function

$$U(x) = \int_{-1}^{1} g(x,z) z V(z)\, dz$$

solves

$$I(U)' = zV \quad \text{in } (-1,1).$$

We can thus rewrite Eq. (16) as

$$U(x) = \int_{-1}^{1} g(x,z) z U^{1-n}(z) dz, \quad x \in [-1,1]$$

and Eq. (22) as

$$U(x) = 2\int_0^1 g(x,z) U^{-n}\left(zU(z) + \frac{3}{2}\mathcal{U}(z)\right) dz + K\sqrt{1-x^2}, \quad x \in [-1,1].$$

Solutions of these integral equations will be obtained via a fixed point argument in an appropriate functional space. One of the main difficulty in developing this fixed point argument is the fact that for $n > 1$ (see Remark 2.1), the function U^{1-n} is singular at the endpoints ± 1. Another difficulty will be to show that the solution has the appropriate behavior at ± 1. These two difficulties are in fact clearly related, and both will require us to have a very precise knowledge of the behavior of the Green function g as x and z approach ± 1. This will be the goal of the next section.

3. Properties of the Green Function

In this section, we are going to derive the formula for the Green function $g(x, z)$ solution of (24) and study its properties (in particular its behavior near the endpoints ± 1).

3.1. Green function for $(-\Delta)^{1/2}$. First, we recall that the Green function for the square root of the Laplacian in $[-1, 1]$ with homogeneous Dirichlet conditions, that is the solution of

$$\begin{cases} -I(G) = \delta_y & \text{in } (-1, 1) \\ G = 0 & \text{in } \mathbb{R}\backslash(-1, 1) \end{cases}$$

is given in [8,22] by the formula:

$$G(x, y) := \begin{cases} \pi^{-1} \operatorname{argsinh}(\sqrt{r_0(x, y)}) & \text{if } x, y \in (-1, 1), \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

with

$$r_0(x, y) = \frac{(1 - x^2)(1 - y^2)}{(x - y)^2}.$$

Equivalently, we have the following formula for $x, y \in (-1, 1)$,

$$G(x, y) = \pi^{-1} \ln\left(\frac{1 - xy + \sqrt{(1 - x^2)(1 - y^2)}}{|x - y|} \right). \tag{26}$$

(Equation (26) follows from (25) using the relation $\operatorname{argsinh}(u) = \ln(u + \sqrt{u^2 + 1})$)).

We give the following lemma for the reader's sake (see also [22] and [8, Corollary 4]):

Lemma 3.1 (Green function of $(-\Delta)^{\frac{1}{2}}$). *The function G defined above satisfies, for all $y \in (-1, 1)$,*

$$-I(G(\cdot, y)) = \delta(\cdot - y) \quad \text{in } \mathcal{D}'((-1, 1)).$$

In particular, for any function $f : (-1, 1) \to \mathbb{R}$ satisfying

$$f(x) \le C(1 - x^2)^b \tag{27}$$

for some $b > -\frac{3}{2}$, the function defined by

$$u(x) = \int_{-1}^{1} G(x, y) f(y)\, dy \tag{28}$$

is continuous in $(-1, 1)$ and it satisfies

$$-I(u) = f \quad \text{in } \mathcal{D}'(-1, 1).$$

Proof. Computations were first made in [22]. The validity of formulas in the one-dimensional setting were established in [8]. So we just want to prove that the integral (28) is finite for all $x \in (-1, 1)$ under condition (27). The rest of the proof follows as in [8].

For that purpose, we fix $x \in [0, 1)$ (the case $x \in (-1, 0]$ would be treated similarly) and denote $\varepsilon = \frac{1-x}{2}$. We then write:

$$|u(x)| \leq \left| \int_{-1+\varepsilon}^{1-\varepsilon} G(x, y) f(y)\, dy \right| + \left| \int_{1-\varepsilon}^{1} G(x, y) f(y)\, dy \right| + \left| \int_{-1}^{-1+\varepsilon} G(x, y) f(y)\, dy \right|$$

To bound the first term, we use formula (26) to get

$$\begin{aligned}
\left| \int_{-1+\varepsilon}^{1-\varepsilon} G(x, y) f(y)\, dy \right| &\leq C(1 + \varepsilon^b) \int_{-1+\varepsilon}^{1-\varepsilon} |G(x, y)|\, dy \\
&\leq C(1 + \varepsilon^b) \int_{-1+\varepsilon}^{1-\varepsilon} (|\ln \varepsilon| + |\ln |x - y||)\, dy \\
&\leq C(1 + \varepsilon^b)(|\ln \varepsilon| + 1)
\end{aligned}$$

where we used the fact that

$$\varepsilon \leq 1 - xy \leq 1 - xy + \sqrt{(1 - x^2)(1 - y^2)} \leq 3 \quad \forall |y| \leq 1 - \varepsilon, \text{ with } x = 1 - 2\varepsilon.$$

In order to bound the last two terms, we use formula (25) and the fact that $\operatorname{argsinh}(u) \leq u$ for all $u \geq 0$ to get

$$\begin{aligned}
\int_{1-\varepsilon}^{1} G(x, y) f(y)\, dy &\leq C \int_{1-\varepsilon}^{1} \sqrt{r_0(x, y)} f(y)\, dy \\
&\leq C \frac{\sqrt{1 - x^2}}{\varepsilon} \int_{1-\varepsilon}^{1} \sqrt{(1 - y^2)} f(y)\, dy \\
&\leq C \varepsilon^{b+1}.
\end{aligned}$$

We have thus showed that

$$|u(x)| \leq h(1 - x) < \infty \quad \text{for all } x \in (-1, 1)$$

for some function h which satisfies in particular $h(y) \leq (1 + y^b)(|\ln y|)$ (this inequality is far from optimal, as we will see later on). □

3.2. Green function for Equation (24). We now claim that the Green function $g(x, z)$, solution of (24), is given by

$$g(x, z) = \frac{1}{2} \left[(z - x) G(x, z) + (z + x) G(x, -z) \right]. \tag{29}$$

More precisely, we have the following proposition.

Proposition 3.2 (A Green function for a higher order operator). *For all* $z \in (-1, 1)$, *the function* $x \mapsto g(x, z)$ *defined by* (29) *is the unique solution of*

$$\begin{cases} I(g(\cdot, z))' = \frac{1}{2}\Big[\delta_z - \delta_{-z}\Big] & \text{in } \mathcal{D}'(-1, 1), \\ g(x, z) = 0 & \text{for } x \in \mathbb{R}\backslash(-1, 1), \\ g(x, z) = o((1 - x^2)^{\frac{1}{2}}) & \text{when } x \to \pm 1. \end{cases} \tag{30}$$

Before proving this result, we give two simple but useful lemmas.

Lemma 3.3. *The partial derivatives of G are given by the following formulas*

$$\frac{\partial G}{\partial x}(x, y) = \pi^{-1}\frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}}\frac{1}{y - x},$$
$$\frac{\partial G}{\partial y}(x, y) = \pi^{-1}\frac{\sqrt{1 - x^2}}{\sqrt{1 - y^2}}\frac{1}{x - y}.$$

Proof of Lemma 3.3. Remark that $G(x, y) = G(y, x)$; hence, it is enough to prove one of the two formulas. To prove the first one, simply write

$$\pi\frac{\partial G}{\partial x} = \frac{\partial_x(1 - xy + \sqrt{1 - x^2}\sqrt{1 - y^2})}{1 - xy + \sqrt{1 - x^2}\sqrt{1 - y^2}} - \frac{1}{x - y}.$$

A rather long but straightforward computation gives the desired result. □

Furthermore, with a simple integration by parts using Lemma 3.3 (see Appendix for details) we get the following lemma.

Lemma 3.4. *For all* $z \in (-1, 1)$ *and* $x \in (-1, 1)$, *we have*

$$\int_{-z}^{z} G(x, y)\, dy = (z - x)G(x, z) + (z + x)G(x, -z) + \frac{2}{\pi}\sqrt{1 - x^2}\arcsin(z). \tag{31}$$

We now turn to the proof of Proposition 3.2.

Proof of Proposition 3.2. We will actually derive formula (29): integrating the equation

$$I(g(\cdot, z))' = \frac{1}{2}\Big[\delta_z - \delta_{-z}\Big]$$

with respect to x, we find that the function $x \mapsto g(x, z)$ must solve

$$I(g(\cdot, z)) = \frac{1}{2}\Big[H(x - z) - H(x + z)\Big] + a(z) \tag{32}$$

for some $a(z)$, where H is the Heaviside function satisfying $H(x) = 1$ for $x \geq 0$, $H(x) = 0$ otherwise.

Together with the boundary condition $g(x, z) = 0$ for $x \notin (-1, 1)$, (32) has a unique solution given by Lemma 3.1:

$$g(x, z) = -\int_{-1}^{1} G(x, y)\left[\frac{1}{2}\Big[H(y - z) - H(y + z)\Big] + a(z)\right] dy$$

$$= \frac{1}{2}\int_{-z}^{z} G(x,y)\,dy - a(z)\int_{-1}^{1} G(x,y)\,dy. \tag{33}$$

Note that since $G(x,1) = G(x,-1) = 0$, (31) with $z \to 1$ gives

$$\int_{-1}^{1} G(x,y)\,dy = \frac{2}{\pi}\sqrt{1-x^2}\arcsin(1) = \sqrt{1-x^2}$$

and (33) thus gives

$$g(x,z) = \frac{1}{2}\left[(z-x)G(x,z) + (z+x)G(x,-z)\right] + b(z)\sqrt{1-x^2} \tag{34}$$

with $b(z) = -\frac{1}{\pi}[a(z)\pi - \arcsin(z)]$.

Finally, the function $a(z)$ (and thus $b(z)$) will be determined uniquely using the last boundary condition in (30). Indeed, using the fact that $\operatorname{argsinh}(u) \sim u$ as $u \to 0$, we get

$$G(x,z) \sim \pi^{-1}\sqrt{r_0(x,z)}$$

when either $x \to \pm 1$ with z fixed, or when $z \to \pm 1$ with x fixed. We deduce

$$\begin{aligned} g(x,z) &\sim \frac{1}{2\pi}\left[(z-x)\sqrt{r_0(x,z)} + (z+x)\sqrt{r_0(x,-z)}\right] + b(z)\sqrt{1-x^2} \\ &\sim \frac{1}{2\pi}\left[\frac{(z-x)}{|z-x|} + \frac{(z+x)}{|z+x|}\right]\sqrt{1-x^2}\sqrt{1-z^2} + b(z)\sqrt{1-x^2}. \end{aligned}$$

Hence, g satisfies

$$g(x,z) = o((1-x^2)^{\frac{1}{2}}) \quad \text{when } x \to \pm 1$$

for all $z \in (-1,1)$ if and only if we choose $b(z) = 0$ (that is $a(z) = \frac{1}{\pi}\arcsin(z)$). The proof of the proposition is now complete. □

3.3. Further properties of $g(x,z)$. The following proposition summarizes the properties of g that will be needed for the proof of our main result.

Proposition 3.5 (Properties of the function g). *We have*:

(1) *The function g is continuous on $(-1,1)^2$ and for all $x, z \in (-1,1)$ with $x \neq z$ and $x \neq -z$, we have*

$$\frac{\partial g}{\partial x}(x,z) = -\frac{1}{2}\left[G(x,z) - G(x,-z)\right] \tag{35}$$

$$= -\frac{1}{2\pi}\operatorname{argsinh}\left(\frac{2xz\sqrt{(1-x^2)(1-z^2)}}{|x-z||x+z|}\right). \tag{36}$$

In particular, $x \mapsto g(x,z)$ is decreasing on $(0,1)$ for all $z \in (0,1)$.

(2) *We have*

$$g(x,-z) = -g(x,z) \quad \textit{and} \quad g(-x,z) = g(x,z) \quad \textit{for all } (x,z) \in \mathbb{R}^2$$

so the function $z \mapsto g(x,z)$ *is odd and the function* $x \mapsto g(x,z)$ *is even. Furthermore,* g *satisfies*

$$g(x,z) > 0 \quad \textit{for all } x \in (-1,1) \textit{ and for all } z \in (0,1)$$

(*and so* $g(x,z) < 0$ *for all* $x \in (-1,1)$ *and for all* $z \in (-1,0)$).

(3) *For all* $x, z \in (-1,1)$,

$$|g(x,z)| \le \frac{1}{\pi}\sqrt{1-x^2}\sqrt{1-z^2}, \tag{37}$$

$$\left|\frac{\partial g}{\partial x}(x,z)\right| \le \frac{1}{2\pi}\ln\left(1 + \frac{4\sqrt{1-x^2}\sqrt{1-z^2}}{|z^2-x^2|}\right). \tag{38}$$

(4) *For all* $x \in (-1,1)$,

$$\int_x^1 z g(x,z)dz \ge C(1-x^2)^2 \tag{39}$$

for some $C > 0$.

Proof of Proposition 3.5. 1. The continuity of g is easy to check. Indeed, the only singularity for the function $G(x,z)$ occurs when $x = z$, and since it is a logarithmic singularity, it is clear that the function $(z-x)G(x,z)$ is continuous everywhere. Next, we have

$$\frac{\partial g}{\partial x}(x,z) = \frac{1}{2}\left[-G(x,z) + G(x,-z)\right] + \frac{1}{2}\left[(x+z)\frac{\partial G}{\partial x}(x,-z) - (x-z)\frac{\partial G}{\partial x}(x,z)\right]$$

and using Lemma 3.3, we find

$$(x-z)\frac{\partial G}{\partial x}(x,z) = (x+z)\frac{\partial G}{\partial x}(x,-z) = -\frac{\sqrt{1-z^2}}{\sqrt{1-x^2}}.$$

We deduce

$$\frac{\partial g}{\partial x}(x,z) = -\frac{1}{2}\left[G(x,z) - G(x,-z)\right].$$

The last formula follows from the identity

$$\operatorname{argsinh}(u) - \operatorname{argsinh}(v) = \operatorname{argsinh}\left(u\sqrt{1+v^2} - v\sqrt{1+u^2}\right).$$

2. The fact that $z \mapsto g(x,z)$ is odd and $x \mapsto g(x,z)$ is even is a direct consequence of the formulas (29) and (25). The positivity of g follows from the monotonicity and the fact that $g(1,z) = 0$ for all z (see (37) for instance).

3. Since $\operatorname{argsinh}(u) \le u$ for all $u \ge 0$, we have

$$G(x,z) \le \sqrt{r_0(x,z)}$$

and so

$$\begin{aligned}|g(x,z)| &\le \frac{1}{2\pi}\left[|z-x|\sqrt{r_0(x,z)} + |z+x|\sqrt{r_0(x,-z)}\right]\\ &\le \frac{1}{\pi}\sqrt{1-x^2}\sqrt{1-z^2}.\end{aligned}$$

To prove (38), we use the fact that for $u \ge 0$, we have $\sqrt{1+u^2} \le 1+u$, and so

$$\operatorname{argsinh}(u) = \ln(u+\sqrt{1+u^2}) \le \ln(1+2u).$$

Inequality (38) now follows from (36) for $x,\ z \in (0,1)$. The symmetries of g then give the result for $x,\ z \in (-1,1)$.

4. In order to prove (39), we write for $x \in [0,1)$,

$$\begin{aligned}\int_x^1 zg(x,z)dz &= \int_x^1 [z(x+z)G(x,-z) + z(z-x)G(x,z)]dz\\ &\ge \int_x^1 z(z-x)G(x,z)\,dz.\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}\int_x^1 z(z-x)G(x,z)dz &= \frac{1}{3}\int_x^1 \partial_z G(x,z)(z-x)^2\left(z+\frac{x}{2}\right)dz\\ &\quad + \frac{1}{3}\left(G(x,z)(z-x)^2\left(z+\frac{x}{2}\right)\right)\Big|_{z=x}^{z=1}.\end{aligned}$$

Keeping in mind that $(z-x)G(x,z)$ is not singular and vanishes when $z=x$ and using the formulas for partial derivatives of G (Lemma 3.3), we obtain

$$\int_x^1 z(z-x)G(x,z)dz = \frac{\sqrt{1-x^2}}{3\pi}\int_x^1 (z-x)\left(z+\frac{x}{2}\right)\frac{dz}{\sqrt{1-z^2}}$$

where

$$\begin{aligned}\int_x^1 (z-x)\left(z+\frac{x}{2}\right)\frac{dz}{\sqrt{1-z^2}} &= \frac{1}{2}(1-x^2)(\pi/2-\arcsin(x))\\ &= \frac{1}{2}(1-x^2)\arccos(x).\end{aligned}$$

Since $\arccos(x) \ge \sqrt{1-x^2}$ the result follows. □

3.4. Application: solving the linear problem. In this subsection, we use the Green function $g(x,z)$ introduced above to find a solution to the linear equation

$$I(U)' = f \quad \text{in } (-1,1), \quad U = 0 \quad \text{in } \mathbb{R}\setminus(-1,1) \tag{40}$$

and to study the behavior of this solution U as $x \to \pm 1$.

We note that the function $V(x) = \sqrt{(1-x^2)_+}$ solves $I(V)' = 0$ in $(-1,1)$, and so given one solution U_0 of (40), we can find all solutions in the form $U_0 + KV(x)$ (and there is a unique solution to (40) if we add a boundary condition such as (5)).

Now, we start with the following result.

Proposition 3.6. *Let $f : (-1,1) \to \mathbb{R}$ be a function satisfying*

$$|f(z)| \le C_f (1-z^2)^a \tag{41}$$

for some $a > -\frac{3}{2}$. Then the function

$$U(x) = \begin{cases} \int_{-1}^{1} g(x,z) f(z)dz & \text{for } x \in (-1,1) \\ 0 & \text{for } x \notin (-1,1) \end{cases} \tag{42}$$

is continuous in $\mathbb{R}$, C^1 in $(-1,1)$ and satisfies

$$|U(x)| \le CC_f\sqrt{1-x^2} \quad \forall x \in (-1,1) \tag{43}$$

for some constant C depending on a.

Furthermore, if f is odd, then U solves

$$I(U)' = I(U') = f \quad \text{in } \mathcal{D}'(-1,1). \tag{44}$$

Proof of Proposition 3.6. First of all, Eq. (37) implies (for $x \in (-1,1)$):

$$\begin{aligned} |U(x)| &\le \sqrt{1-x^2} \int_{-1}^{1} \sqrt{1-z^2}|f(z)|dz \\ &\le C_f\sqrt{1-x^2} \int_{-1}^{1} (1-z^2)^{\frac{1}{2}+a} dz \end{aligned}$$

where this last integral is clearly convergent for $a > -\frac{3}{2}$. We deduce (43) which gives in particular the continuity of U at ± 1 (the continuity of U in $(-1,1)$ is clear).

Furthermore, we have (using (35)):

$$\begin{aligned} U'(x) &= \int_{-1}^{1} \frac{\partial g}{\partial x}(x,z) f(z)dz \\ &= -\frac{1}{2}\int_{-1}^{1} [G(x,z) - G(x,-z)] f(z)\, dz \end{aligned} \tag{45}$$

and Lemma 3.1 implies that U' is continuous in $(-1,1)$ and satisfies

$$I(U')(x) = \frac{1}{2}[f(x) - f(-x)] \quad \text{in } \mathcal{D}'(-1,1).$$

In particular, if f odd, we deduce

$$I(U') = f \quad \text{in } \mathcal{D}'(-1,1).$$

Finally, Proposition 3.2 also implies that

$$I(U)'(x) = \frac{1}{2}[f(x) - f(-x)] = f(x) \quad \text{in } \mathcal{D}'(-1,1).$$

□

In the proof of our main result, we will need to further characterize the behavior of the function U near the end points $x = \pm 1$. We thus prove the following proposition.

Proposition 3.7. *Consider an odd function $f: (-1,1) \to \mathbb{R}$ satisfying (41) for some $a > -\frac{3}{2}$.*

Then the function U defined by (42) satisfies

$$|U'(x)| \le C C_f F(1-x^2) \tag{46}$$

with

$$F(y) = \begin{cases} y^{a+1} & \text{if } -\frac{3}{2} < a < -\frac{1}{2} \\ y^{\frac{1}{2}} \ln\left(\frac{1}{y}\right) & \text{if } a = -\frac{1}{2} \\ y^{\frac{1}{2}} & \text{if } a > -\frac{1}{2}. \end{cases} \tag{47}$$

Together with the fact that $U(\pm 1) = 0$ (which follows from (43)), this proposition gives

$$U(x) \le C \begin{cases} (1-x^2)^{a+2} & \text{if } -\frac{3}{2} < a < -\frac{1}{2} \\ (1-x^2)^{\frac{3}{2}} \ln\left(\frac{1}{1-x^2}\right) & \text{if } a = -\frac{1}{2} \\ (1-x^2)^{\frac{3}{2}} & \text{if } a > -\frac{1}{2}. \end{cases} \tag{48}$$

In particular, we have

$$U(x) = o(\sqrt{1-x^2}) \quad \text{as } x \to \pm 1.$$

Remark 3.8. We will apply estimate (46) twice in the proof of our main result. It will be used with $a = 2/n - 2$ in the zero toughness case and $a = 1/2 - n/2$ in the finite toughness case. We remark that in both cases, the condition $a > -3/2$ requires that $n < 4$.

Remark 3.9. In terms of Sobolev regularity, we note that (46) implies that under the assumption of Proposition 3.7, U belongs to $H^1(\mathbb{R})$. In particular, $I(U)$ is a function in $L^2(\mathbb{R})$, and (44) implies that $I(U)' \in L^\infty_{loc}(-1,1)$. We can thus write that U satisfies

$$I(U)'(x) = f(x) \quad \text{a.e. in } (-1,1).$$

Proof of Proposition 3.7. First of all, we note that it is enough to consider x close to 1 (or -1). So we will always assume that $\frac{3}{4} \le x < 1$. Using the fact that $z \mapsto f(z)$ and $z \mapsto g(x,z)$ are odd, we can write

$$U'(x) = 2\int_0^1 \frac{\partial g}{\partial x}(x,z) f(z)\, dz$$

where we recall that $\frac{\partial g}{\partial x}(x, y)$ is given by (36).

In order to get a bound on $U'(x)$, we first write

$$\begin{aligned} U'(x) &= 2\int_0^{\frac{1}{2}} \frac{\partial g}{\partial x}(x, z) f(z)\, dz + 2\int_{1/2}^1 \frac{\partial g}{\partial x}(x, z) f(z)\, dz \\ &= I_1 + I_2. \end{aligned} \tag{49}$$

To bound the first integral, we use (36) which gives

$$I_1 = -\frac{1}{\pi}\int_0^{\frac{1}{2}} \operatorname{argsinh}\left(\frac{2xz\sqrt{(1-x^2)(1-z^2)}}{|x^2 - z^2|}\right) f(z)\, dz$$

and using the fact that f is bounded in $(0, 1/2)$, that $\operatorname{argsinh} u \le u$ for $u \ge 0$ and that $x - z \ge 1/4$, we deduce

$$\begin{aligned} |I_1| &\le C\int_0^{\frac{1}{2}} \frac{2xz\sqrt{(1-x^2)(1-z^2)}}{|x^2 - z^2|}\, dz \\ &\le C\int_0^{\frac{1}{2}} \sqrt{1 - z^2}\, dz\sqrt{1 - x^2} \\ &\le C\sqrt{1 - x^2}. \end{aligned} \tag{50}$$

In order to estimate I_2, we use (38) and (41) (and the fact that $z > 1/2$) to write

$$\begin{aligned} |I_2| &\le \frac{1}{\pi}\int_{1/2}^1 \ln\left(1 + \frac{4\sqrt{1-x^2}\sqrt{1-z^2}}{|z^2 - x^2|}\right)(1 - z^2)^a\, dz \\ &\le \frac{1}{\pi}\int_{1/2}^1 \ln\left(1 + \frac{4\sqrt{1-x^2}\sqrt{1-z^2}}{|z^2 - x^2|}\right)(1 - z^2)^a\, z\, dz \end{aligned}$$

Now, the change of variables $u = (1 - x^2)^{-1}(1 - z^2)$, gives

$$|I_2| \le C(1 - x^2)^{a+1}\int_0^{\frac{3/4}{1-x^2}} \ln\left(1 + \frac{4\sqrt{u}}{|1 - u|}\right) u^a\, du.$$

We note that the integral

$$\int_0^\infty \ln\left(1 + \frac{4\sqrt{u}}{|1 - u|}\right) u^a\, du$$

has an integrable singularity at $u = 1$; it is convergent at $u = 0$ for all $a > -\frac{3}{2}$; it is convergent at $u = \infty$ for all $a < -\frac{1}{2}$. In particular, we deduce that

$$|I_2| \le C(1 - x^2)^{a+1} \quad \text{if} \quad -\frac{3}{2} < a < -\frac{1}{2}. \tag{51}$$

When $a \ge -\frac{1}{2}$, we find that for x close enough to 1, we have

$$|I_2| \le C(1 - x^2)^{a+1}\left(1 + \int_2^{\frac{3/4}{1-x^2}} \ln\left(1 + \frac{4\sqrt{u}}{|1 - u|}\right) u^a\, du\right)$$

$$\leq C(1-x^2)^{a+1}\left(1+\int_2^{\frac{3/4}{1-x^2}} u^{a-\frac{1}{2}}du\right).$$

When $a > -\frac{1}{2}$, this implies

$$\begin{aligned}|I_2| &\leq C(1-x^2)^{a+1}\left(1+(1-x^2)^{-a-\frac{1}{2}}\right)\\ &\leq C\sqrt{1-x^2}.\end{aligned} \tag{52}$$

While when $a = -\frac{1}{2}$, we get

$$\begin{aligned}|I_2| &\leq C(1-x^2)^{\frac{1}{2}}\left(1+|\ln(1-x^2)|\right)\\ &\leq C\sqrt{1-x^2}|\ln(1-x^2)|.\end{aligned} \tag{53}$$

Putting together (49)–(53), we deduce

$$|U'(x)| \leq \begin{cases} C(1-x^2)^{a+1} & \text{if } -\frac{3}{2} < a < -\frac{1}{2} \\ C\sqrt{1-x^2}|\ln(1-x^2)| & \text{if } a = -\frac{1}{2} \\ C\sqrt{1-x^2} & \text{if } a > -\frac{1}{2} \end{cases}$$

which gives the result. □

To conclude this subsection concerning the linear equation (40), we are going to prove that we can improve estimate (46) and derive the precise asymptotic behavior of $U'(x)$ when $f(z)$ has a particular form.

Proposition 3.10. *Assume that*

$$f(z) = z(1-z^2)^a h(z)$$

where $h(z) \geq 0$ is a bounded even function on $(-1, 1)$ and $a > -\frac{3}{2}$. If $a \leq -\frac{1}{2}$, we further assume that $h(1) = \lim_{x\to 1} h(x)$ exists.

Then the function U defined by (42) satisfies

$$U'(x) = \begin{cases} -C_0(1-x^2)^{a+1} + o((1-x^2)^{a+1}) & \text{if } -\frac{3}{2} < a < -\frac{1}{2} \\ -C_0(1-x^2)^{\frac{1}{2}}\ln\left(\frac{1}{(1-x^2)}\right) + \mathcal{O}((1-x^2)^{\frac{1}{2}}) & \text{if } a = -\frac{1}{2} \\ -C_0(1-x^2)^{\frac{1}{2}} + \mathcal{O}((1-x^2)^{a+1}) & \text{if } a > -\frac{1}{2}. \end{cases} \tag{54}$$

where the constant C_0 is given by

$$C_0 = \begin{cases} c_a h(1) & \text{when } -\frac{3}{2} < a \leq -\frac{1}{2} \\ \frac{1}{2\pi}\int_0^1 2f(\sqrt{1-v})v^{-1/2}dv & \text{when } a > -\frac{1}{2} \end{cases} \tag{55}$$

for some constant c_a depending only on a.

Proof. We recall the formula (using the fact that $z \mapsto f(z)$ is odd and the formula (36)):

$$U'(x) = 2\int_0^1 \frac{\partial g}{\partial x}(x,z) f(z)\,dz$$
$$= -\frac{1}{\pi}\int_0^1 \operatorname{argsinh}\left(\frac{2xz\sqrt{(1-x^2)(1-z^2)}}{|x^2-z^2|}\right)(1-z^2)^a h(z) z\,dz.$$

The change of variables $u = (1-x^2)^{-1}(1-z^2)$ yields

$$\frac{U'(x)}{(1-x^2)^{a+1}} = -\frac{1}{2\pi}\int_0^{\frac{1}{1-x^2}} \Theta(x,u)du$$

where the integrand $\Theta(x,u)$ is given by

$$\Theta(x,u) = \operatorname{argsinh}\left(\frac{2x\sqrt{1-(1-x^2)u}\sqrt{u}}{|1-u|}\right)u^a h(\sqrt{1-(1-x^2)u}).$$

Note that $\Theta(x,u)$ is bounded (uniformly in x) by

$$C||h||_\infty \operatorname{argsinh}\left(\frac{2\sqrt{u}}{|1-u|}\right)u^a$$

which is integrable on $(0,\infty)$ provided $-\frac{3}{2} < a < -\frac{1}{2}$. So Lebesgue dominated convergence theorem implies

$$\lim_{x\to 1}\int_0^{\frac{1}{1-x^2}} \Theta(x,u)du = \int_0^{+\infty}\Theta(1,u)du = \int_0^\infty \operatorname{argsinh}\left(\frac{2\sqrt{u}}{|1-u|}\right)u^a h(1)du$$

which gives (54) and (55) in the case $-\frac{3}{2} < a < -\frac{1}{2}$ (and we see that this limit is strictly positive as soon as $h(1) > 0$).

When $a \geq -\frac{1}{2}$, we write, for $\frac{1}{1-x^2} \geq 2$:

$$\frac{U'(x)}{\sqrt{1-x^2}} = -\frac{1}{2\pi}(1-x^2)^{a+\frac{1}{2}}\int_0^{\frac{1}{1-x^2}}\Theta(x,u)du$$
$$= -\frac{1}{2\pi}(1-x^2)^{a+\frac{1}{2}}\int_0^2\Theta(x,u)du - \frac{1}{2\pi}(1-x^2)^{a+\frac{1}{2}}\int_2^{\frac{1}{1-x^2}}\Theta(x,u)du$$
$$= I_1 + I_2.$$

The first term satisfies

$$|I_1| \leq C(1-x^2)^{a+\frac{1}{2}}||h||_\infty\int_0^2 \operatorname{argsinh}\left(\frac{\sqrt{u}}{|1-u|}\right)u^a du$$

and so

$$\lim_{x\to 1}|I_1| = 0 \quad \text{if } a > -\frac{1}{2}$$
$$|I_1| \leq C \quad \text{if } a = -\frac{1}{2}.$$

For the second term, we recall that $|\operatorname{argsinh}(w) - w| \leq Cw^3$, and so for all $2 \leq u \leq \frac{1}{1-x^2}$, we have

$$\left|\operatorname{argsinh}\left(\frac{2x\sqrt{1-(1-x^2)u}\sqrt{u}}{|1-u|}\right) - \frac{2x\sqrt{1-(1-x^2)u}\sqrt{u}}{|1-u|}\right| \leq \frac{C}{u^{\frac{3}{2}}},$$

which also yields

$$\left|\operatorname{argsinh}\left(\frac{2x\sqrt{1-(1-x^2)u}\sqrt{u}}{|1-u|}\right) - \frac{2x\sqrt{1-(1-x^2)u}}{\sqrt{u}}\right| \leq \frac{C}{u^{\frac{3}{2}}}.$$

We deduce

$$\begin{aligned} I_2 &= -\frac{1}{2\pi}(1-x^2)^{a+\frac{1}{2}} \int_2^{\frac{1}{1-x^2}} 2x\sqrt{1-(1-x^2)u}\,h(\sqrt{1-(1-x^2)u})u^{a-\frac{1}{2}}\,du + R \\ &= -\frac{1}{2\pi}\int_{2(1-x^2)}^1 2x\sqrt{1-v}\,h(\sqrt{1-v})v^{a-\frac{1}{2}}\,dv + R \end{aligned}$$

where

$$\begin{aligned} R &\leq C||h||_\infty (1-x^2)^{a+\frac{1}{2}} \int_2^{\frac{1}{1-x^2}} u^{a-\frac{3}{2}}\,du \\ &= \mathcal{O}((1-x^2) + (1-x^2)^{a+\frac{1}{2}}). \end{aligned}$$

When $a > -\frac{1}{2}$, we deduce that

$$\lim_{x\to 1} I_2 = -\frac{1}{2\pi}\int_0^1 2\sqrt{1-v}\,h(\sqrt{1-v})v^{a-\frac{1}{2}}\,dv,$$

which implies (54) and (55) in that case (note that $\sqrt{1-v}\,h(\sqrt{1-v})v^a = f(\sqrt{1-v})$).

When $a = -\frac{1}{2}$, we use L'Hospital's Rule to prove that

$$\lim_{x\to 1-}\frac{I_2}{\ln(1-x^2)} = \frac{1}{\pi}h(1)$$

which gives (54) and (55) in the case $a = -\frac{1}{2}$ and completes the proof. □

4. Proof of the Main Result

We are now ready to prove our main result, that is the existence of self-similar solutions for (3)–(5). As shown in Sect. 2.1, the proof of Theorem 1.1 reduces to the proving Propositions 2.2 and 2.3, which is the goal of this section.

4.1. The zero toughness case: proof of Proposition 2.2. In this section, we will prove Proposition 2.2, that is the existence of a solution $U(x)$ of (16) satisfying (17).

Remark 4.1. We already mentioned that for $n = 1$, the function $U(x) = \frac{4}{9}(1-x^2)_+^{\frac{3}{2}}$ is a solution of (16) (see Lemma A.1 in Appendix). In the sequel, we will thus always assume that $n \in (1, 4)$.

We recall that, using the Green function $g(x, z)$ introduced in Section 3, we can rewrite, formally at least, Eq. (16) as the following integral equality:

$$U(x) = \int_{-1}^{1} zg(x, z)\,(U(z))^{1-n}\,dz. \tag{56}$$

The fact that a solution of (56) actually solves (16) will follow from Proposition 3.6 once we have established appropriate estimates on U (more precisely, we will need to control the behavior of $(U(z))^{1-n}$ near $z = \pm 1$).

Now, we will find a solution of (56) by a fixed point argument. However, when $n > 1$, the integrand is singular whenever $U(z) = 0$, so we first construct approximate solutions of (56) as follows:

Lemma 4.2 (Construction of an approximate solution). *For any $n \in (1, 4)$ and for all $k \in \mathbb{N}$, there exists a continuous function $U_k : \mathbb{R} \to \mathbb{R}$ such that*

$$\begin{cases} U_k(x) = \displaystyle\int_{-1}^{1} zg(x, z)\left(\frac{1}{k} + U_k(z)\right)^{1-n} dz & \text{for } x \in (-1, 1) \\ U_k(x) = 0 & \text{for } x \notin (-1, 1). \end{cases} \tag{57}$$

Furthermore, U_k is non-negative in $\mathbb{R}$ and is C^1 in $(-1, 1)$.

Proof. We will first construct the solution in the interval $[-1, 1]$ (we then extend U_k by zero outside $[-1, 1]$). For that, we consider the following closed convex set of $C([-1, 1])$

$$S = \{V \in C([-1, 1]) : V(x) = V(-x), 0 \le V \le A \text{ in } [-1, 1]\}$$

(for a positive number $A > 0$ to be fixed later) and the operator $\mathcal{T} : S \to C([-1, 1])$ which maps $V \in S$ to the function

$$U(x) = \int_{-1}^{1} zg(x, z)\left(\frac{1}{k} + V(z)\right)^{1-n} dz.$$

Proposition 3.5-(2) implies that $z \mapsto zg(x, z)$ is even and positive on $(-1, 1)$ for all $x \in (-1, 1)$, so

$$U(x) \ge 0 \quad \text{in } (-1, 1).$$

Proposition 3.5-(2) also implies that $x \mapsto U(x)$ is even. Next, Proposition 3.6 and the fact that $\left(\frac{1}{k} + V(z)\right)^{1-n} \le k^{n-1}$ imply that $U \in C^1(-1, 1)$ and satisfies (see (43))

$$|U(x)| \le Ck^{n-1}\sqrt{1-x^2} \quad \forall x \in (-1, 1). \tag{58}$$

Finally, the bound (37) gives in particular $|g(x, z)| \le 1$ for all $x,\ z \in (-1, 1)$. Hence

$$0 \le U(x) \le 2k^{n-1} \quad \text{for all } x \in (-1, 1).$$

Choosing $A = 2k^{n-1}$, we deduce that

$$\mathcal{T}(S) \subset S.$$

Moreover, Proposition 3.7 (see (46) with $a = 0$) implies

$$|U'(x)| \leq Ck^{n-1}\sqrt{1-x^2} \quad \forall x \in (-1, 1)$$

and so $\mathcal{T}(S)$ is equi-Lipschitz continuous. Using Ascoli–Arzelà's theorem, we deduce that $\mathcal{T}(S)$ is a compact subset of $C([-1, 1])$. Finally, using once again the fact that $|g(x, z)| \leq 1$ together with Lebesgue dominated convergence Theorem, it is easy to show that $\mathcal{T}$ is a continuous operator. We can thus use Schauder's fixed point Theorem and deduce that $\mathcal{T}$ has a fixed point U_k. We can now define $U_k(x) = 0$ for $x \notin [-1, 1]$. Using (58), the resulting function is indeed continuous in $\mathbb{R}$. □

In order to pass to the limit $k \to \infty$, we now need to derive some estimates on U_k which do not depend on the parameter k.

Lemma 4.3 (Uniform estimates). *For $n \in (1, 4)$, there exists $C > 0$ such that for all $k \in \mathbb{N}$, the function U_k constructed in Lemma 4.2 satisfies, for all $x \in (-1, 1)$:*

$$\frac{1}{k} + U_k(x) \geq C^{-1}(1-x^2)^{\frac{2}{n}} \tag{59}$$

$$|U_k(x)| \leq C\sqrt{1-x^2} \tag{60}$$

$$|U_k'(x)| \leq C \begin{cases} (1-x^2)^{\frac{2}{n}-1} & \text{if } n \in \left(\frac{4}{3}, 4\right) \\ \sqrt{1-x^2}\ln\frac{1}{\sqrt{1-x^2}} & \text{if } n = \frac{4}{3} \\ \sqrt{1-x^2} & \text{if } n \in [1, \frac{4}{3}). \end{cases} \tag{61}$$

Proof. In view of (36), the function $x \mapsto zg(x, z)$ is decreasing on the interval $[0, 1]$, for all $z \in (-1, 1)$. The definition of U_k, (57), thus implies that $x \mapsto U_k(x)$ is non-increasing on the interval $[0, 1]$. Using (39), we deduce that for $x \in (0, 1)$ we have

$$U_k(x) \geq \left(\frac{1}{k} + U_k(x)\right)^{1-n} \int_x^1 zg(x, z)dz \geq C\left(\frac{1}{k} + U_k(x)\right)^{1-n}(1-x^2)^2$$

which yields (59), and, in turns, gives

$$f(z) = z\left(\frac{1}{k} + U_k(z)\right)^{1-n} \leq \left(1-z^2\right)^{\frac{2}{n}-2}.$$

We note that for $n < 4$, we have $a = \frac{2}{n} - 2 > -3/2$, so Proposition 3.6 gives (60) and Proposition 3.7 (note that $z \mapsto f(z)$ is odd) implies (61). □

We can now pass to the limit $k \to \infty$ in (57) and complete the proof of Proposition 2.2.

Proof of Proposition 2.2. Thanks to estimates (60)–(61), Ascoli–Arzelà's Theorem implies that there exists a subsequence, denoted U_p, of U_k and a function $U(x)$ defined in $(-1, 1)$ such that $U_p(x) \longrightarrow U(x)$ as $p \to \infty$, locally uniformly in $(-1, 1)$. Moreover, Eq. (60) implies that

$$|U(x)| \leq C\sqrt{1-x^2} \quad \text{in } (-1, 1),$$

so we can define $U(x) = 0$ for $x \notin (-1, 1)$ and get a continuous function in $\mathbb{R}$. Finally, Eq. (59) implies

$$U(x) \geq C^{-1}(1 - x^2)^{\frac{2}{n}} \quad \text{in } (-1, 1). \tag{62}$$

Furthermore, we note that the sequence of functions

$$f_p(x) = x\left(\frac{1}{p} + U_p(x)\right)^{1-n}$$

converges locally uniformly to $f(x) = xU(x)^{1-n}$ and satisfies (using (59))

$$|f_p(x)| \leq C^{-1}(1 - x^2)^{\frac{2}{n}-2}.$$

Since $\frac{2}{n} - 2 > -\frac{3}{2}$ for $n \in (1, 4)$, and in view of Proposition 3.6, we can pass to the limit in (57) and deduce that U satisfies (56), that is

$$U(x) = \int_{-1}^{1} g(x, z)\,(U(z))^{1-n}\, z dz.$$

Proposition 3.6 also implies that U is in $C^1(-1, 1)$ and solves

$$\begin{aligned} I(U)' = xU^{1-n} &\quad \text{in } \mathcal{D}'((-1, 1)), \\ U = 0 &\quad \text{for } x \notin (-1, 1). \end{aligned}$$

Note that this implies in particular for that $I(U)'$ in $L^\infty_{loc}(-1, 1)$ and that

$$U^n I(U)' = xU \quad \text{for all } x \in (-1, 1).$$

It remain to prove (17) which now follows from Proposition 3.10. Indeed U is given by

$$U(x) = \int_{-1}^{1} g(x, z) f(z) dz$$

with

$$f(z) = z\,(U(z))^{1-n} = z\left(\frac{U(z)}{(1 - z^2)^{2/n}}\right)^{1-n}(1 - z^2)^{\frac{2}{n}-2}.$$

We can thus apply Proposition 3.10 with $h(z) = \left(\frac{U(z)}{(1-z^2)^{2/n}}\right)^{1-n}$ and $a = \frac{2}{n} - 2$ (note that the function $h(z)$ is in particular non-negative, bounded and even). We deduce

$$U'(x) = \begin{cases} -C_0(1 - x^2)^{\frac{1}{2}} + \mathcal{O}\left(|1 - x^2|^{\frac{2}{n}-1}\right) & \text{if } n \in [1, \frac{4}{3}) \\ -C_0(1 - x^2)^{\frac{1}{2}} \ln\left(\frac{1}{(1-x^2)}\right) + \mathcal{O}\left((1 - x^2)^{\frac{1}{2}}\right) & \text{if } n = \frac{4}{3} \\ -C_0(1 - x^2)^{\frac{2}{n}-1} + o\left(|1 - x^2|^{\frac{2}{n}-1}\right) & \text{if } n \in \left(\frac{4}{3}, 4\right) \end{cases} \tag{63}$$

and (17) follows (using the fact that $U(\pm 1) = 0$). Note in particular that (62) implies that $C_0 \neq 0$ in the case $n \in (\frac{4}{3}, 4)$, while formula (55) gives $C_0 \neq 0$ in the case $n \in [1, \frac{4}{3})$.

In the critical case $n = \frac{4}{3}$, however, we can show that $C_0 = 0$. Indeed, in that case, we have

$$h(z) = \left(\frac{U(z)}{(1-z^2)^{3/2}}\right)^{-\frac{1}{3}}$$

and so (63) implies that $h(1) = 0$ and in turn, formula (55) gives $C_0 = 0$. We thus need to work some more to derive the correct behavior as $x \to \pm 1$, namely

$$U'(x) \sim -C_0(1-x^2)^{\frac{1}{2}} \ln\left(\frac{1}{(1-x^2)}\right)^{3/4}.$$

The interested reader will find the proof of this fact in Appendix B. □

4.2. The finite toughness case: proof of Proposition 2.3. We now consider the case of positive toughness $K \neq 0$. As shown in Sect. 2.2, the proof of Theorem 1.1 in this case is equivalent to proving Proposition 2.3, that is the existence of a solution $U(x)$ to Eq. (22) satisfying (23).

We recall (see Sect. 2.3) that Eq. (22) can be (formally) written as the following integral equality:

$$U(x) = \int_{-1}^{1} g(x,z) U^{-n}\left(zU(z) + \frac{3}{2}\mathcal{U}(z)\right) dz + K\sqrt{1-x^2}, \quad x \in [-1,1] \tag{64}$$

with

$$\mathcal{U}_k(z) = \begin{cases} \displaystyle\int_z^1 U_k(y)\,dy & \text{for } z > 0 \\ \displaystyle -\int_{-1}^z U_k(y)\,dy & \text{for } z < 0. \end{cases}$$

As we did in the zero toughness case, we will solve (64) by a fixed point argument. But we first need to solve an approximate problem to avoid the singularity in (64) when $U = 0$. Because of the term $\mathcal{V}(z)$, the approximation that we use here is slightly different from that of the previous section:

Lemma 4.4 (Construction of an approximate solution). *For all $k \in \mathbb{N}$, there exists a continuous function $U_k\colon [-1,1] \to]0,+\infty[$ such that for all $x \in (-1,1)$,*

$$U_k(x) = \frac{1}{k} + \int_{-1}^{1} g(x,z)\,(U_k(z))^{-n}\left(zU_k(z) + \frac{3}{2}\mathcal{U}_k(z)\right) dz + K\sqrt{\frac{1}{k} + 1 - x^2}. \tag{65}$$

Furthermore, U_k is non-negative in $\mathbb{R}$ and is C^1 in $(-1,1)$.

Proof. The proof follows that of Lemma 4.2 with minor modifications. We consider the closed convex set of $C([-1,1])$

$$S = \{V \in C([-1,1])\colon \frac{1}{k} \le V \le A \text{ in } [-1,1],$$
$$x \mapsto V(x) \text{ even in } [-1,1] \text{ and non-increasing in } [0,1]\}$$

(for a positive number $A > 0$ to be fixed later) and the operator $\mathcal{T}: S \to C([-1,1])$ which maps $V \in S$ to the function

$$U(x) = \frac{1}{k} + \int_{-1}^{1} g(x,z)\,(V(z))^{-n}\left(zV(z) + \frac{3}{2}\mathcal{V}(z)\right) dz + K\sqrt{\frac{1}{k} + 1 - x^2}.$$

Note that since $x \mapsto g(x,z)$ is even (see Proposition 3.5-2)), so is the function U, and using the fact that $z \mapsto g(x,z)$ is odd, we can rewrite this equality as

$$U(x) = \frac{1}{k} + 2\int_{0}^{1} g(x,z)\,(V(z))^{-n}\left(zV(z) + \frac{3}{2}\mathcal{V}(z)\right) dz + K\sqrt{\frac{1}{k} + 1 - x^2}.$$

Proposition 3.5-(2) implies that the integrand is non-negative in $(0, 1)$, and so it is readily seen that

$$U(x) \geq \frac{1}{k}.$$

Using now Proposition 3.5-(1) implies that $x \mapsto U(x)$ is non-increasing on $(0, 1)$. Next, we note that for $V \in S$, we have

$$\mathcal{V}(z) = \int_z^1 V(y)\,dy \leq (1-z)V(z)$$

and so, for $z \in (0, 1)$,

$$zV(z) + \frac{3}{2}\mathcal{V}(z) \leq \frac{3}{2}V(z). \tag{66}$$

We thus have (using (37))

$$\begin{aligned} U(x) &\leq \frac{1}{k} + 3\int_0^1 g(x,z)\,(V(z))^{1-n}\,dz + K\sqrt{1-x^2} \\ &\leq \frac{1}{k} + 3k^{n-1} + K\sqrt{1+\frac{1}{k}}, \end{aligned}$$

so we choose

$$A = \frac{1}{k} + 3k^{n-1} + K\sqrt{1+\frac{1}{k}}$$

and we deduce

$$\mathcal{T}(S) \subset S.$$

Moreover, Proposition 3.6 (see (46) with $a = 0$) implies that U' is C^1 in $(-1, 1)$ and

$$|U'(x)| \leq C(k, A)\sqrt{1-x^2} + 2K\left(\frac{1}{k} + 1 - x^2\right)^{-\frac{1}{2}} \quad \forall x \in (-1, 1)$$

and so $\mathcal{T}(S)$ is equi-Lipschitz continuous. Hence $\mathcal{T}(S)$ is compact. Finally, using once again the fact that $|g(x,z)| \leq 1$ together with Lebesgue dominated convergence Theorem, it is easy to show that $\mathcal{T}$ is a continuous operator. We can thus use Schauder's fixed point Theorem and deduce that $\mathcal{T}$ has a fixed point U_k. □

We then derive uniform (with respect to k) estimates for these approximate solutions.

Lemma 4.5 (Uniform estimates). *Let $K > 0$ and assume $n \in [1, 4)$. There exists a constant $C > 0$ depending only on n such that for all $k \in \mathbb{N}$, the function U_k constructed in Lemma 4.4 satisfies*

$$U_k(x) \geq K(1 - x^2)^{\frac{1}{2}} \quad \text{for all } x \in (-1, 1) \tag{67}$$

$$|U_k(x)| \leq \frac{C}{k} + C(K^{1-n} + K)(1 - x^2)^{\frac{1}{2}} \quad \text{for all } x \in (-1, 1) \tag{68}$$

$$|U_k'(x)| \leq C(K^{1-n} + K)(1 - x^2)^{-\frac{1}{2}} \quad \text{for all } x \in (-1, 1). \tag{69}$$

Proof. Estimate (67) follows immediately from (65) (note that the first two terms in the right hand side are non-negative). Next, we note (using (66)), that the odd function

$$f_k(z) = (U_k(z))^{-n} \left(zU_k(z) + \frac{3}{2}\mathcal{U}_k(z) \right)$$

satisfies

$$f_k(z) \leq \frac{3}{2} U_k^{1-n}(z) \leq CK^{1-n}(1 - z^2)^{\frac{1-n}{2}}.$$

In particular, f_k satisfies the condition of Propositions 3.6 and 3.7 with $a = \frac{1-n}{2} > -\frac{3}{2}$ provided $n < 4$. Proposition 3.6 now implies (68), and Proposition 3.7 gives

$$\begin{aligned} |U_k'(x)| &\leq CK^{1-n} F(\sqrt{1 - x^2}) + K \frac{1}{\sqrt{1 - x^2}} \\ &\leq C(K^{1-n} + K) \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

(recall that F is given by (47)), which is exactly (69). □

We can now pass to the limit $k \to \infty$ and complete the proof of Proposition 2.3.

Proof of Proposition 2.3. Estimates from Lemma 4.5 together with the fact that $U_k(\pm 1) = \frac{1}{k}$ implies that we can extract a subsequence U_p which converges locally uniformly in $(-1, 1)$ towards a continuous function U which vanishes at ± 1.

First, we can pass to the limit in (65) by using (67), (37) and Legesgue dominated convergence theorem (note that $1 - \frac{n}{2} > -1$ when $n < 4$). We deduce that U satisfies (64), and Proposition 3.6 implies that U solves

$$I(U)' = U^{-n} \left(zU + \frac{3}{2}\mathcal{U} \right) \quad \text{in } (-1, 1).$$

In order to study the behavior of U near $x = \pm 1$, we write

$$U(x) = W(x) + K\sqrt{1 - x^2}$$

with

$$W(x) = \int_{-1}^{1} g(x,z)U^{-n}\left(zU(z) + \frac{3}{2}\mathcal{U}(z)\right)dz$$
$$= \int_{-1}^{1} g(x,z)f(z)dz$$

where (proceeding as above), we see that $f(z)$ satisfies

$$f(z) \le \frac{3}{2}U_k^{1-n}(z) \le CK^{1-n}(1-z^2)^{\frac{1-n}{2}}.$$

Proposition 3.7 with $a = \frac{1-n}{2}$ (see also (48)) thus implies that there exists a constant C such that

$$|W(x)| \le \begin{cases} C(1-x^2)^{\frac{3}{2}} & \text{if } 1 < n < 2 \\ C(1-x^2)^{\frac{3}{2}} \ln\left(\frac{1}{1-x^2}\right) & \text{if } n = 2 \\ C(1-x^2)^{\frac{5-n}{2}} & \text{if } 2 < n < 4 \end{cases}$$

and the proof is now complete. □

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A. Some Technical Results

A.1. An explicit solution.

Lemma A.1 (The special case $n = 1$). *For $U(x) = \frac{4}{9}(1-x^2)_+^{\frac{3}{2}}$, we have*

$$I(U)' = x \quad \textit{for} \quad x \in [-1, 1].$$

Proof. We first compute the Riesz potential $\mathcal{I}_\beta(U)$ for $\beta \in (0,1)$ by using [7, Lemma 4.1] and get

$$\mathcal{I}_\beta(U) = \frac{4}{9}C_{3,\beta,1} \times {}_2F_1\left(\frac{1-\beta}{2}, -2; 1; x^2\right).$$

Hence differentiating and using the fact that ${}_2F_1(a, -1; c; z) = 1 - \frac{a}{c}z$, we get

$$(\mathcal{I}_\beta(U))' = -\frac{4}{9}C_{3,\beta,1}(1-\beta) \times {}_2F_1\left(\frac{3-\beta}{2}, -1; 2; x^2\right)(2x)$$
$$= D_\beta\left(\frac{3-\beta}{4}x^2 - 1\right)2x = D_\beta\frac{3-\beta}{2}x^3 - 2D_\beta x$$

where

$$D_\beta = \frac{4}{9}\cdot\frac{(1-\beta)\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1-\beta}{2}\right)}{2^\beta\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{3+\beta}{2}\right)} = \frac{4}{9}\cdot\frac{2^{1-\beta}\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3-\beta}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{3+\beta}{2}\right)}.$$

Then

$$D_\beta \to \frac{4}{9} \cdot \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)\Gamma(3)} = \frac{1}{6}.$$

Hence

$$(\mathcal{I}_1(U))' = \frac{1}{6}x^3 - \frac{1}{3}x$$

and

$$(I(U))' = (\mathcal{I}_1(U))''' = x.$$

□

A.2. Proof of Lemma 3.4.

Proof of Lemma 3.4. When $x \geq z$ or $x \leq -z$, the $y \mapsto G(x, y)$ has no singularities in the interval $(-z, z)$, and a simple integration by parts yields

$$\begin{aligned}\int_{-z}^{z} G(x, y)\, dy &= (y-x)G(x, y)\Big|_{y=-z}^{y=z} - \int_{-z}^{z} (y-x)\partial_y G(x, y)\, dy \\ &= (z-x)G(x, z) + (z+x)G(x, -z) + \int_{-z}^{z} (x-y)\partial_y G(x, y)\, dy.\end{aligned}$$

Lemma 3.3 implies

$$\int_{-z}^{z} (x-y)\partial_y G(x, y)\, dy = \frac{1}{\pi}\sqrt{1-x^2}[\arcsin(z) - \arcsin(-z)] = \frac{2}{\pi}\sqrt{1-x^2}\arcsin(z)$$

and (31) follows.
When $-z \leq x \leq z$, we need to split the integral:

$$\int_{-z}^{z} G(x, y)\, dy = \int_{-z}^{x} G(x, y)\, dy + \int_{x}^{z} G(x, y)\, dy.$$

We then proceed as before to evaluate those integrals, after noticing that the function $y \mapsto (y-x)G(x, y)$ vanishes for $y = x$:

$$\begin{aligned}\int_{-z}^{x} G(x, y)\, dy &= (y-x)G(x, y)\Big|_{y=-z}^{y=x} - \int_{-z}^{x} (y-x)\partial_y G(x, y)\, dy \\ &= (z+x)G(x, -z) + \int_{-z}^{x} (x-y)\partial_y G(x, y)\, dy \\ &= (z+x)G(x, -z) + \frac{1}{\pi}\int_{-z}^{x} \frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}\, dy \\ &= (z+x)G(x, -z) + \frac{1}{\pi}\sqrt{1-x^2}\big[\arcsin(x) + \arcsin(z)\big]\end{aligned}$$

and

$$\begin{aligned}\int_x^z G(x,y)\,dy &= (y-x)G(x,y)\Big|_{y=x}^{y=z} - \int_{-z}^{x}(y-x)\partial_y G(x,y)\,dy\\ &= (z-x)G(x,z) - \int_x^z (y-x)\partial_y G(x,y)\,dy\\ &= (z-x)G(x,z) + \frac{1}{\pi}\int_x^z \frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}\,dy\\ &= (z-x)G(x,z) + \frac{1}{\pi}\sqrt{1-x^2}\big[\arcsin(z)-\arcsin(x)\big].\end{aligned}$$

The result follows. □

B. Boundary Behavior in the Critical Case

In this section, we complete the proof of Proposition 2.2 by deriving the boundary behavior of the function $U(x)$ in the critical case $n = \frac{4}{3}$ when $K = 0$. More precisely, we will show that in that case we have

$$U(x) \sim (2/9\pi)^{\frac{3}{4}}(1-x^2)^{\frac{3}{2}}|\ln(1-x^2)|^{\frac{3}{4}}, \tag{70}$$

when $x \to \pm 1$.
Since U is even, it is enough to look at the case $x \to 1$, and we recall that

$$U(x) = \int_{-1}^{1} g(x,z)f(z)dz$$

where

$$f(z) = z\,(U(z))^{-\frac{1}{3}} = z\left(\frac{U(z)}{(1-z^2)^{3/2}}\right)^{-\frac{1}{3}}(1-z^2)^{\frac{2}{3}}.$$

We thus denote

$$h(x) = \left(\frac{U(x)}{(1-x^2)^{\frac{3}{2}}}\right)^{-\frac{1}{3}}.$$

Inequality (62) and the inequality above imply

$$0 < c \le \frac{U(x)}{(1-x^2)^{\frac{3}{2}}} \le C|\ln(1-x^2)|$$

and so

$$\frac{c}{|\ln(1-x^2)|^{\frac{1}{3}}} \le h(x) \le C.$$

Proceeding as in the proof of Proposition 3.10 (in the case $a = -\frac{1}{2}$), we can prove the following lemma:

Lemma B.1. *In the case $n = \frac{4}{3}$, we have*

$$\frac{U'(x)}{\sqrt{1-x^2}} = -\frac{1}{2\pi}\int_{2(1-x^2)}^{1} \frac{h(\sqrt{1-v})}{v} dv + \tilde{R}(x)$$

where $\tilde{R}(x)$ is a bounded function for $x \geq 0$ and

$$h(x) = \left(\frac{U(x)}{(1-x^2)^{\frac{3}{2}}}\right)^{-\frac{1}{3}}.$$

Postponing the proof of this lemma to the end of this section, we now define $H(y) = h(\sqrt{1-y})$ and

$$E(y) = \int_y^1 \frac{H(\tau)}{2\pi\tau} d\tau.$$

Remark that E is differentiable and satisfies

$$E'(y) = -\frac{H(y)}{2\pi y}. \tag{71}$$

We also have

$$E(y) \geq \int_y^1 \frac{c}{2\pi |\ln(\tau)|^{\frac{1}{3}}} \cdot \frac{d\tau}{\tau}$$

and in particular,

$$E(y) \to +\infty \quad \text{as } y \to 0. \tag{72}$$

$$U'(x) \sim -(1-x^2)^{\frac{1}{2}} E(2(1-x^2)) \quad \text{as } x \to 1^-.$$

We claim that this implies

$$U(x) \sim \frac{2}{3}(1-x^2)^{\frac{3}{2}} E(2(1-x^2)) \quad \text{as } x \to 1^-. \tag{73}$$

Indeed,

$$\int_x^1 U'(y)dy \sim -\int_x^1 (1-y^2)^{\frac{1}{2}} E(2(1-y^2))dy$$

implies

$$U(x) \sim \int_0^{1-x^2} z^{\frac{1}{2}} E(2z)dz.$$

Integrating by parts, we get,

$$\begin{aligned}\int_0^{1-x^2} z^{\frac{1}{2}} E(2z)dz &= \frac{2}{3}\left[E(2z)z^{\frac{3}{2}}\right]_0^{1-x^2} - \frac{4}{3}\int_0^{1-x^2} E'(2z)z^{\frac{3}{2}}dz \\ &= \frac{2}{3}(1-x^2)^{\frac{3}{2}} E(2(1-x^2)) + \frac{2}{3\pi}\int_0^{1-x^2} H(z)z^{\frac{1}{2}}dz\end{aligned}$$

$$= \frac{2}{3}(1-x^2)^{\frac{3}{2}} E(2(1-x^2)) + \mathcal{O}((1-x^2)^{\frac{3}{2}})$$

(where we used (71) and the fact that h is bounded). In view of (72), it follows that (73) indeed holds true.
Now, Eq. (73) implies that the function H satisfies

$$H(y) \sim \left(\frac{2}{3}E(2y)\right)^{-\frac{1}{3}} \quad \text{as } y \to 0.$$

Furthermore, L'Hospital's rule implies

$$\lim_{y\to 0} \frac{E(2y)}{E(y)} = \lim_{y\to 0} \frac{H(2y)}{H(y)} = \left(\lim_{y\to 0} \frac{E(2y)}{E(y)}\right)^{-\frac{1}{3}}$$

and so

$$E(2y) \sim E(y).$$

We can thus write

$$H(y) \sim \left(\frac{2}{3}E(y)\right)^{-\frac{1}{3}} \quad \text{as } y \to 0.$$

In view of (71), this implies

$$-4\pi y E'(y) \sim \left(\frac{2}{3}E(Ay)\right)^{-\frac{1}{3}} \quad \text{as } y \to 0,$$

or

$$(E^{\frac{4}{3}})'(y) \sim -\frac{1}{3\pi y} \quad \text{as } y \to 0.$$

This finally gives

$$E(y) \sim (3/2)^{\frac{1}{4}} (3\pi)^{-\frac{3}{4}} |\ln y|^{\frac{3}{4}},$$

and (73) implies finally

$$U(x) \sim (2/9\pi)^{\frac{3}{4}} (1-x^2)^{\frac{3}{2}} |\ln(1-x^2)|^{\frac{3}{4}}.$$

The proof of Proposition 2.2 is now complete.

Proof of Lemma B.1. We proceed as in the proof of Proposition 3.10 (in the case $a = -\frac{1}{2}$). First, we have

$$\frac{U'(x)}{(1-x^2)^{\frac{1}{2}}} = -\frac{1}{2\pi} \int_0^{\frac{1}{1-x^2}} \Theta(x,u)du$$

where the integrand $\Theta(x,u)$ is given by

$$\Theta(x,u) = \operatorname{argsinh}\left(\frac{2x\sqrt{1-(1-x^2)u}\sqrt{u}}{|1-u|}\right) u^{-\frac{1}{2}} h(\sqrt{1-(1-x^2)u}).$$

Next, we write, for $\frac{1}{1-x^2} \geq 2$:

$$\begin{aligned}\frac{U'(x)}{\sqrt{1-x^2}} &= -\frac{1}{2\pi}\int_0^{\frac{1}{1-x^2}} \Theta(x,u)du \\ &= -\frac{1}{2\pi}\int_0^{2} \Theta(x,u)du - \frac{1}{2\pi}\int_2^{\frac{1}{1-x^2}} \Theta(x,u)du \\ &= I_1 + I_2.\end{aligned}$$

where the first term satisfies

$$|I_1| \leq C||h||_\infty \int_0^2 \text{argsinh}\left(\frac{\sqrt{u}}{|1-u|}\right) u^{-\frac{1}{2}}du \leq C||h||_\infty.$$

and the second term can be written as

$$\begin{aligned}I_2 &= -\frac{1}{2\pi}\int_2^{\frac{1}{1-x^2}} 2x\sqrt{1-(1-x^2)u}h(\sqrt{1-(1-x^2)u})u^{-1}du + R \\ &= -\frac{1}{2\pi}\int_{2(1-x^2)}^{1} 2x\sqrt{1-v}h(\sqrt{1-v})v^{-1}dv + R\end{aligned}$$

where

$$\begin{aligned}R &\leq C||h||_\infty \int_2^{\frac{1}{1-x^2}} u^{-2}du \\ &\leq C.\end{aligned}$$

Finally, we write

$$\begin{aligned}&-\frac{1}{2\pi}\int_{2(1-x^2)}^{1} 2x\sqrt{1-v}h(\sqrt{1-v})v^{-1}dv \\ &\quad= -\frac{1}{2\pi}\int_{2(1-x^2)}^{1} 2\sqrt{1-v}h(\sqrt{1-v})v^{-1}dv \\ &\quad\quad+ \frac{1}{2\pi}\int_{2(1-x^2)}^{1} 2(1-x)\sqrt{1-v}h(\sqrt{1-v})v^{-1}dv\end{aligned}$$

where the second term is bounded as $x \to 1$ (because $\frac{(1-x)}{v} \leq C$ for $v \geq 2(1-x^2)$), and

$$\begin{aligned}&-\frac{1}{2\pi}\int_{2(1-x^2)}^{1} 2\sqrt{1-v}h(\sqrt{1-v})v^{-1}dv \\ &\qquad= -\frac{1}{2\pi}\int_{2(1-x^2)}^{1} 2h(\sqrt{1-v})v^{-1}dv \\ &\qquad\quad+ \frac{1}{2\pi}\int_{2(1-x^2)}^{1} 2(1-\sqrt{1-v})h(\sqrt{1-v})v^{-1}dv\end{aligned}$$

where the second term is again bounded as $x \to 1$ (because $\frac{(1-\sqrt{1-v})}{v} \leq C$). The lemma follows. □

C. Derivation of the Pressure Law

We recall here the main step of the derivation of the pressure law from linear elasticity equations in the particular geometry of a crack of plain strain. These computations can be found elsewhere [9, 17] and are recalled here for the reader's sake.

C.1. Linear elasticity equations. The *strain tensor* $\boldsymbol{\epsilon}$ is related to the *displacement* $\mathbf{u}$ through the following equality

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) \tag{74}$$

where $\nabla \mathbf{u}$ denotes the Jacobian matrix of u. The *stress tensor* is denoted by $\boldsymbol{\sigma}$.
We next recall the equations of linear elasticity.

- *Force equilibrium* considerations show that the components of the stress tensor must satisfy the equations

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{F} = 0$$

  where $\mathbf{F}$ denotes body forces (such as gravity).
- The *stress–strain* relations for an isotropic linearly elastic material can be written in the form:

$$\boldsymbol{\epsilon} = \frac{1}{E}(\boldsymbol{\sigma} - \nu[\mathrm{trace}(\boldsymbol{\sigma}) I - \boldsymbol{\sigma}]) \tag{75}$$

  where E is Young's modulus and ν is Poisson's ratio.

C.2. 2D plane-strain problems.

- The components of the symmetric 2-tensor $\boldsymbol{\sigma}$ are denoted by σ_{xx}, σ_{yy}, σ_{zz}, $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$ and $\sigma_{yz} = \sigma_{zy}$.
- The components of the vector field $\mathbf{u}$ are denoted by u_x, u_y and u_z.
- The components of the vector field $\boldsymbol{\epsilon}$ are denoted by ϵ_{xx}, ϵ_{yy}, ϵ_{zz}, $\epsilon_{xy} = \epsilon_{yx}$, $\epsilon_{xz} = \epsilon_{zx}$ and $\epsilon_{yz} = \epsilon_{zy}$.

If the solid is in a state of plain strain (parallel to the xy plane), then $u_z = 0$ and the components u_x and u_y of the displacement are independent of the z coordinate. As a consequence, the strain tensor components ϵ_{zz}, $\epsilon_{xz} = \epsilon_{zx}$ and $\epsilon_{yz} = \epsilon_{zy}$ are zero, and the remaining components are independent of z.
We note that the three remaining strain components are defined in terms of two displacements. This implies that they cannot be specified independently. In fact, we can easily verify that if the displacement are continuously differentiable, then the strain tensor components must satisfy the following *compatibility condition*

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}. \tag{76}$$

Furthermore, the third equation in (75) implies

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

and the last two equations give $\sigma_{xz} = \sigma_{yz} = 0$.

The stress–strain relations (75) can thus be rewritten as:

$$\begin{cases} \epsilon_{xx} = \frac{1}{2G}[\sigma_{xx} - \nu(\sigma_{xx} + \sigma_{yy})] \\ \epsilon_{yy} = \frac{1}{2G}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{yy})] \\ \epsilon_{xy} = \frac{1}{2G}\sigma_{xy} \end{cases} \tag{77}$$

where $G = \frac{1}{2}\frac{E}{1+\nu}$ is the shear modulus and the equilibrium conditions (without body forces inside the solid, $F \equiv 0$) yield

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \\ \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0. \end{cases} \tag{78}$$

At this point, we note that (77) and (78) provide 5 equations with 5 unknowns (σ_{xx}, σ_{yy}, σ_{xy}, u_x, u_y).

The Airy stress function. The equilibrium Eq. (78) imply the existence of a function $U(x, y)$ (the Airy stress function) such that the three components of the stress tensor can be written as

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}.$$

Furthermore, the compatibility condition (76) and equation (77) imply

$$\frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} = 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$$

which yields:

$$\Delta^2 U = 0$$

(so U is biharmonic).
Recalling that ϵ_{xx}, ϵ_{yy} and ϵ_{xy} are defined in terms of the displacements u_x and u_y by (74), we finally rewrite (77) as follows:

$$\begin{cases} \frac{\partial u_x}{\partial x} = \frac{1}{2G}\left[\frac{\partial^2 U}{\partial y^2} - \nu\left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial x^2}\right)\right] \\ \frac{\partial u_y}{\partial y} = \frac{1}{2G}\left[\frac{\partial^2 U}{\partial x^2} - \nu\left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial x^2}\right)\right] \\ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -\frac{1}{G}\frac{\partial^2 U}{\partial x \partial y}. \end{cases} \tag{79}$$

We now have reduced the problem to finding a biharmonic potential $U(x, y)$ and the displacements $u_x(x, y)$, $u_y(x, y)$ such that (79) holds (together with some appropriate boundary conditions).

C.3. Derivation of the pressure law for a 2-D crack on an infinite domain. We consider a fracture of opening w in an infinite solid occupying the whole space $\mathbb{R}^2$. The fracture is assumed to be symmetric with respect to the $y = 0$ axis, so that we only need to consider the problem in the upper half $\{y > 0\}$. Along $y = 0$, we have the following boundary conditions:

$$\sigma_{xy}(x,0) = 0 \quad \text{and} \quad u_y(x,y) = \frac{1}{2}w(x) \quad \text{for all } x \in \mathbb{R}$$

and we assume

$$\sigma_{ij} \longrightarrow 0 \quad \text{as } |(x,y)| \to \infty.$$

Our goal is to determine the pressure

$$p(x) = -\sigma_{yy}(x,0).$$

The main result of this section is the following:

Theorem C.1. *The pressure $p(x)$ satisfies*

$$p(x) = \frac{E}{4(1-\nu^2)}(-\Delta)^{1/2}w(x) \quad \textit{for} \quad x \in \mathbb{R}.$$

Proof. We use the Fourier transform with respect to x. Denoting

$$\widehat{U}(k,y) = \int_{\mathbb{R}} U(x,y)e^{-ikx}\,dx,$$

the biharmonic equation yields

$$\left(\frac{d^2}{dy^2} - k^2\right)^2 \widehat{U}(k,y) = 0$$

and so (using the conditions as $|y| \to \infty$)

$$\widehat{U}(k,y) = (A(k) + B(k)y)e^{-|k|y} \quad \text{for all } k \in \mathbb{R},\ y > 0.$$

Next, Eq. (79) implies

$$\begin{cases} -ik\widehat{u_x}(k,y) = \frac{1}{2G}[(1-\nu)\frac{\partial^2\widehat{U}}{\partial y^2}(k,y) + \nu k^2\widehat{U}(k,y)] \\ \frac{\partial\widehat{u_y}}{\partial y}(k,y) = \frac{1}{2G}[-k^2(1-\nu)\widehat{U}(k,y) - \nu\frac{\partial^2\widehat{U}}{\partial y^2}] \\ \frac{\partial\widehat{u_x}}{\partial y}(k,y) - ik\widehat{u_y}(k,y) = \frac{ik}{G}\frac{\partial\widehat{U}}{\partial y}(k,y). \end{cases} \tag{80}$$

The first equation yields

$$-i\widehat{u_x} = \frac{1}{2G}\left[(1-\nu)\frac{1}{k}\frac{\partial^2\widehat{U}}{\partial y^2} + \nu k\widehat{U}\right]$$

and the last equation then implies

$$\begin{aligned} \widehat{u_y}(k,y) &= -\frac{i}{k}\frac{\partial\widehat{u_x}}{\partial y} - \frac{1}{G}\frac{\partial\widehat{U}}{\partial y} \\ &= \frac{1}{2G}\left[(1-\nu)\frac{1}{k^2}\frac{\partial^3\widehat{U}}{\partial y^3} + (\nu-2)\frac{\partial\widehat{U}}{\partial y}\right]. \end{aligned}$$

A simple computation gives

$$\frac{\partial \widehat{U}}{\partial y}(k, y) = (B(k) - |k|A(k) - |k|B(k)y)e^{-|k|y}$$
$$\frac{\partial^2 \widehat{U}}{\partial y^2}(k, y) = (-2|k|B(k) + k^2 A(k) + k^2 B(k)y)e^{-|k|y}$$
$$\frac{\partial^3 \widehat{U}}{\partial y^3}(k, y) = (3k^2 B(k) - |k|^3 A(k) - |k|^3 B(k)y)e^{-|k|y}.$$

We recall that $\sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}$ and so

$$\widehat{\sigma_{xy}} = ik \frac{\partial \widehat{U}(k, y)}{\partial y}.$$

In particular, the condition $\sigma_{xy}(x, 0) = 0$ for all $x \in \Omega$ implies $\frac{\partial \widehat{U}}{\partial y}(k, 0) = 0$ for all $k \in \mathbb{N}$ and so

$$B(k) = |k|A(k) \quad \text{for all } k \in \mathbb{R}.$$

The condition $u_y = \frac{1}{2} w$ then gives

$$\frac{1}{2G}(1 - \nu) \frac{1}{k^2} \frac{\partial^3 \widehat{U}}{\partial y^3}(k, 0) = \frac{1}{2} \widehat{w}(k) \quad \text{for all } k \in \mathbb{R},$$

which implies

$$\frac{2(1 - \nu)}{G} |k| A(k) = \widehat{w}(k) \quad \text{for all } k \in \mathbb{R}.$$

We deduce

$$\widehat{U}(k, 0) = A = \frac{G}{2(1 - \nu)} \frac{1}{|k|} \widehat{w}(k) = \frac{E}{4(1 - \nu^2)} \frac{1}{|k|} \widehat{w}(k) \quad \text{for all } k \in \mathbb{R},$$

and so

$$\widehat{p}(k) = -\widehat{\sigma_{yy}}(k, 0) = k^2 \widehat{U}(k, 0) = \frac{E}{4(1 - \nu^2)} |k| \widehat{w}(k) \quad \text{for all } k \in \mathbb{R},$$

which is the Fourier transform of the equation

$$p(x) = \frac{E}{4(1 - \nu^2)} (-\Delta)^{1/2} w(x) \quad \text{for } x \in \mathbb{R}.$$

□

D. Proof of Lemma 1.3

In this section, we give the proof of Lemma 1.3, which relates the behavior of u and p at the tip of the fracture. For that purpose, we rewrite (12) as

$$-I(u) = \frac{4(1-\nu^2)}{E} p(x). \tag{81}$$

Proof of Lemma 1.3. We first prove (14). For that we use (81) and Lemma (3.1) to write

$$u(x) = \frac{4(1-\nu^2)}{E} \int_{-1}^{1} G(x,y)p(y)\,dy \quad \text{for } x \in (-1,1) \tag{82}$$

and so using Lemma 3.3, we get

$$u'(x) = \frac{4(1-\nu^2)}{\pi E} \int_{-1}^{1} \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \frac{p(y)}{y-x}\,dy.$$

We deduce

$$u'(x)\sqrt{1-x} = \frac{4(1-\nu^2)}{\pi E} \frac{1}{\sqrt{1+x}} \int_{-1}^{1} \frac{\sqrt{1-y^2}}{y-x} p(y)\,dy$$

hence

$$\begin{aligned}
\lim_{x\to 1^-} u'(x)\sqrt{1-x} &= \frac{4(1-\nu^2)}{\pi E} \frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{\sqrt{1-y^2}}{y-1} p(y)\,dy \\
&= -\frac{4(1-\nu^2)}{\pi E} \frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{\sqrt{1+y}}{\sqrt{1-y}} p(y)\,dy
\end{aligned}$$

which is (14).

We now turn to the proof of (13). First, we recall that the square root of the Laplacian can also be represented by a singular integral:

$$(-\Delta)^{1/2}(u) = \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^2}\,dy.$$

In view of the pressure law (12), we deduce:

$$p(x) = \frac{E}{4(1-\nu^2)} (-\Delta)^{1/2} u = \frac{E}{4(1-\nu^2)} \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{+\infty} \frac{u(x)-u(y)}{|x-y|^2}\,dy$$

for all $x \in \mathbb{R}$. In particular, using the fact that $\operatorname{supp} u = (-1,1)$, we deduce that for $x > 1$, we have

$$p(x) = -\frac{E}{4(1-\nu^2)} \frac{1}{\pi} \int_{-1}^{1} \frac{u(y)}{|x-y|^2}\,dy \quad \text{for all } x > 1$$

(note that the principal value is no longer necessary here). Using (82) in this last expression, we get

$$p(x) = -\frac{1}{\pi} \int_{-1}^{1} \frac{1}{|x-y|^2} \left\{ \int_{-1}^{1} G(y,z)p(z)\,dz \right\} dy \quad \text{for all } x > 1$$

and so

$$-\sqrt{x-1}\,p(x) = \frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{\sqrt{x-1}G(y,z)}{|x-y|^2}p(z)\,dy\,dz \quad \text{for all } x>1.$$

Using the change of variable $t=\frac{1-y}{x-1}$ we have

$$\begin{aligned}\int_{-1}^{1}\frac{\sqrt{x-1}G(y,z)}{|x-y|^2}\,dy &= \int_0^{\frac{2}{x-1}}\frac{\sqrt{x-1}G(1-(x-1)t,z)}{(x-1)^2|1+t|^2}(x-1)dt\\ &= \int_0^{\frac{2}{x-1}}\frac{G(1-(x-1)t,z)}{\sqrt{x-1}}\frac{dt}{|1+t|^2}\end{aligned}$$

and formula (25) implies that for $y\in(-1,1)$, $z\in(-1,1)$,

$$\lim_{x\to 1^+}\frac{G(1-(x-1)t,z)}{\sqrt{x-1}} = \frac{1}{\pi}\frac{\sqrt{2t}\sqrt{1-z^2}}{|1-z|} = \frac{1}{\pi}\frac{\sqrt{2t}\sqrt{1+z}}{\sqrt{1-z}}.$$

We deduce (arguing as in Sect. 3 to justify exchanging limits and integrals)

$$\begin{aligned}\lim_{x\to 1^+} -\sqrt{x-1}p(x) &= \frac{1}{\pi}\int_{-1}^{1}\int_0^{\infty}\frac{1}{\pi}\frac{\sqrt{2t}\sqrt{1+z}}{\sqrt{1-z}}\frac{dt}{|1+t|^2}p(z)\,dz\\ &= \frac{1}{\pi^2}\int_{-1}^{1}\int_0^{\infty}\frac{\sqrt{2t}}{|1+t|^2}dt\frac{\sqrt{1+z}}{\sqrt{1-z}}p(z)\,dz.\end{aligned}$$

The result now follows using the fact that $\int_0^\infty \frac{\sqrt{2t}}{|1+t|^2}dt = \frac{\pi}{\sqrt{2}}$. □

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